# Instantaneous Control of the linear wave equation

Nils Altmüller, Lars Grüne, and Karl Worthmann

***Abstract*— We are concerned with the one dimensional linear wave equation with Dirichlet boundary condition and Neumann boundary control. It has been shown numerically that this hyperbolic partial differential equation can be stabilized by instantaneous control, i.e. model predictive control with the shortest feasible prediction and optimization horizon. Our contribution is the complete theoretical analysis.**

## I. INTRODUCTION

In this paper we consider the one dimensional linear wave equation with Dirichlet boundary condition and Neumann boundary control. Exact controllability in finite time of this hyperbolic partial differential equation has been shown, cf. [5]. However, an optimization horizon of at least $2L/c$ is required in this approach. Here $L$ denotes the length of the domain and $c$ the propagation speed of the wave. Since the complexity and – as a consequence – the numerical effort of the corresponding optimal control problem grows rapidly with the horizon length, we pursue a receding horizon approach in order to reduce this horizon and thus the numerical effort significantly.

Model predictive control (MPC) – often also termed receding horizon control – relies on an iterative online solution of finite horizon optimal control problems. To this end, a performance criterion is optimized over the predicted trajectories of the system, cf. [10]. Typically, this optimization based technique is used in order to deal with optimal control problems on an infinite horizon. Hence, stabilizing terminal constraints or terminal costs are introduced in order to ensure stability, cf. [7]. However, the construction of an appropriate Lyapunov–function which can be used as a terminal cost remains a challenging task. In order to avoid this drawback we consider unconstrained model predictive control which seems to be predominant in practical – and in particular industrial – applications, cf. [9].

Moreover, using unconstrained MPC is motivated by numerical results which indicate that MPC performs well in this setting, [6]. Indeed, this observation even holds for the shortest feasible prediction horizon in our appraoch – a special case which is also termed instantaneous control in the literatur. Our contribution is the complete theoretical analysis of this observation. To be more precise, we exploit the stability and performance analysis which is presented in [1] in order to prove stability of the closed loop rigorously. However, in constrast to prior results for finite dimensional linear systems which are based on the explicit knowledge of the optimal value function, cf. [8], [11], our results are derived from a controllability condition which is easier to verify – especially in view of generalizations to nonlinear and/or infinite dimensional systems, cf. [3], [4].

The paper is organised as follows. In Section II we give a precise problem formulation and shortly summarize the needed results from [1]. In the ensuing section we construct suitable stage costs and explain why it is not possible to obtain comparable results which employ the classical energy norm of the system in the MPC cost functional. In Section IV we deduce our central controllability condition which allows for concluding stability of the closed loop. In the following section we illustrate the results by means of a numerical example and show that the derived bounds are tight. In Section VI we conclude the paper and give a short outlook on future work.

## II. PROBLEM FORMULATION AND PRELIMINARIES

We are concerned with the one dimensional linear wave equation with homogeneous Dirichlet boundary condition on the left and Neumann boundary control on the right boundary of the domain $\Omega = (0, L)$:

$$y_{tt}(x,t) - c^2 y_{xx}(x,t) = 0 \qquad \text{on } \Omega \times (0,\infty) \tag{1}$$

$$y(0,t) = 0 \qquad \text{on } (0,\infty) \tag{2}$$

$$y_x(L,t) = u(t) \qquad \text{on } (0,\infty) \tag{3}$$

Here $c \neq 0$ denotes the propagation speed of the wave. The initial data are given by

$$y(x,0) = y_0(x) \qquad \text{and} \qquad y_t(x,0) = y_1(x)$$

with $(y_0, y_1) \in C([0,L]) \times L^2([0,L])$. Moreover, the solution space is given by

$$X = \{y : y \in L^2(0,t^*; H^1([0,L])) \text{ with } y_t \in L^2(0,t^*; L^2([0,L])),\ \forall t^* > 0\}$$

and $u \in L^\infty([0,\infty])$. Our goal consists of steering the system to the origin, which is the unique equilibrium. To this end, we consider unconstrained model predictive control (MPC) with the cost functional

$$J_N(y(\cdot,0),u(\cdot)) := \sum_{n=0}^{N-1} \int_0^L \varrho(y_x(x,nT), y_t(x,nT))\,dx + \lambda \int_0^{NT} u(t)^2\,dt. \tag{4}$$

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An obvious choice for the function $\rho(\cdot,\cdot)$ in (4) is given by

$$\varrho(y_x(\cdot,t), y_t(\cdot,t)) = y_x(\cdot,t)^2 + (y_t(\cdot,t)/c)^2 \qquad (5)$$

which corresponds to measuring the energy of the system at each multiple of the given time parameter $T$. The second term in our cost functional penalizes the control effort with regularization parameter $\lambda \geq 0$.

In order to prove stability of the receding horizon closed loop and deduce tight performance estimates, we apply results from [2], [3] which are formulated in discrete time. Thus, we rewrite (1) as

$$z(n+1) = f(z(n), u(n)) \qquad (6)$$

with state $z(n) \in Z := \mathcal{H}_1(\Omega)$ and control $u(n) \in U := \mathcal{L}^\infty([0,T),\mathbb{R})$. Here the discrete time $n$ corresponds to the continuous time $nT$ which implies $z(n) = y(\cdot, nT)$. We denote the solution trajectory for a given control sequence $u : \mathbb{N}_0 \to U$ by $z_u(\cdot)$.

This allows for preserving the cost functional (4) in the discrete time setting by suitably chosen stage costs $l : Z \times U \to \mathbb{R}_0^+$:

$$J_N(z(0), u) = \sum_{n=0}^{N-1} l(z_u(n), u(n)).$$

Moreover, we define the optimal value function

$$V_N(z(0)) := \inf_{u \in \mathcal{U}} J_N(z(0), u(\cdot)) \qquad (7)$$

which is the truncated sum induced by the optimal value function $V_\infty(z(0)) := \inf_{u\in\mathcal{U}} \sum_{n=0}^{\infty} l(z_u(n), u(n))$ on the infinite time horizon. Hence, our goal consists of finding a feedback map $\mu_N : Z \to U$ such that the feedback controlled system

$$z_\mu(n+1) = f(z_\mu(n), \mu(z_\mu(n))) \qquad (8)$$

is asymptotically stable. To this end, we briefly summarize some stability and suboptimality results which can be found in [1].

*Proposition 1:* Assume that there exists $\alpha \in (0,1]$ such that for all $z \in Z$ the relaxed Lyapunov inequality

$$V_N(z) \geq V_N(f(z, \mu_N(z))) + \alpha l(z, \mu_N(z)) \qquad (9)$$

holds. Then for all $z \in Z$ the estimate

$$\alpha V_\infty(z) \leq \alpha J_\infty(z, \mu_N) \leq V_N(z) \leq V_\infty(z) \qquad (10)$$

holds. If in addition, there exist $z^* \in Z$ and $\mathcal{K}_\infty$-functions[1] $\alpha_1, \alpha_2$ such that the inequalities

$$l^*(z) := \min_{u\in U} l(z,u) \;\geq\; \alpha_1(d(z,z^*)) \quad \text{and} \qquad (11)$$
$$V_N(z) \;\leq\; \alpha_2(d(z,z^*)) \qquad (12)$$

hold for all $z \in Z$, then $z^*$ is a globally asymptotically stable equilibrium for (8) with feedback $\mu = \mu_N$ and Lyapunov function $V_N$. Here $d(\cdot,\cdot)$ is an arbitrary metric on $Z$.

[1] A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, and unbounded with $\alpha(0) = 0$

A metric on $Z$ which is suitable for our analysis will be specified in Section IV-B.

In order to estimate $\alpha$ in (9) we require the following controllability property.

*Definition 1:* We call the system (6) exponentially controllable with respect to the running cost $l$ if there exist an overshoot bound $C \geq 1$ and a decay rate $\sigma \in (0,1)$ such that for each $z \in Z$ there exists $u_z \in \mathcal{U}$ satisfying

$$l(z_{u_z}(n), u_z(n)) \leq C\sigma^n l^*(z) \qquad (13)$$

for $l^*(\cdot)$ as defined in (11).

*Remark 1:* Note that exponential controllability with respect to the running costs is not as restrictive as it may seem. Since the running costs can be used as a design parameter, this includes even systems which are merely asymptotically but not exponentially controllable, cf. [3].

Based on this controllability condition and Bellman's optimality principle a formula which enables us to explicitly calculate a lower bound – depending on the overshoot $C$ and the decay rate $\sigma$ – for the suboptimality degree $\alpha$ in (9) is introduced in [3, Theorem 5.3].

*Theorem 1:* Assume that the sytem (6) and $l$ satisfy the controllability condition from Definition 1 and let the optimization horizon $N$ be given. Then the suboptimality degree $\alpha_N$ from (9) is given by

$$\alpha_N := \alpha = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)} \qquad (14)$$

with $\gamma_i = C(1-\sigma^i)/(1-\sigma)$.

*Remark 2:* Theorem 1 is strict in the following sense. For negative $\alpha$ there exist a system (6) and running costs $l$ which satisfy (11) and (12) but for which the closed loop system with $\mu = \mu_N$ is not asymptotically stabilizable, cf. [2].

As a consequence of Theorem 1 the closed loop (8) is asymptotically stable and the suboptimality estimate (10) holds whenever $\alpha$ is positive and (11), (12) are satisfied.

## III. Design of the stage costs $\rho(\cdot,\cdot)$

Our goal consists of showing instantaneous controllability of the linear wave equation (1)–(3), i.e., controllability of the respective MPC closed loop with optimization horizon $N = 2$. Since the results from Section II are based on the relaxed Lyapunov Inequality (9), we have to construct suitable stage costs which allow for deducing this estimate. To this end, we consider (1)–(3) with parameters $L = c = 1$ and $\lambda = 10^{-3}$ numerically. Let the initial data be specified by

$$y_0(x) := \begin{cases} +2x - 0.5 & : \quad 0.25 < x \leq 0.50 \\ -2x + 1.5 & : \quad 0.50 < x \leq 0.75 \\ 0 & : \quad \text{otherwise} \end{cases}$$

and $y_1(x) \equiv 0$. For solving the finite horizon optimal control problems we discretize the spatial domain with discretization parameter $\Delta x = 0.001$ and employ the software package PCC[2] in order to solve the resulting tasks. Moreover, we set the sampling time $T = 0.025$. Our numerical computations

[2] see http://www.nonlinearmpc.com/

indicate that model predictive control stabilizes these initial data with the stage cost based on (5), cf. the dashed line in Figure 1. However, one observes plateaus, i.e., areas on which the optimal value function $V_N(\cdot)$ from Equality (7) exhibits constant values. Thus, the cost functional which is based solely on the energy of the system does not provide a strictly decreasing function for arbitrary initial data and – as a result – cannot be used as a Lyapunov function in order to conclude asymptotic stability. As a consequence, the corresponding solution does not satisfy (9) which is an essential requisite in order to deduce instantaneous controllability.

This problem is closely related with the finite propagation speed of the wave. Since the energy of the considered initial data is located in the middle of our domain $\Omega$ it can not be reduced by means of our boundary control until $T$, i.e., during the first sampling period. This explains why it is not possible to maintain a strict decrease on this short time interval. As a remedy we consider stage costs based on

$$\begin{aligned}\varrho(y_x(\cdot,t),y_t(\cdot,t)) &= \omega_1(\cdot)(y_x(\cdot,t)+(y_t(\cdot,t)/c))^2\\ &+ \omega_2(\cdot)(y_x(\cdot,t)-(y_t(\cdot,t)/c))^2\end{aligned}$$

with weight functions

$$\omega_1(x) := 1+L+x \qquad \text{and} \qquad \omega_2(x) := 1+L-x. \quad (15)$$

This enables us to employ our cost functional for the desired purpose. The weight functions $\omega_i : [0,L] \to \mathbb{R}_0^+$, $i = 1,2$, measure the distance to the right boundary and take the direction of movement into account, i.e., they measure the time we have to wait until we are able to influence the energy. Note that this approach contains the prior one ($\omega_1 = \omega_2 \equiv 1$). Figure 1 depicts the optimal value function $V_2$ along the closed loop trajectories for $\omega_1 = \omega_2 \equiv 1$, i.e., the classical energy norm (dashed line), in comparison to its counterpart based on the weight functions defined above (solid line). Apparently each of these two curves is monotonically decreasing, yet only the value along the trajectory corresponding to (15) is strictly decreasing.

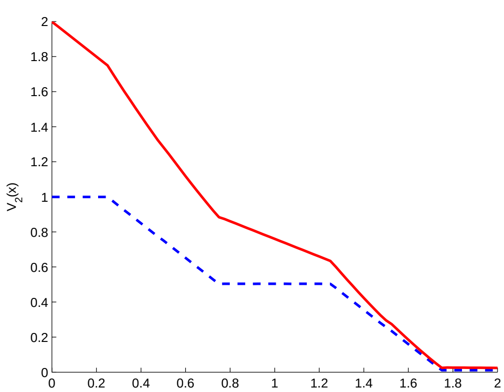


Fig. 1. Comparison of the optimal value function $V_2(\cdot)$ for different weight functions in the running costs $l(\cdot,\cdot)$. The dashed curve corresponds to the classical energy norm, i.e., $\omega_1 = \omega_2 \equiv 1$. Whereas the continuous curve is based on our weighted energy from (15).

## IV. Controllability of the wave equation

The goal of this section consists in deducing an overshoot bound $C$ and a decay rate $\sigma$ which satisfy the proposed controllability condition from Definition 1. This enables us to apply Theorem 1 in order to verify the assumptions of Proposition 1 and – as a consequence – to conclude stability of the closed loop (8), i.e., that the MPC feedback steers the system to its equilibrium.

To this end, we use a control $u^\star(\cdot)$ which avoids reflections on the right boundary. This way of proceeding simplifies the involved calculations significantly. In order to perform this step we employ the fact that the control sequence in (13) does not need to be optimal – a key characteristic of our approach. Moreover, since we focus on instantaneous control, i.e., $N = 2$, Formula (14) is given by

$$\alpha := \alpha_2 = 1-(C(1+\sigma)-1)^2. \quad (16)$$

In order to estimate the parameters $C$ and $\sigma$ from 13 we choose the particular control

$$u(n) := \frac{1}{2}\left(y_x(L-ct,nT)-\frac{y_t(L-ct,nT)}{c}\right) \quad (17)$$

which ensures that there do not occur any reflections on the right boundary. By using this control the solution of (1) coincides with the uncontrolled solution of the wave equation on a semi-infinite interval $[0,\infty)$. The corresponding solution can be calculated by D'Alembert's method, cf. [12],

$$\begin{aligned}y(x,t) &= \frac{1}{2}(y_0(x+ct)+y_0(x-ct))\\ &+ \frac{1}{2c}\int_{x-ct}^{x+ct} y_1(s)\,ds \quad \text{for} \quad x>ct,\\ y(x,t) &= \frac{1}{2}(y_0(ct+x)-y_0(ct-x))\\ &+ \frac{1}{2c}\int_{ct-x}^{ct+x} y_1(s)\,ds \quad \text{for} \quad x<ct. \quad (18)\end{aligned}$$

In order to prove this fact, we consider $y_x(L,t)$. This yields

$$\begin{aligned}y_x(L,t) &= \left[y_0'(L-ct)-y_1(L-ct)/c\right]/2\\ &= \left[y_x(L-ct,0)-y_t(L-ct,0)/c\right]/2 = u(0).\end{aligned}$$

Thus, the initial conditions as well as the boundary conditions coincide. As a consequence the solution of the controlled wave equation coincides with the solution of the uncontrolled wave equation on an unbounded domain. Iterative application of this argument shows the assertion on $[0,iT)$ for all $i \in \mathbb{N}_{\geq 1}$.

### A. Overshoot bound $C$ and decay rate $\sigma$

We begin with estimating the overshoot constant $C$ from (13) for the running costs defined in (15) and the control which is specified in (17). To this end, we estimate the control effort which is caused by (17). Since the chosen control function is a certain fraction of the cost induced by the current state we obtain the estimate

$$\begin{aligned}&\frac{\lambda}{4}\int_0^T \left[y_x(L-ct,nT)-y_t(L-ct,nT)/c\right]^2\ dt\\ =\ &\frac{\lambda}{4c}\int_{L-cT}^L \left[y_x(x,nT)-y_t(x,nT)/c\right]^2\ dt\\ \leq\ &\lambda l^*(y(nT))/c.\end{aligned}$$

Here we have used the property $\omega_i \geq 1$, $i=1,2$, of our weight functions from (15). Using this estimate we obtain

$$l(y(n),u(n)) \leq (1+\lambda/c)\, l^*(y(nT)) = Cl^*(y(nT)) \quad (19)$$

with $C := (1+\lambda/c)$. Again, note that we have not assumed optimality of the control in our approach.

Next we show the inequality

$$l^*(y(i+1)) \leq \sigma l^*(y(i)) \quad (20)$$

with decay rate $\sigma \in (0,1)$. Inequality (20) is equivalent to $(1-\eta)l^*(y(i)) \geq l^*(y(i+1))$ with $\eta := 1-\sigma$. Hence, it suffices to establish the inequality

$$l^*(y(i)) - l^*(y(i+1)) \geq \eta l^*(y(i)) \quad (21)$$

in order to show the desired inequality. The decisive tools in order to establish this relation are the explicit formulas given from (18) for the control defined in (17). Using these formulas we perform the calculations

$$\begin{aligned}&\int_0^L \omega_1(x)\left[y_x(x,T)+y_t(x,T)/c\right]^2 dx\\ =\ &\int_0^{cT} \omega_1(x)\left[y_0'(cT+x)+y_1(cT+x)/c\right]^2 dx\\ &+\int_{cT}^L \omega_1(x)\left[y_0'(x+cT)+y_1(x+cT)/c\right]^2 dx\\ =\ &\int_{cT}^L \omega_1(x-cT)\left[y_0'(x,T)+y_1(x)/c\right]^2 dx\end{aligned}$$

and

$$\begin{aligned}&\int_0^L \omega_2(x)\left[y_x(x,T)-y_t(x,T)/c\right]^2 dx\\ =\ &\int_0^{cT} \omega_2(x)\left[y_0'(cT-x)+y_1(cT-x)/c\right]^2 dx\\ &+\int_{cT}^L \omega_2(x)\left[y_0'(x-cT)-y_1(x-cT)/c\right]^2 dx\\ =\ &\int_0^{cT} \omega_1(x-cT)\left[y_0'(x)+y_1(x)/c\right]^2 dx\\ &+\int_0^{L-cT} \omega_2(x+cT)\left[y_0'(x)-y_1(x)/c\right]^2 dx\end{aligned}$$

in order to simplify the following arguments where we have used $\omega_2(cT-x) = \omega_1(x-cT)$ in the last inequality. Combining these equalities leads to

$$\begin{aligned}l^*(y(1)) &= \int_0^L \rho(y_x(x,T),y_t(x,T))\, dx\\ &= \int_0^L \omega_1(x-cT)\left[y_0'(x,T)+y_1(x)/c\right]^2 dx\\ &\quad+\int_0^{L-cT} \omega_2(x+cT)\left[y_0'(x)-y_1(x)/c\right]^2 dx.\end{aligned}$$

This equality allows for deducing an appropriate estimate in order to derive our controllability condition. In consideration of the equality

$$\begin{aligned}l^*(y(0)) &= \int_0^L \omega_1(x)\left[y_0'(x)+y_1(x)/c\right]^2 dx\\ &\quad+\int_0^L \omega_2(x)\left[y_0'(x)-y(x)/c\right]^2 dx,\end{aligned}$$

the nonnegativity of $\omega_2(\cdot)$ on $\Omega$, $\omega_1(x)-\omega_1(x-cT) = cT$ and $\omega_2(x)-\omega_2(x+cT) = cT$ we obtain Inequality (21) with $\eta = cT/(1+2L)$:

$$\begin{aligned}&l^*(y(0)) - l^*(y(1))\\ \geq\ &cT\int_0^L \left[y_0'(x)+y_1(x)/c\right]^2+\left[y_0'(x)-y(x)/c\right]^2 dx\\ \geq\ &\frac{cT}{1+2L}l^*(y(0))\end{aligned}$$

Consequently, this leads to $\sigma = 1-cT/(1+2L)$. This implies – in combination with our estimate for the overshoot $C$ – exponential controllability in terms of the running costs, i.e.,

$$l(y(n),u(n)) \leq Cl^*(y(n)) \leq C\sigma^n l^*(y(0)).$$

Hence, we have shown the validity of the controllability condition given in Definition 1.

*Remark 3:* The decrease reflected by $\sigma$ depends only on the chosen weight functions. In addition, there occurs an energy loss in the amount of

$$\int_{L-cT}^L \omega_2(x+cT)[y_0'(x)-y_1(x)/c]^2\ dx$$

This represents the energy which is removed by means of the boundary control.

### B. Stability of the closed loop

Since we have deduced explicit expressions for the overshoot $C$ and the decay rate $\sigma$ in (13), we are able to utilize Theorem 1 in order to show stability of the receding horizon feedback for optimization horizon $N=2$, i.e., instantaneous controllability. To this end, we need $\alpha_2 > 0$ with $\alpha_2$ from (16). Thus, Theorem 1 ensures stability for

$$T > \frac{(2+4L)\lambda}{c(c+\lambda)}. \quad (22)$$

For $L = c = 1$ this yields the estimate $T > 6\lambda/(1+\lambda)$. Hence, the sampling interval has to be sufficiently large in order to allow for compensating the control effort which is reflected by the overshoot constant $C$. However, choosing a small weight in the control penalization, e.g. $\lambda = 10^{-3}$, this results in a very short optimization horizon compared to the time $\bar{T} = 2L/c = 2$ required for finite time controllability, cf. [5]. It remains to establish (11) and (12). To this end we choose $\alpha_1(r) = r$ and define the metric $d(z,z') := l^*(z-z')$ which is well defined due to (2) and obviously satisfies (11). Since $\omega_i(\cdot)$, $i=1,2$, is uniformly bounded by $1+L$ similar arguments show that (12) is satisfied for $\alpha_2(r) = 2(1+L)r$. Hence, Proposition 1 ensures the desired stability of the closed loop.

## V. Numerical Results

In this section we revisit the example considered in Section III in order to show that the derived bounds with respect to the decay rate $\sigma$ are tight.

We have already seen that using the weighted energy norm with the weight functions defined in (15) exactly encounters the problem which occurs for the classical energy. Hence, this approach enables us to employ $V_2$ as a Lyapunov function which satisfies the relaxed Lyapunov Inequality (9). However, the deduced decay rate $\sigma$ seems to be pessimistic at first glance. In order to investigate this issue more accurately we calculate the corresponding $\sigma$–values for the example from Section III. In order to visualize our theoretically calculated estimate we have drawn a horizontal line at $1-T/3$ in Figure 2 which shows that the calculated values for the classical energy are arbitrarily close to one and exceed our estimated bound whereas the values corresponding to the stage costs which incorporate (15) are smaller than $1-T/3$ which confirms our theoretical results. Moreover,

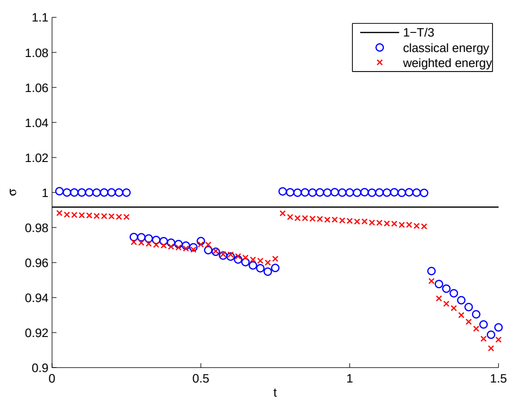


Fig. 2. We depict the corresponding $\sigma$–values with respect to the classical (○) and the weighted energy (15, x) from Inequality (20). Moreover, we have drawn a horizontal line at $1-T/3$ in order to indicate our theoretically derived bound.

the considered example shows that a further improvement of the deduced estimate is not possible.

The solution trajectory of the instantaneous controlled wave equation is depicted in Figure 3. Indeed, it even coincides with the solution trajectory corresponding to an optimization horizon of length $2L/c = 2$ which is needed in order to show finite time controllability. Hence, model predictive control with $N = 2$ performs very well for the stabilization task in consideration. The computing time for solving the instantaneous control problem on the time interval $[0, 2]$ is less than one second even for a fine spatial discretization.

## VI. Outlook

We have proven the instantaneous controllability of the one dimensional linear wave equation (1)–(3) rigorously. Numerical results indicate that MPC also works well for the

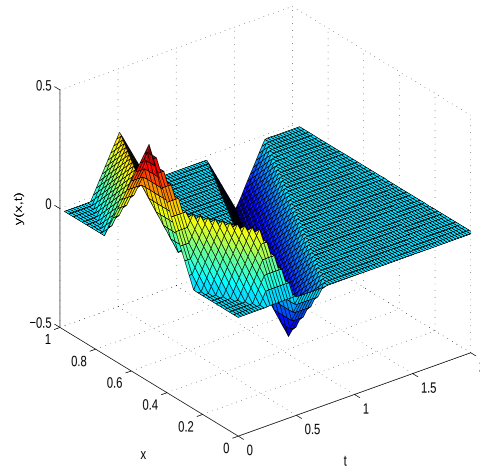


Fig. 3. The solution trajectory for the instantaneous controlled wave equation (MPC with optimization horizon $N = 2$) for the initial data given in Subsection III.

two dimensional wave equation. Hence, our one of our future goals consists of proving this fact.

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