# Hamiltonian Evolution Equations of inductionless Magnetohydrodynamics

Andreas Siuka, Markus Schöberl and Kurt Schlacher

***Abstract*— The objective of this contribution is to find a coordinate independent Hamiltonian representation of the governing equations of inductionless Magnetohydrodynamics, where we are interested in analysing the relevant energy flows in a purely geometric fashion also taking dissipative effects into account. We especially treat the boundary conditions in an extraordinary manner and we define control inputs which may act on the system boundary. Finally, the Port-Controlled Hamiltonian system representation, well-known in the lumped parameter case, is also reflected in the infinite dimensional case which is crucial particularly with regard to control theoretic aspects.**

## I. INTRODUCTION

Magnetohydrodynamics - abbreviated MHD - is a well-established and mainly challenging discipline since it combines two main field theories in physics; fluid mechanics, mostly represented by the Navier-Stokes equations, on the one hand and electrodynamics described by Maxwell's equations on the other hand, both linked together via Ohm's law and Lorentz forces, of course. This contribution is dedicated to the so-called inductionless MHD (iMHD for short) case and its variational formulation. Roughly speaking, in the iMHD case we consider the macroscopic behavior of an electrically conducting fluid in the presence of external electromagnetic fields, where it is assumed that the dynamic of the additionally induced electromagnetic parts can be neglected (low magnetic Reynold's number). As this is the case for many industrial applications, see, e.g., [14] for more detailed information. Based on the Lagrangian formulation of the governing partial differential equations of iMHD we intend to represent these equations from the Hamiltonian point of view in a purely geometric fashion, where it is worth noting that for the distributed-parameter case there exist, in general, different Hamiltonian representations. In the lumped-parameter case the Hamiltonian formalism is well-known, where the equations may be derived from a symplectic form or from the corresponding Poisson brackets, see, e.g., [4], or where the equations are represented in an evolutionary first-order form which are connected with the equations of the Lagrangian formalism via a regularity assumption on the Legendre transformation. From a control theoretic point of view, the resulting system equations describe, in general, an autonomous, lossless system, where the Hamiltonian serves as a conserved quantity. In order to obtain a more general system class it is possible to take dissipative effects into account and to define system inputs leading to the so-called Port-Controlled Hamiltonian system representation with dissipation (PCHD for short) which is the basis for several control methods, see [15], for instance. With respect to the extension of this formalism to the distributed-parameter case there exist several approaches; the polysymplectic approach going back to DeDonder/Weyl (e.g. [5]), a concept based on Dirac structures (see [16]) and the so-called evolutionary approach, see, e.g., [8], [10], [11] and references therein, which maintains the evolutionary description from the lumped-parameter case. In this contribution we confine ourselves to the evolutionary point of view for first-order Hamiltonian field theory and its application to iMHD, where we mainly focus our interests on a clear geometric description in a coordinate independent manner, on a clear treatment of the boundary terms and on the consideration of dissipative effects.

The contribution is organised as follows: In the second section we will introduce the mathematical notation and analyse in detail the geometric objects which are necessary for a coordinate independent description of the governing equations of iMHD. In the third part of this contribution we will briefly illustrate the electromagnetic body forces in a geometric fashion and the Lagrangian point of view will be presented in the fourth section. The fifth section deals with the introduction of the Hamiltonian evolutionary approach and its application to the iMHD case, where we will investigate two cases; first, we will neglect dissipative effects in order to formulate a Hamiltonian density which serves as a conserved quantity (section six) and, finally, we will extend the presented formalism to include dissipative effects and to define system inputs in order to obtain a PCHD system representation of the governing equations of iMHD (section seven). Some remarks on further extensions of the presented framework and on possible applications will close this contribution.

Furthermore, standard tensor notation and especially Einstein's convention on sums will be used to keep the formulas short and readable. The interested reader is referred to standard books dealing with differential geometry and jet bundles such as [5], [9] for detailed information.

## II. THE GEOMETRY OF MAGNETOHYDRODYNAMICS

This section deals with the main notions of differential geometry and illustrates the geometric objects which will be used in the sequel. Let us introduce the trivial reference

A. Siuka, M. Schöberl and K. Schlacher are with the Institute of Automatic Control and Control Systems Technology, Johannes Kepler University, Altenberger Str. 69, 4040 Linz, Austria.
andreas.siuka@jku.at
markus.schoeberl@jku.at
kurt.schlacher@jku.at

ISBN 978-963-311-370-7

bundle $\mathcal{M} = \mathcal{I} \times \mathcal{M}_x \to \mathcal{I}$ equipped with coordinates $(t^0, X^i) \to (t^0)$, $i = 1, ..., n_x$ and the trivial configuration bundle $\mathcal{X} = \mathcal{I} \times \mathcal{X}_q \to \mathcal{I}$ with coordinates $(t^0, q^\alpha) \to (t^0)$, $\alpha = 1, ..., n_q$ in a standard manner, where the so-called material coordinates $X^i$ correspond to the spatial coordinates on the reference manifold (material picture) and $q^\alpha$ denote the spatial coordinates on the configuration manifold (spatial picture). Therefore, the product structure of these manifolds can be characterised by the tensor $\Lambda = \mathrm{d}t^0 \otimes \partial_0$ (trivial reference frame) with $\partial_0 = \frac{\partial}{\partial t^0}$. Furthermore, we are able to construct the bundle $\mathcal{E} \to \mathcal{M}$ with $\mathcal{E} = \mathcal{M} \times_{\mathcal{I}} \mathcal{X}$ equipped with coordinates $(t^0, X^i, q^\alpha) \to (t^0, X^i)$ and introduce the first jet manifold $\mathcal{J}^1(\mathcal{E})$ which possesses the coordinates $(t^0, X^i, q^\alpha, q_0^\alpha, q_i^\alpha)$. A section $\Phi : \mathcal{M} \to \mathcal{E}$ is given by $\Phi = (t^0, X^i, \Phi^\alpha(t^0, X^i))$, where we have $q^\alpha \circ \Phi = \Phi^\alpha(t^0, X^i)$ which is called a motion in the Lagrangian setting. Roughly speaking, for fluid dynamics $\mathcal{M}_x$ can be interpreted as the reference configuration and, therefore, all subsequent configurations of the fluid are described by the section $\Phi$ which characterises the particle paths leading to a particle displacement field. Thus, for a material point $X$ with coordinates $X^i$ the motion $q^\alpha \circ \Phi = \Phi^\alpha(t^0, X^i)$ represents the position of the fluid particle $X$ at time $t^0$. The first jet of this section is given by $j^1\Phi : \mathcal{M} \to \mathcal{J}^1(\mathcal{E})$ which leads to the components of the so-called material velocity $q_0^\alpha \circ j^1\Phi = \partial_0 \Phi^\alpha = V_0^\alpha$ and to the definition of the components of the deformation gradient $q_i^\alpha \circ j^1\Phi = \partial_i \Phi^\alpha = F_i^\alpha$, $\partial_i = \frac{\partial}{\partial X^i}$, see [6], for instance. Additionally, we consider the case of Riemannian manifolds. Therefore, we introduce a (positive definite) vertical metric on the reference manifold denoted by $G = G_{ij}\,\mathrm{d}X^i \otimes \mathrm{d}X^j$ with $G_{ij} = G_{ji} \in C^\infty(\mathcal{M}_x)$ as well as a volume form

$$\mathrm{VOL} = \sqrt{\det G}\,\mathrm{d}X^1 \wedge ... \wedge \mathrm{d}X^{n_x} = \sqrt{\det G}\,\Omega\,.$$

Additionally, we introduce a (positive definite) vertical metric on the configuration manifold given by $g = g_{\alpha\beta}\,\mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta$ with $g_{\alpha\beta} = g_{\beta\alpha} \in C^\infty(\mathcal{X}_q)$ and the volume form

$$\mathrm{vol} = \sqrt{\det g}\,\mathrm{d}q^1 \wedge ... \wedge \mathrm{d}q^{n_q}\,.$$

It is worth noting that the corresponding metric coefficients are time independent since we consider inertial frames. Furthermore, we are able to introduce the tangent bundle $\mathcal{T}(\mathcal{X}) \to \mathcal{X}$ with coordinates $(t^0, q^\alpha, \dot{t}^0, \dot{q}^\alpha) \to (t^0, q^\alpha)$ and holonomic base $\{\partial_0, \partial_\alpha\}$ on $\mathcal{T}(\mathcal{X})$, $\partial_\alpha = \frac{\partial}{\partial q^\alpha}$, as well as the vertical tangent bundle $\mathcal{V}(\mathcal{X}) \to \mathcal{X}$ equipped with coordinates $(t^0, q^\alpha, \dot{q}^\alpha) \to (t^0, q^\alpha)$ and holonomic base $\{\partial_\alpha\}$ on $\mathcal{V}(\mathcal{X})$. The corresponding connection which splits $\mathcal{V}(\mathcal{X}) \to \mathcal{X}$, see, e.g., [5], [12], is of the form

$$\Lambda_c = \mathrm{d}t^0 \otimes \partial_0 + \mathrm{d}q^\alpha \otimes \left(\partial_\alpha - \gamma_{\alpha\delta}^\beta \dot{q}^\delta \dot{\partial}_\beta\right)$$

with $\dot{\partial}_\beta = \frac{\partial}{\partial \dot{q}^\beta}$ including the Christoffel symbols of second kind given by $\gamma_{\alpha\varepsilon}^\beta = \frac{1}{2} g^{\beta\delta}(\partial_\alpha g_{\varepsilon\delta} + \partial_\varepsilon g_{\alpha\delta} - \partial_\delta g_{\alpha\varepsilon})$ with $g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha$, where $\delta_\gamma^\alpha$ denotes the components of the Kronecker symbol. For the following let $\mathcal{B}$ be an arbitrary reference state with $\mathcal{B} \subset \mathcal{M}_x$ and let $\rho \in C^\infty(\mathcal{X})$ denote the spatial mass density. Then, the mass of a continuum is defined as $\int_{\Phi(t^0, \mathcal{B})} \rho\,\mathrm{vol} \in \mathbb{R}^+$, where this integral has to be evaluated at a fixed time $t^0$ and $\Phi(t^0, \mathcal{B}) \subset \mathcal{X}_q$. In consideration of the motion this expression is equivalent to

$$\Phi^*\left(\int_{\Phi(t^0,\mathcal{B})} \rho\,\mathrm{vol}\right) = \int_{\mathcal{B}} \rho_{\mathcal{R}}\,\mathrm{VOL}\,, \tag{1}$$

with the material mass density $\rho_{\mathcal{R}} \in C^\infty(\mathcal{M})$ which meets $\rho_{\mathcal{R}} = (\rho \circ \Phi) J$, since we have $\Phi^*(\mathrm{vol}) = J\,\mathrm{VOL}$ with

$$J = \breve{J} \circ j^1\Phi = \det F \frac{\sqrt{\det g \circ \Phi}}{\sqrt{\det G}}\,. \tag{2}$$

For the following part we consider the case $n_x = n_q = n$ and, therefore, we have $\dim \mathcal{M} = \dim \mathcal{X}$. Furthermore, we assume that the motion is regular, meaning that for an open set $\Phi(t^0, \mathcal{B})$ we are able to define the inverse of the motion by the map $\hat{\Phi} : \Phi(t^0, \mathcal{B}) \to \mathcal{B}$, see, e.g., [6] for detailed information. As a consequence, we have $0 < J < \infty$, since $J$ also describes the ratio of an elementary volume in the configuration to its initial volume on the reference as stated before. Moreover, we are able to define the velocity $v$ as a vertical vector field which can be interpreted as a section of the bundle $(\varrho_0^1)^*(\mathcal{V}(\mathcal{X})) \to \mathcal{J}^1(\mathcal{X})$ where $\varrho_0^1 : \mathcal{J}^1(\mathcal{X}) \to \mathcal{X}$ and follows to $v = v^\alpha \partial_\alpha = q_0^\alpha \partial_\alpha$. If we restrict the velocity to the motion we are able to introduce the components of the so-called spatial velocity corresponding to $v^\alpha = V_0^\alpha \circ \hat{\Phi} \in C^\infty(\mathcal{X})$. Finally, by evaluating the total time change of (1) the conservation of mass principle leads directly to the equation of continuity $\partial_0 \rho_{\mathcal{R}} = 0$ in the material picture as well as $\partial_0 \rho + \mathrm{div}(\rho v^\alpha) = 0$ in the spatial picture with $\mathrm{div}(v^\alpha) = \partial_\alpha v^\alpha + \gamma_{\alpha\gamma}^\alpha v^\gamma$ and, therefore, we actually have $\rho_{\mathcal{R}} \in C^\infty(\mathcal{M}_x)$. A similar result can be obtained for the charge density. Let the charge of the continuum be defined as $\int_{\Phi(t^0,\mathcal{B})} \mu\,\mathrm{vol} \in \mathbb{R}$ with the spatial charge density $\mu \in C^\infty(\mathcal{X})$. As before, we are able to introduce the material charge density $\mu_{\mathcal{R}} \in C^\infty(\mathcal{M})$ which certainly meets $\mu_{\mathcal{R}} = (\mu \circ \Phi) J$. In general, a current density can be represented by the form $j = j^\alpha \partial_\alpha \rfloor \mathrm{vol}$ with $j^\alpha \in C^\infty(\mathcal{X})$, where $\rfloor$ denotes the standard contraction of vector fields and tensors. In MHD we are usually concerned with two types of current densities: On the one hand we take the convective transport of charge into account, meaning that the so-called convective current density possesses the components $j^\alpha = \mu v^\alpha$, see, e.g., [4], [14], and on the other hand for the case of finite electrical conductivity a conductive current density is induced due the interaction of the external electromagnetic fields with the conducting fluid. Consequently, conservation of charge in the material picture is equivalent to $\partial_0 \mu_{\mathcal{R}} = 0$ and, therefore, $\mu_{\mathcal{R}} \in C^\infty(\mathcal{M}_x)$ as well as $\partial_0 \mu + \mathrm{div}(\mu v^\alpha) = 0$ in the spatial picture which is the correct conservation law in the case of a convective current density only, meaning that the equation of continuity takes the form of $\partial_0 \mu + \mathrm{div}(j^\alpha) = 0$ with $j^\alpha = \mu v^\alpha$.

For the following let us introduce the Cauchy stress form $\sigma = \sigma^{\alpha\beta} \partial_\alpha \rfloor \mathrm{vol} \otimes \partial_\beta$ with $\sigma^{\alpha\beta} = \sigma^{\beta\alpha} \in C^\infty(\mathcal{X})$, see [4], for instance. In consideration of the motion we can pull back the form part leading to the definition of the first Piola-Kirchhoff stress form

$$P = \Phi^* \left( \sigma^{\alpha\beta} \partial_\alpha \rfloor \mathrm{vol} \right) \otimes \partial_\beta = P^{i\beta} \partial_i \rfloor \mathrm{VOL} \otimes \partial_\beta \,, \quad (3)$$

with $P^{i\beta} = J \left( \hat{F}^i_\alpha \sigma^{\beta\alpha} \right) \circ \Phi \in C^\infty(\mathcal{M})$, $\hat{F}^i_\alpha F^\alpha_j = \delta^i_j$, where we have $\hat{F}^i_\alpha = \partial_\alpha \hat{\Phi}^i$. The second Piola-Kirchhoff stress form is obtained by

$$S = \Phi^* \left( \sigma^{\alpha\beta} \partial_\alpha \rfloor \mathrm{vol} \otimes \partial_\beta \right) = S^{ij} \partial_i \rfloor \mathrm{VOL} \otimes \partial_j \,, \quad (4)$$

where the relation $S^{ij} F^\beta_j = P^{i\beta}$ is easily verified. Furthermore, the Cauchy Green tensor is obtained by the pull back of the metric tensor $g$, see [6], resulting in

$$C = \Phi^*(g) = C_{ij} \mathrm{d}X^i \otimes \mathrm{d}X^j \,, \quad (5)$$

with $C_{ij} = (g_{\alpha\beta} \circ \Phi) F^\alpha_i F^\beta_j$. By neglecting viscous stresses as well as electro - and magnetostrictive effects, see, e.g., [3], the Cauchy stress form for a conducting fluid reads as

$$\sigma = -\mathcal{P}_\mathcal{K} \, g^{\alpha\beta} \, \partial_\alpha \rfloor \mathrm{vol} \otimes \partial_\beta \,, \quad (6)$$

which includes the spatial hydrostatic pressure function $\mathcal{P}_\mathcal{K} \in C^\infty(\mathcal{X})$. Therefore, the so-called material pressure $\mathcal{P}_\mathcal{R} \in C^\infty(\mathcal{M})$ can be introduced which meets $\mathcal{P}_\mathcal{K} \circ \Phi = \mathcal{P}_\mathcal{R}$, see [7] for more detailed information. According to [6], we assume the existence of a stored energy function $E_{st} \in C^\infty\left(\mathcal{J}^1(\mathcal{E})\right)$ which meets

$$S^{ij} = 2\rho_\mathcal{R} \frac{\partial \left( E_{st} \circ j^1\Phi \right)}{\partial C_{ij}} \,. \quad (7)$$

For Newtonian fluid dynamics the stored energy function solely depends on $J$ and, consequently, we have $E_{st} = E_{st}\left(\breve{J}\right)$, see [1], [7]. There, it is assumed that the motion is such that the pressure and the density are directly related (so-called barotropic fluids). Therefore, with respect to the components of (5) it is easily seen that $\det C = (\det g \circ \Phi)(\det F)^2$ is valid which allows to reparametrise (2) leading to $J = \sqrt{\frac{\det C}{\det G}}$. Consequently, we are able to derive

$$S^{ij} = \rho_\mathcal{R} \frac{\partial \left( E_{st} \circ j^1\Phi \right)}{\partial J} J \, C^{ij} = J \hat{F}^i_\alpha \hat{F}^j_\beta \left( \sigma^{\alpha\beta} \circ \Phi \right)$$

via the components of (3) and (4) and where we have used $\frac{\partial J}{\partial C_{ij}} = \frac{1}{2} J \, C^{ij}$ with $C^{ij} C_{jk} = \delta^i_k$. In consideration of the components of (5) and (6) we, finally, end up with the result

$$\mathcal{P}_\mathcal{R} = -\rho_\mathcal{R} \frac{\partial \left( E_{st} \circ j^1\Phi \right)}{\partial J} \,. \quad (8)$$

## III. Electromagnetic Body Forces

By neglecting polarisation and magnetisation effects, see [3], the relevant electromagnetic forces for the iMHD case are given by the electrostatic body force, a body force due to the convective transport of charge and an electromagnetic force density resulting from the induced conductive current in connection with the external magnetic field (conducting fluid with finite conductivity). First, we intend to give a geometric interpretation of the electrostatic body force and the body force caused by the convective transport of charge. Let us introduce the electromagnetic field strength two form $F : \mathcal{X} \to \wedge^2 \mathcal{T}^*(\mathcal{X})$ corresponding to

$$F = E_{0\alpha} \mathrm{d}q^\alpha \wedge \mathrm{d}t^0 + \frac{1}{2} B_{\alpha\beta} \mathrm{d}q^\alpha \wedge \mathrm{d}q^\beta \,, \quad (9)$$

with the electric field strength $E = E_{0\alpha} \mathrm{d}q^\alpha$, $E_{0\alpha} \in C^\infty(\mathcal{X})$ and the magnetic flux density $B = \frac{1}{2} B_{\alpha\beta} \mathrm{d}q^\alpha \wedge \mathrm{d}q^\beta$, $B_{\alpha\beta} \in C^\infty(\mathcal{X})$, where $\mathrm{d}F = 0$ is met, see, e.g., [4] and references therein. If the Lemma of Poincaré can be applied it is possible to introduce the electromagnetic potential $A : \mathcal{X} \to \mathcal{T}^*(\mathcal{X})$ which reads as

$$A = A_0 \mathrm{d}t^0 + A_\alpha \mathrm{d}q^\alpha \quad (10)$$

with $A_0$, $A_\alpha \in C^\infty(\mathcal{X})$ and which meets $F = \mathrm{d}A$ leading to the parametrisation

$$E_{0\alpha} = \partial_\alpha A_0 - \partial_0 A_\alpha \,, \quad B_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \,, \quad (11)$$

where $A_0$ denotes the electrostatic potential and $A_\alpha$ the components of the vector potential. Furthermore, in consideration of $f = -(\partial_0 + v^\alpha \partial_\alpha) \rfloor F$ the force density can be defined as $f^L = \mu \, \mathrm{vol} \otimes \gamma_c \rfloor f$ with $\gamma_c = \mathrm{d}q^\beta \otimes \partial_\beta$ and takes the form of

$$f^L = \mu \, \mathrm{vol} \otimes \left( E_{0\beta} - v^\gamma B_{\gamma\beta} \right) \mathrm{d}q^\beta \,. \quad (12)$$

Thus, it is easily seen that the force density consists of the sum of the electrostatic force density and the resulting force density caused by the convective transport of charge. In the case of convective currents only (vanishing conductivity) the corresponding force density with regard to the material picture reads as

$$F^L = \mu_\mathcal{R} \mathrm{VOL} \otimes \left( E_{0\beta} \circ \Phi - V^\gamma_0 B_{\gamma\beta} \circ \Phi \right) \mathrm{d}q^\beta \,. \quad (13)$$

For the case of finite conductivity an electromagnetic force density is caused by the induced conductive current in connection with the external electromagnetic field. This force density may be interpreted as a body force which counteracts the motion of the fluid (according to Lenz's law) leading to dissipative effects, see [14] for detailed information. Before this electromagnetic force will be introduced we must analyse the constitutive relation for the conductive current density which is given by Ohm's law, where we neglect thermoelectric effects, see [3]. There, special care must be taken since the constitutive relations in electrodynamics are valid only in the fluid frame (frame which is attached to the fluid), see, e.g., [2] for a more general discussion. To overcome this problem we transform the electromagnetic two form in a frame, whose coordinate lines are fixed to the deforming medium (convected picture, see [1], [6]) in order to obtain the right result. Then we use the classical relations and transform them back to obtain the corresponding law in the spatial picture in order to define the electromagnetic force. First, we consider a bundle morphism without time reparametrisation on the configuration bundle $\mathcal{X} \to \mathcal{I}$ of the form

$$\begin{aligned} \bar{t}^{\bar{0}} &= \delta^{\bar{0}}_0 t^0 \,, \quad t^0 = \delta^0_{\bar{0}} \bar{t}^{\bar{0}} \\ \bar{q}^{\bar{\alpha}} &= \varphi^{\bar{\alpha}}\left(q^\alpha, t^0\right) \,, \quad q^\alpha = \hat{\varphi}^\alpha\left(\bar{q}^{\bar{\alpha}}, \bar{t}^{\bar{0}}\right) \end{aligned} \quad (14)$$

with a diffeomorphism $\varphi$ for the spatial coordinates on the configuration manifold and, furthermore, we successively obtain

$$\bar{q}_{\bar{0}}^{\bar{\alpha}} = \delta_{\bar{0}}^{0}\left(\partial_\alpha \varphi^{\bar{\alpha}} q_0^\alpha + \partial_0 \varphi^{\bar{\alpha}}\right) . \quad (15)$$

*Remark 1:* Applying the change of coordinates (14) with respect to the diffeomorphism $\varphi : \mathcal{X} \to \bar{\mathcal{X}}$ the reference frame in the new coordinates reads as

$$\bar{\Lambda} = \mathrm{d}\bar{t}^{\bar{0}} \otimes \left(\partial_{\bar{0}} + \delta_{\bar{0}}^{0}\left(\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}\right)\partial_{\bar{\alpha}}\right)$$

since $\partial_0 \varphi^{\bar{\alpha}} \neq 0$ and, therefore, the product structure of the configuration manifold is not preserved any more as it is the case for an inertial system. The corresponding velocity field may be interpreted as a section $\bar{v} : \mathcal{J}^1\left(\bar{\mathcal{X}}\right) \to \left(\bar{\varrho}_0^1\right)^*\left(\mathcal{V}\left(\bar{\mathcal{X}}\right)\right)$, where $\bar{\varrho}_0^1 : \mathcal{J}^1\left(\bar{\mathcal{X}}\right) \to \bar{\mathcal{X}}$, and takes the form of $\bar{v} = \left(\bar{q}_{\bar{0}}^{\bar{\alpha}} - \delta_{\bar{0}}^{0}\left(\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}\right)\right)\partial_{\bar{\alpha}}$ including the resulting transition functions of the corresponding connection coefficients, see [12], for instance.

In order to obtain a frame, that has coordinate lines fixed to the deforming medium (convected coordinates), we consider the special bundle morphism

$$\bar{t}^{\bar{0}} = \delta_0^{\bar{0}} t^0 , \quad \bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}\left(q^\alpha, t^0\right) = \hat{\Phi}^{\bar{\alpha}}\left(q^\alpha, t^0\right) \quad (16)$$

involving the inverse of the motion which leads to $\bar{q}^{\bar{\alpha}} \circ \Phi = \delta_i^{\bar{\alpha}} X^i$ with regard to the material picture. This condition results in $\bar{q}_{\bar{0}}^{\bar{\alpha}} \circ j^1\Phi = 0$ and, when applied to (15), we end up with

$$\partial_0 \varphi^{\bar{\alpha}} = -\partial_\alpha \varphi^{\bar{\alpha}}\left(V_0^\alpha \circ \hat{\Phi}\right) = -\delta_i^{\bar{\alpha}} \hat{F}_\alpha^i v^\alpha \quad (17)$$

in consideration of (16) with $\partial_\alpha \varphi^{\bar{\alpha}} = \delta_i^{\bar{\alpha}} \hat{F}_\alpha^i$. With regard to Remark (1) the corresponding velocity field follows to $\bar{v} = \delta_{\bar{0}}^{0}\delta_i^{\bar{\alpha}}\left(\hat{F}_\alpha^i v^\alpha \circ \hat{\varphi}\right)\partial_{\bar{\alpha}}$ and is equivalent to the so-called convective velocity, see [6] for more detailed information. Furthermore, with the help of

$$\mathrm{d}\bar{q}^{\bar{\alpha}} = \partial_0 \varphi^{\bar{\alpha}} \mathrm{d}t^0 + \partial_\alpha \varphi^{\bar{\alpha}} \mathrm{d}q^\alpha ,$$

obtained via (16), we are able to compute the expression of $g$ in the convected picture via $\varphi^*\left(\bar{g}\right) = g$ resulting in

$$\bar{g} = \bar{g}_{\bar{\alpha}\bar{\beta}}\left(\mathrm{d}\bar{q}^{\bar{\alpha}} - \left(\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}\right)\mathrm{d}t^0\right) \otimes \left(\mathrm{d}\bar{q}^{\bar{\beta}} - \left(\partial_0 \varphi^{\bar{\beta}} \circ \hat{\varphi}\right)\mathrm{d}t^0\right)$$

with components $\bar{g}_{\bar{\alpha}\bar{\beta}} = \left(g_{\alpha\beta} \circ \hat{\varphi}\right)\partial_{\bar{\alpha}}\hat{\varphi}^\alpha \partial_{\bar{\beta}}\hat{\varphi}^\beta$. With respect to (16) it can be shown that these components can be formally identified with the components of (5). The appropriate volume form is derived similarly and reads as

$$\overline{\mathrm{vol}} = \sqrt{\det \bar{g}}\left(\mathrm{d}\bar{q}^{\bar{1}} - \left(\partial_0 \varphi^{\bar{1}} \circ \hat{\varphi}\right)\mathrm{d}t^0\right) \wedge \ldots$$
$$\ldots \wedge \left(\mathrm{d}\bar{q}^{\bar{n}} - \left(\partial_0 \varphi^{\bar{n}} \circ \hat{\varphi}\right)\mathrm{d}t^0\right) .$$

It is worth mentioning that the metric and the volume form in the convected picture are explicitly time dependent in contrast to the usage of the initial system for the spatial picture.

Consequently, the electromagnetic two form $F$ in the convected picture is obtained by $\varphi^*\left(\bar{F}\right) = F$ where the components read as

$$\begin{aligned} \bar{E}_{0\bar{\alpha}} &= \partial_{\bar{\alpha}}\hat{\varphi}^\alpha \left(E_{0\alpha} - v^\gamma B_{\gamma\alpha}\right) \circ \hat{\varphi} \\ \bar{B}_{\bar{\alpha}\bar{\beta}} &= \left(B_{\alpha\beta} \circ \hat{\varphi}\right)\partial_{\bar{\alpha}}\hat{\varphi}^\alpha \partial_{\bar{\beta}}\hat{\varphi}^\beta , \end{aligned} \quad (18)$$

in consideration of (17). The electrical conductivity form can be introduced as a vector valued form corresponding to

$$\bar{\kappa} = \bar{\kappa}^{\bar{\alpha}\bar{\beta}}\partial_{\bar{\alpha}} \rfloor \overline{\mathrm{vol}} \otimes \partial_{\bar{\beta}} .$$

Thus, Ohm's Law can be formulated in the convected picture which follows to

$$\bar{j} = -\bar{\kappa} \rfloor \left(\partial_0 \rfloor \bar{F}\right) = \bar{\kappa}^{\bar{\alpha}\bar{\beta}} \bar{E}_{0\bar{\beta}}\,\partial_{\bar{\alpha}} \rfloor \overline{\mathrm{vol}} = \bar{j}^{\bar{\alpha}}\partial_{\bar{\alpha}} \rfloor \overline{\mathrm{vol}} ,$$

where the conductive current density $\bar{j}$ results from the electric field strength which the medium actually receives. For the equivalent expression in the spatial picture we have to evaluate $\varphi^*\left(\bar{j}\right) = j$ which reads as

$$j = j^\alpha \partial_\alpha \rfloor \mathrm{vol} = \kappa^{\alpha\beta}\left(E_{0\beta} - v^\gamma B_{\gamma\beta}\right)\partial_\alpha \rfloor \mathrm{vol} , \quad (19)$$

where we also have $\varphi^*\left(\bar{\kappa}\right) = \kappa$, with $\kappa^{\alpha\beta} = \left(\bar{\kappa}^{\bar{\alpha}\bar{\beta}}\partial_{\bar{\alpha}}\hat{\varphi}^\alpha \partial_{\bar{\beta}}\hat{\varphi}^\beta\right) \circ \varphi$, and where the relation $\partial_{\bar{\alpha}} \to \partial_{\bar{\alpha}}\hat{\varphi}^\alpha \partial_\alpha$ has been used. If the Hall current and ion-slip effects are neglected (reasonable approximation for conducting liquids, see [14]), Ohm's Law in classical MHD is of this simple form usually with components $\bar{\kappa}^{\bar{\alpha}\bar{\beta}} = \eta\,\bar{g}^{\bar{\alpha}\bar{\beta}}$, $\eta \in \mathbb{R}^+$, or equivalently $\kappa^{\alpha\beta} = \eta\, g^{\alpha\beta}$ due to the components of $\bar{g}$.

*Remark 2:* The former introduced components of the Cauchy stress form (6) including the hydrostatic pressure are still valid if this constitutive relation would be introduced in the convected picture and, afterwards, transformed back to obtain the equivalent expression in spatial coordinates, see, e.g., [1] for a more general discussion.

Finally, the electromagnetic force density caused by the conductive current reads as

$$f^D = -j \rfloor B = -\mathrm{vol} \otimes \left(j^\alpha B_{\alpha\beta}\right)\mathrm{d}q^\beta , \quad (20)$$

since the conductive current density is represented by a vector valued form which is isomorphic to $\mathrm{vol} \otimes j^\alpha \partial_\alpha$, see [4]. Finally, we are able to derive

$$\begin{aligned} F^D &= J\left(\kappa^{\beta\delta} E_{0\delta} B_{\alpha\beta}\right) \circ \Phi\, \mathrm{VOL} \otimes \mathrm{d}q^\alpha \\ &\quad - J V_0^\gamma \left(\kappa^{\beta\delta} B_{\gamma\delta} B_{\alpha\beta}\right) \circ \Phi\, \mathrm{VOL} \otimes \mathrm{d}q^\alpha , \end{aligned} \quad (21)$$

with respect to the material picture.

*Remark 3:* In MHD the electrostatic body force and the force density caused by the convective transport of charge are usually negligible with respect to the body force caused by the conductive current due to the quasi-neutrality of the fluid, see [3], [14], for instance. Therefore, in MHD the charge density is usually set to zero in the governing equations (MHD approximation) and, consequently, the equation of continuity - $\partial_0 \mu + \mathrm{div}\left(\mu v^\alpha + j^\alpha\right) = 0$ in the spatial picture containing the total current density (sum of convective and conductive current density) - degenerates to $\mathrm{div}\left(j^\alpha\right) = 0$.

## IV. The Euler-Lagrange Equations of iMHD

In this section we focus our interests on the Lagrangian point of view in order to derive the governing equations of iMHD without dissipative effects from a first-order Lagrangian density. Therefore, we consider the case of a vanishing conductivity, meaning that the induced conductive current may be neglected and thus, we deal with a convective current density only. For this case we propose a first-order Lagrangian density $\mathcal{L} \in C^\infty\left(\mathcal{J}^1(\mathcal{E})\right)$ of the form

$$\begin{aligned}\mathcal{L} &= e_{kin} - e_{pot} - e_{mag} \\ &= \frac{1}{2}\rho_\mathcal{R} g_{\alpha\beta} q_0^\alpha q_0^\beta \sqrt{\det G} - \rho_\mathcal{R} E_{st} \sqrt{\det G} \\ &\quad + \mu_\mathcal{R}\left(A_0 + q_0^\alpha A_\alpha\right)\sqrt{\det G}\end{aligned} \tag{22}$$

which includes the kinetic and the associated potential energy density, $e_{kin}$ and $e_{pot}$. Additionally, the relevant electromagnetic energy density meets $e_{mag} \circ j^1\Phi = -\mu_\mathcal{R}\partial_0 \rfloor \Phi^*(A)$ which can be interpreted as the material electrostatic potential. Therefore, the appropriate first-order Lagrangian $L : \mathcal{J}^1(\mathcal{E}) \to \left(\lambda^1\right)^*\left(\wedge^{n+1}\mathcal{T}^*(\mathcal{M})\right)$ meets $L = \mathcal{L}\,\Omega \wedge \mathrm{d}t^0$ with regard to $\lambda^1 : \mathcal{J}^1(\mathcal{E}) \to \mathcal{M}$. Then it is well-known that the partial differential equations for a first-order Lagrangian follow, in general, to

$$\delta_\alpha \mathcal{L} = 0\,, \quad \delta_\alpha = \partial_\alpha - d_0\partial_\alpha^0 - d_i\partial_\alpha^i \tag{23}$$

with $\partial_\alpha^0 = \frac{\partial}{\partial q_0^\alpha}$, $\partial_\alpha^i = \frac{\partial}{\partial q_i^\alpha}$ and the corresponding total derivatives are of the form

$$\begin{aligned} d_0 &= \partial_0 + q_0^\alpha\partial_\alpha + q_{00}^\alpha\partial_\alpha^0 + q_{0i}^\alpha\partial_\alpha^i\,, \\ d_i &= \partial_i + q_i^\alpha\partial_\alpha + q_{ij}^\alpha\partial_\alpha^j + q_{i0}^\alpha\partial_\alpha^0\,. \end{aligned}$$

This procedure is a standard result in variational calculus, see, e.g., [5], [8], [9] and references therein. At this point we do not discuss the boundary terms, since this topic will be treated in detail in the following sections. In consideration of the former presented explanations (7), (8) concerning the stored energy we conclude that $\rho_\mathcal{R}\partial_\alpha^i E_{st} = -\mathcal{P}\breve{J}\hat{F}_\alpha^i$ is fulfilled with respect to $\partial_\alpha^i\breve{J} = \breve{J}\hat{F}_\alpha^i$ and, formally, $\mathcal{P} \circ j^1\Phi = \mathcal{P}_\mathcal{R}$. Thus, the governing equations of iMHD follow from (23) with respect to (22) which, finally, read as

$$q_{00}^\beta + \gamma_{\gamma\varepsilon}^\beta q_0^\gamma q_0^\varepsilon = \frac{1}{\rho} g^{\alpha\beta}\hat{F}_\alpha^i d_i\mathcal{P} + \frac{\mu}{\rho} g^{\alpha\beta}\left(E_{\alpha 0} - q_0^\gamma B_{\gamma\alpha}\right) \tag{24}$$

in consideration of the parametrisations (11). By plugging in a section $\Phi$ theses equations are partial differential equations in the unknown functions $\Phi^\alpha$. Furthermore, the impact of the electrostatic body force and the body force caused by the convective transport of charge in (24) is easily seen.

## V. Hamiltonian Evolutionary Approach

This section is dedicated to a brief introduction of the evolutionary approach for first-order Hamiltonian field theory based on, e.g., [8], [10], [11] and references therein. Let us introduce the bundle $\pi : \mathcal{D} \to \mathcal{B}$ equipped with coordinates $\left(X^i, x^\alpha\right) \to \left(X^i\right)$. A section is given by $\Phi : \mathcal{B} \to \mathcal{D}$. Furthermore, we have a vertical bundle $\mathcal{V}(\mathcal{D}) \to \mathcal{D}$ with coordinates $\left(X^i, x^\alpha, \dot{x}^\alpha\right)$ on $\mathcal{V}(\mathcal{D})$ and introduce the first jet bundle $\pi_0^1 : \mathcal{J}^1(\mathcal{D}) \to \mathcal{D}$ where $\mathcal{J}^1(\mathcal{D})$ possesses the coordinates $\left(X^i, x^\alpha, x_i^\alpha\right)$. The second jet bundle is given by $\pi_0^2 : \mathcal{J}^2(\mathcal{D}) \to \mathcal{D}$ with coordinates $\left(X^i, x^\alpha, x_i^\alpha, x_{ij}^\alpha\right)$ on $\mathcal{J}^2(\mathcal{D})$. In addition, we introduce the generalised Hamiltonian vector field $v_H = v_H^\alpha\partial_\alpha$ as a section $v_H : \mathcal{J}^2(\mathcal{D}) \to \left(\pi_0^2\right)^*(\mathcal{V}(\mathcal{D}))$. Its first prolongation reads as $j^1(v_H) = v_H^\alpha\partial_\alpha + d_i(v_H)\partial_\alpha^i$ with $\partial_\alpha^i = \frac{\partial}{\partial x_i^\alpha}$, see [8], where we have used the total derivative with respect to the first-order case which meets $d_i = \partial_i + x_i^\alpha\partial_\alpha + x_{ij}^\alpha\partial_\alpha^j$ and which satisfies $(d_i f) \circ j^2\Phi = \partial_i\left(f \circ j^1\Phi\right)$ for $f \in C^\infty\left(\mathcal{J}^1(\mathcal{D})\right)$, see [9]. We consider a first-order Hamiltonian density $\mathcal{H} \in C^\infty\left(\mathcal{J}^1(\mathcal{D})\right)$ and the corresponding Hamiltonian functional of the form $\int_\mathcal{B}\left(j^1\Phi\right)^*(\mathcal{H}\,\Omega)$. The total time change of the functional reads as

$$\int_\mathcal{B}\left(j^2\Phi\right)^*\left(L_{j^1(v_H)}(\mathcal{H}\,\Omega)\right) = \int_\mathcal{B}\left(j^2\Phi\right)^*\left(j^1(v_H)\rfloor \mathrm{d}(\mathcal{H}\,\Omega)\right)$$

where we have $j^1(v_H)\rfloor\mathrm{d}(\mathcal{H}\Omega) = v_H^\alpha\partial_\alpha\mathcal{H} + d_i\left(v_H^\alpha\right)\partial_\alpha^i\mathcal{H}$. As usual, the expression $L_w(\cdot)$ denotes the Lie derivative of $(\cdot)$ along the vector field $w$. Via integration by parts this result can be written as

$$\int_\mathcal{B}\left(j^2\Phi\right)^*\left(v_H^\alpha\delta_\alpha\mathcal{H}\,\Omega + \mathrm{d}_h\left(v_H^\alpha\partial_\alpha^i\mathcal{H}\,\partial_i\rfloor\Omega\right)\right)\,,$$

including the variational derivative $\delta_\alpha = \partial_\alpha - d_i\partial_\alpha^i$. Additionally, we have introduced the horizontal derivative $\mathrm{d}_h = \mathrm{d}X^i \wedge L_{\mathrm{d}_i}$ which satisfies $\left(j^2\Phi\right)^* \circ \mathrm{d}_h = \mathrm{d} \circ \left(j^1\Phi\right)^*$, see [9] for more detailed information. Using Stokes' theorem we are able to derive

$$\begin{aligned}\int_\mathcal{B}\left(j^2\Phi\right)^*\left(L_{j^1(v_H)}(\mathcal{H}\,\Omega)\right) &= \int_\mathcal{B}\left(j^2\Phi\right)^*\left(v_H^\alpha\delta_\alpha\mathcal{H}\,\Omega\right) \\ &\quad + \int_{\partial\mathcal{B}}\left(j^1\Phi\right)^*\left(v_H^\alpha\partial_\alpha^i\mathcal{H}\,\partial_i\rfloor\Omega\right)\,,\end{aligned} \tag{25}$$

where it is worth mentioning that the Hamiltonian functional serves as a conserved quantity if the resulting terms inside the domain as well as on the boundary vanish. For this case, it is obvious to represent the evolution equations in the form $\dot{x}^\alpha = \mathfrak{J}^{\alpha\beta}\delta_\beta\mathcal{H}$ with a skew-symmetric map $\mathfrak{J}$ which meets $\mathfrak{J}^{\alpha\beta} = -\mathfrak{J}^{\beta\alpha} \in C^\infty\left(\mathcal{J}^2(\mathcal{D})\right)$. According to [10], [11], this procedure can be extended for the general case such that dissipative effects and the definition of system inputs acting on the domain as well as on the boundary can be taken into account. Therefore, we introduce the PCHD system representation in the form

$$\dot{x}^\alpha = \left(\mathfrak{J}^{\alpha\beta} - \mathfrak{R}^{\alpha\beta}\right)\delta_\beta\mathcal{H} + \mathfrak{G}_\xi^\alpha u^\xi\,, \quad y_\xi = \mathfrak{G}_\xi^\alpha\delta_\alpha\mathcal{H}\,, \tag{26}$$

where $\mathfrak{R}$ is a symmetric, positive semidefinite map with components $\mathfrak{R}^{\alpha\beta} = \mathfrak{R}^{\beta\alpha} \in C^\infty\left(\mathcal{J}^2(\mathcal{D})\right)$. Furthermore, we have introduced the input map $\mathfrak{G} : \mathcal{U} \to \left(\pi_0^2\right)^*(\mathcal{V}(\mathcal{D}))$, where $\mathcal{U}$ denotes the input space with coordinates $u^\xi$, $\xi = 1, \ldots, m$, and the choice for the so-called collocated output $y_\xi$. Therefore, the output space is given by $\mathcal{Y} = \mathcal{U}^*$. In general, these maps can also be differential operators, see [8], which will be the objective in the following part.

## VI. Hamiltonian Evolution Equations of iMHD: The Case of negligible Conductivity

This section presents the derivation of the Hamiltonian evolution equations of iMHD in consideration of the case of a vanishing electrical conductivity. Therefore, we intend to represent the equations of (24), via the Legendre transformation concerning the momenta with respect to time, as an autonomous, lossless system, where we will see that the Hamiltonian functional serves (for this case) as a conserved quantity provided that a suitable choice of the boundary conditions is met. In the following we have to combine the geometric considerations of the last sections. Therefore, the base manifold $\mathcal{B}$ possesses the coordinates of the spatial part $X^i$ of the reference manifold and $\mathcal{D}$ is equipped with coordinates $\left(X^i, q^\alpha, p_\alpha\right)$ including the momenta with respect to time. Consequently, the generalised Hamiltonian vector field reads as $v_H = \dot{q}^\alpha \partial_\alpha + \dot{p}_\alpha \partial^\alpha$ with $\partial^\alpha = \frac{\partial}{\partial p_\alpha}$. In order to cope with the former presented results it is worth mentioning that we have the identification $\dot{q}^\alpha = q_0^\alpha$ since the time coordinate $t^0$ now has the interpretation of the flow parameter (evolution parameter). Obviously, we have $G_{ij}, \rho_\mathcal{R}, \mu_\mathcal{R} \in C^\infty(\mathcal{B})$, $g_{\alpha\beta} \in C^\infty(\mathcal{D})$ and $E_{st} = E_{st}\left(\breve{J}\right) \in C^\infty\left(\mathcal{J}^1(\mathcal{D})\right)$, of course. Moreover, for the present we make the assumption of quasi-stationary electromagnetic fields, since we have not defined any system inputs yet. Therefore, we have $A_0, A_\alpha \in C^\infty(\mathcal{D})$ in order to deal with the recently introduced bundle structure. For this special case the Lagrangian density of (22) is time-independent and the Hamiltonian density reads as

$$\mathcal{H} = \left(p_\alpha \dot{q}^\alpha - \mathcal{L}\right) \circ \hat{\tau}\,, \tag{27}$$

see, e.g., [13], where we have introduced the momenta with respect to time resulting from a Legendre transformation $\tau : (q^\alpha, \dot{q}^\alpha) \to (q^\alpha, p_\alpha)$ of the form

$$p_\alpha = \dot{\partial}_\alpha \mathcal{L} = \sqrt{\det G}\left(\rho_\mathcal{R} g_{\alpha\beta}\dot{q}^\beta + \mu_\mathcal{R} A_\alpha\right)\,, \tag{28}$$

which consist of the sum of the mechanical momenta and of electromagnetic parts. The inverse $\hat{\tau}$ exist due to the positive definiteness of the metric resulting in

$$\dot{q}^\beta = \frac{1}{\rho_\mathcal{R}} g^{\alpha\beta}\left(\frac{1}{\sqrt{\det G}} p_\alpha - \mu_\mathcal{R} A_\alpha\right)\,. \tag{29}$$

In consideration of (22), (27) and (29) the Hamiltonian density for the iMHD case with vanishing conductivity reads as

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2\rho_\mathcal{R}} g^{\alpha\beta}\left(\frac{1}{\sqrt{\det G}} p_\alpha - \mu_\mathcal{R} A_\alpha\right)\cdot \\
&\quad \left(\frac{1}{\sqrt{\det G}} p_\beta - \mu_\mathcal{R} A_\beta\right)\sqrt{\det G} \\
&\quad + \rho_\mathcal{R} E_{st}\sqrt{\det G} - \mu_\mathcal{R} A_0 \sqrt{\det G} \\
&= e_{kin} + e_{pot} - \mu_\mathcal{R} A_0 \sqrt{\det G}\,.
\end{aligned} \tag{30}$$

Applying the machinery illustrated before the evolution equations are of the form

$$\dot{q}^\alpha = \delta^\alpha \mathcal{H} = \partial^\alpha \mathcal{H}\,, \quad \dot{p}_\alpha = -\delta_\alpha \mathcal{H}$$

or equivalently

$$\begin{bmatrix} \dot{q}^\alpha \\ \dot{p}_\alpha \end{bmatrix} = \mathfrak{J}\begin{bmatrix} \delta_\beta \mathcal{H} \\ \delta^\beta \mathcal{H} \end{bmatrix} = \begin{bmatrix} 0 & \delta^\alpha_\beta \\ -\delta^\beta_\alpha & 0 \end{bmatrix}\begin{bmatrix} \delta_\beta \mathcal{H} \\ \delta^\beta \mathcal{H} \end{bmatrix}. \tag{31}$$

The equations concerning the velocities are given by (29) and the equations for the momenta with respect to time, finally, read as

$$\begin{aligned}
\dot{p}_\alpha &= -\frac{1}{2\rho_\mathcal{R}}\frac{1}{\sqrt{\det G}}\left(\partial_\alpha g^{\gamma\beta}\right)\left(p_\beta - \mu_\mathcal{R}\sqrt{\det G} A_\beta\right)\cdot \\
&\qquad \left(p_\gamma - \mu_\mathcal{R}\sqrt{\det G} A_\gamma\right) \\
&\quad + \frac{1}{\rho_\mathcal{R}} g^{\gamma\beta}\left(p_\beta - \mu_\mathcal{R}\sqrt{\det G} A_\beta\right)\mu_\mathcal{R}\left(\partial_\alpha A_\gamma\right) \\
&\quad + \mu_\mathcal{R}\left(\partial_\alpha A_0\right)\sqrt{\det G} - \sqrt{\det G} J \hat{F}^i_\alpha d_i \mathcal{P}\,,
\end{aligned}$$

which are the counterparts of (24) with regard to (29).

In the end, we have to analyse the boundary conditions. With regard to the total time change of the Hamiltonian functional (25) the domain term vanishes completely due to (31) and the term on the boundary reads as

$$\mathcal{Q}^0_\partial = \int_{\partial\mathcal{B}} \left(j^1\Phi\right)^*\left(v^\alpha_H \partial^i_\alpha \mathcal{H}\,\partial_i \rfloor \Omega\right) = \int_{\partial\mathcal{B}} \left(j^1\Phi\right)^*\left(\dot{q}^\alpha \partial^i_\alpha \mathcal{H}\partial_i \rfloor \Omega\right)$$

where in consideration of (7), (8) we have

$$\partial^i_\alpha \mathcal{H} = \rho_\mathcal{R}\partial^i_\alpha E_{st}\sqrt{\det G} = -\mathcal{P}\breve{J}\hat{F}^i_\alpha\sqrt{\det G}\,.$$

Furthermore, we can rewrite this condition with respect to the spatial picture and obtain

$$\begin{aligned}
\mathcal{Q}^0_\partial &= -\int_{\partial\mathcal{B}} J\left(\hat{F}^i_\alpha \circ \Phi\right) V^\alpha_0 \mathcal{P}_\mathcal{R}\,\partial_i \rfloor \mathrm{VOL} \\
&= -\int_{\partial\Phi(t^0,\mathcal{B})} v^\alpha \mathcal{P}_\mathcal{K}\,\partial_\alpha \rfloor \mathrm{vol}\,.
\end{aligned} \tag{32}$$

If the boundary conditions are such that (32) vanishes the Hamiltonian functional (30) serves as a conserved quantity since its total time change vanishes completely.

## VII. Hamiltonian Evolution Equations of iMHD: Considering the Conductive Current

The objective of this section is to take the body force into account which is caused by the conductive current density in connection with the external electromagnetic fields. Therefore, we extend the structure of (31) with regard to dissipative effects and the definition of system inputs to obtain a Port-Controlled Hamiltonian system representation in the form of (26). According to Remark 3, the equation of continuity for the conservation of charge is no more of the form as in the case of convective currents only. Nevertheless, in order to deal with the material picture which has turned out to be crucial for the Hamiltonian evolutionary approach, we take the MHD approximation of Remark 3 into account. For this case the momenta with respect to time read as

$$\widetilde{p}_\alpha = \rho_\mathcal{R} g_{\alpha\beta}\dot{q}^\beta\sqrt{\det G}$$

which are equal to the mechanical momenta and the Hamiltonian density simplifies to

$$\widetilde{\mathcal{H}} = \frac{1}{2\sqrt{\det G}\rho_\mathcal{R}} g^{\alpha\beta}\widetilde{p}_\alpha\widetilde{p}_\beta + \rho_\mathcal{R} E_{st}\sqrt{\det G}\,.$$

Applying the machinery of the evolutionary point of view, we end up with

$$\begin{aligned} \dot{q}_\alpha &= \frac{1}{\rho_{\mathcal{R}}\sqrt{\det G}} g^{\alpha\beta}\widetilde{p}_\alpha \,, \\ \dot{\widetilde{p}}_\alpha &= -\frac{1}{2\rho_{\mathcal{R}}\sqrt{\det G}}\left(\partial_\alpha g^{\beta\delta}\right)\widetilde{p}_\beta\widetilde{p}_\delta - \sqrt{\det G}\breve{J}\hat{F}^i_\alpha d_i \mathcal{P} \,, \end{aligned}$$

or equivalently

$$\begin{bmatrix} \dot{q}^\alpha \\ \dot{\widetilde{p}}_\alpha \end{bmatrix} = \mathfrak{J}\begin{bmatrix} \delta_\beta\widetilde{\mathcal{H}} \\ \delta^\beta\widetilde{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} 0 & \delta^\alpha_\beta \\ -\delta^\beta_\alpha & 0 \end{bmatrix}\begin{bmatrix} \delta_\beta\widetilde{\mathcal{H}} \\ \delta^\beta\widetilde{\mathcal{H}} \end{bmatrix}$$

which are clearly the equations for a vanishing charge density and conductivity. The boundary term is still of the form (32). Before we extend these equations with respect to the body force caused by the conductive current density we intend to define the electrostatic potential as our system input. Therefore, we set $u = A_0\left(q^\alpha, t^0\right)$. It is clear, then, that we have to replace the input space $\mathcal{U}$, necessary for the PCHD structure, by the input bundle $\mathcal{U} \rightarrow \mathcal{D}\times\mathcal{I}$ equipped with coordinates $\left(t^0, X^i, q^\alpha, p_\alpha, u\right) \rightarrow \left(t^0, X^i, q^\alpha, p_\alpha\right)$.

*Remark 4:* The input bundle is introduced such that the time coordinate explicitly appears in order to take the time dependency of the input into account. This fact is represented by the product structure of the base manifold of the input bundle.

In the following part we focus our interests on the properties of the chosen input and its impact on the domain as well as on the boundary. Furthermore, the vector potential still has the components $A_\alpha \in C^\infty(\mathcal{D})$ and causes a quasi-stationary magnetic field in this point of view. The electromagnetic force density of (21) takes the form of

$$\begin{aligned} F^D &= -JV^\gamma_0\left(\kappa^{\beta\delta}B_{\gamma\delta}B_{\alpha\beta}\right)\circ\Phi\,\mathrm{VOL}\otimes\mathrm{d}q^\alpha \\ &\quad + J\left(\kappa^{\beta\delta}B_{\alpha\beta}\partial_\delta A_0\right)\circ\Phi\,\mathrm{VOL}\otimes\mathrm{d}q^\alpha \,, \end{aligned} \quad (33)$$

where this force density splits into two parts; the first part consists of a quadratic term with respect to the magnetic flux density and the second part contains the chosen system input acting on the domain. First, we analyse the quadratic term. In order to consider this expression in the PCHD system representation (26) we have to demand that $\kappa^{\alpha\beta} = \kappa^{\beta\alpha}$ and, additionally, that the conductivity form is a positive semidefinite map. Obviously, for the simple case $\kappa^{\alpha\beta} = \eta\, g^{\alpha\beta}$, illustrated before, these conditions are fulfilled, of course. Next, we analyse the second part of the force density which contains the system input. The components of this part can be rewritten as $\breve{J}\kappa^{\beta\gamma}B_{\alpha\beta}\hat{F}^i_\gamma d_i A_0$ and thus, it is clear that the input map $\mathfrak{G}$ of (26) plays the role of a differential operator. Finally, in consideration of the electromagnetic body force caused by the conductive current density (33) we are able to derive the PCHD system representation for the iMHD case corresponding to

$$\begin{aligned} \begin{bmatrix} \dot{q}^\alpha \\ \dot{\widetilde{p}}_\alpha \end{bmatrix} &= (\mathfrak{J}-\mathfrak{R})\begin{bmatrix} \delta_\beta\widetilde{\mathcal{H}} \\ \delta^\beta\widetilde{\mathcal{H}} \end{bmatrix} + \mathfrak{G}[u] \\ &= \left(\begin{bmatrix} 0 & \delta^\alpha_\beta \\ -\delta^\beta_\alpha & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{R}_{\alpha\beta} \end{bmatrix}\right)\begin{bmatrix} \delta_\beta\widetilde{\mathcal{H}} \\ \delta^\beta\widetilde{\mathcal{H}} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathfrak{G}_\alpha[u] \end{bmatrix} \end{aligned}$$

where the symmetric, positive semidefinite map $\mathfrak{R}$ contains the first part of the body force of (33) with

$$\mathfrak{R}_{\alpha\beta} = \breve{J}\kappa^{\gamma\delta}B_{\alpha\gamma}B_{\beta\delta}\sqrt{\det G} = \mathfrak{R}_{\beta\alpha}\,, \quad [\mathfrak{R}_{\alpha\beta}] \geq 0\,,$$

and, for this case, $\mathfrak{R}_{\alpha\beta} \in C^\infty\left(\mathcal{J}^1(\mathcal{D})\right)$. The input map $\mathfrak{G}$ contains the second part of (33) with the operator

$$\begin{aligned} \mathfrak{G}_\alpha &= \breve{J}\kappa^{\beta\gamma}B_{\alpha\beta}\sqrt{\det G}\hat{F}^i_\gamma d_i \\ &= \kappa^{\beta\gamma}B_{\alpha\beta}\det F\sqrt{\det g}\hat{F}^i_\gamma d_i = \mathfrak{G}^i_\alpha d_i \,, \end{aligned} \quad (34)$$

with $\mathfrak{G}^i_\alpha \in C^\infty\left(\mathcal{J}^1(\mathcal{D})\right)$. Next, we intend to analyse the total time change of the Hamiltonian functional which does not serve as a conserved quantity any longer in detail, since its total time change reads as

$$\begin{aligned} &\int_{\mathcal{B}}\left(j^2\Phi\right)^*\left(j^1(v_H)\rfloor\mathrm{d}\left(\widetilde{\mathcal{H}}\,\Omega\right)\right) \\ &= \int_{\mathcal{B}}\left(j^2\Phi\right)^*\left(-\delta^\alpha\widetilde{\mathcal{H}}\,R_{\alpha\beta}\,\delta^\beta\widetilde{\mathcal{H}} + \delta^\alpha\widetilde{\mathcal{H}}\,\mathfrak{G}_\alpha[u]\right)\Omega + \mathcal{Q}^0_\partial \,, \end{aligned} \quad (35)$$

in consideration of nontrivial boundary conditions (32). Choosing the electrostatic potential as our system input and the corresponding input map playing the role of a differential operator enable us to gain more insight concerning the relevant energy flows inside the domain and over the boundary. Therefore, with respect to the abbreviation

$$\delta^\alpha\widetilde{\mathcal{H}} = \Psi^\alpha\left(X^i, q^\alpha, p_\alpha\right) \in C^\infty(\mathcal{D})\,,$$

we are able to introduce the formal adjoint $\mathfrak{G}^*_\alpha$ of the operator (34) via integration by parts, see, e.g., [8], leading to

$$\int_{\mathcal{B}}\left(j^2\Phi\right)^*\left(\Psi^\alpha\mathfrak{G}_\alpha[u]\,\Omega\right) = \int_{\mathcal{B}}\left(j^2\Phi\right)^*\left(u\,\mathfrak{G}^*_\alpha[\Psi^\alpha]\,\Omega\right) + \mathcal{Q}^{\mathfrak{G}}_\partial \,, \quad (36)$$

where the adjoint operator is given by $\mathfrak{G}^*_\alpha = -\mathfrak{G}^i_\alpha d_i - d_i\left(\mathfrak{G}^i_\alpha\right)$. Hence, the corresponding domain term including the system input in (35) splits into two parts; the first part again is a term acting on the domain containing the adjoint operator and the second part degenerates to a term on the boundary denoted by $\mathcal{Q}^{\mathfrak{G}}_\partial$. This additional boundary term takes the form of

$$\begin{aligned} \mathcal{Q}^{\mathfrak{G}}_\partial &= \int_{\mathcal{B}}\left(j^2\Phi\right)^*\left(\mathrm{d}_h\left(u\,\mathfrak{G}^i_\alpha\Psi^\alpha\partial_i\rfloor\Omega\right)\right) \\ &= \int_{\mathcal{B}}\mathrm{d}\left(JV^\alpha_0\left(j^1\Phi\right)^*\left(u\,\kappa^{\beta\gamma}B_{\alpha\beta}\hat{F}^i_\gamma\partial_i\rfloor\mathrm{VOL}\right)\right) \\ &= -\int_{\partial\Phi(t^0,\mathcal{B})} s^\gamma u\,\partial_\gamma\rfloor\mathrm{vol}\,, \end{aligned}$$

containing the chosen input acting on the system boundary. Furthermore, we have introduced the form $s = s^\gamma\partial_\gamma\rfloor\mathrm{vol}$

with components $s^{\gamma} = -\kappa^{\beta\gamma} v^{\alpha} B_{\alpha\beta}$ which equals the part of the conductive current density caused by the motion only, cf. (19). The expression concerning the adjoint operator in (36) reads as

$$\int_{\mathcal{B}} (j^2\Phi)^* (u\, \mathfrak{G}^*_{\alpha} [\Psi^{\alpha}]\, \Omega) = -\int_{\mathcal{B}} (j^2\Phi)^* \left(u\, d_i \left(\mathfrak{G}^i_{\alpha} \Psi^{\alpha}\right) \Omega\right)$$
$$= \int_{\mathcal{B}} (j^2\Phi)^* \left(u\, d_i \left(\breve{J}\hat{F}^i_{\gamma} s^{\gamma} \sqrt{\det G}\right) \Omega\right)$$
$$= \int_{\Phi(t^0,\mathcal{B})} u \operatorname{div}(s^{\gamma}) \operatorname{vol}.$$

Obviously, for the case

$$s^{\gamma} = -\kappa^{\beta\gamma} v^{\alpha} B_{\alpha\beta} = \text{const.}$$

the electrostatic potential has no impact on the domain and, consequently, the primary domain input could act on the system the same way as a boundary input does.

*Remark 5:* It is worth mentioning that by a rearrangement of the terms in (35) it is possible to show that the total time change of the Hamiltonian functional consists of a term describing the Ohmic power loss and, besides the boundary conditions $\mathcal{Q}^0_{\partial}$, a boundary term which contains the product of the input and the conductive current density (on the boundary), provided that the degenerated continuity equation may be taken into account, see Remark 3.

## VIII. Conclusions and Future Work

This contribution presents a Hamiltonian description of iMHD and its infinite-dimensional PCHD system representation which is crucial with respect to control theoretic aspects. Since the presented approach is treated in a purely geometric fashion functional analytic aspects are completely missing. This fact represents the content of the future work besides the consideration of viscous stresses in the illustrated framework for iMHD and the formulation of specific feedback control problems, mainly for industrial applications. There, the material description, which has turned out to be the basis for the Hamiltonian evolutionary approach, could be an appropriate tool for describing, e.g., recasting or welding processes in remelting furnaces. Furthermore, it is worth mentioning that the (degenerated) continuity equation for the conservation of charge is not directly included in the presented framework which would lead to a restriction equation for the electrostatic potential. This fact should also be treated leading us to the class of infinite-dimensional PCHD systems with constraints.

## IX. Acknowledgments

The second author has been supported by the Austrian Center of Competence in Mechatronics (ACCM).

## References

[1] Aris R., *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Dover Publications NY, 1989.
[2] Burke W.L., *Applied Differential Geometry*, Cambridge University Press, 1994.
[3] Eringen A.C. and Maugin G.A., *Electrodynamics of Continua II: Fluids and Complex Media*, Springer-Verlag NY, 1990;
[4] Frankel T., *The Geometry of Physics*, Cambridge University Press, 2nd ed., 2004.
[5] Giachetta G., Mangiarotti L. and Sardanashvily G., *New Lagrangian and Hamiltonian Methods in Field Theory*, World Scientific Singapore, 1997.
[6] Marsden J.E. and Hughes T.J.R., *Mathematical Foundations of Elasticity*, Dover Publications NY, 1994.
[7] Marsden J.E, Pekarsky S., Shkoller S. and West M., "Variational methods, multisymplectic geometry and continuum mechanics", in *J. of Geometry and Physics*, pp. 253–284, 2001.
[8] Olver P.J., *Applications of Lie Groups to Differential Equations*, Springer-Verlag NY, 1993.
[9] Saunders D.J., *The Geometry of Jet Bundles*, Cambridge University Press, 1989.
[10] Schlacher K., "Mathematical Modeling for Nonlinear Control: A Hamiltonian Approach, in *Mathematics and Computers in Simulation*, vol. 79, No. 4, pp. 829–849, 2008.
[11] Schöberl M., Ennsbrunner H. and Schlacher K., "Modelling of piezoelectric structures - a Hamiltonian approach , in *Mathematical and Computer Modelling of Dynamical Systems*, vol. 14, No. 3, pp. 179–193, 2008.
[12] Schöberl M. and Schlacher K., "Covariant formulation of the governing equations of continuum mechanics in an Eulerian description" , in *J. of Mathematical Physics*, vol. 48, No. 5, pp. 052902–1 - 052902–15, 2007.
[13] Schöberl M. and Schlacher K., "Hamiltonian Field Theory and Mechanics, in Proceedings Vienna Mathmod 2009, No. 35, ISBN: 978-3-901608-35-3, 2009.
[14] Sutton G.W. and Sherman A., *Engineering Magnetohydrodynamics*, Dover Publications NY, 2006.
[15] van der Schaft A.J., *L2-Gain and Passivity Techniques in Nonlinear Control*, Springer-Verlag NY, 2000.
[16] van der Schaft A.J. and Maschke B.M., "Hamiltonian formulation of distributed-parameter systems with boundary energy flow", in *J. of Geometry and Physics*, vol. 42, pp. 166–194, 2002.