

Well-posedness, regularity and exact controllability for the problem of transmission of the Schrödinger equation

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Abstract—In this paper, we shall study the system of transmission of the Schrödinger equation with Dirichlet control and collocated observation. Using the multiplier method, we show that the system is well-posed with input and output space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$. The regularity of the system is also established and the feedthrough operator is found to be zero. Finally, the exact controllability of the open-loop system is obtained by proving the observability inequality of the dual system.

I. INTRODUCTION

In [12], Salamon introduced the class of well-posed linear systems. The aim was to provide a unifying abstract framework to formulate and solve control problems for systems described by functional and partial differential equations. Roughly speaking a well-posed linear system is a linear time invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. This means that every well-posed system has a well defined transfer function $G(s)$. An important subclass of well-posed linear systems is formed by the regular systems. A regular system ([13]) is a well-posed system satisfying the extra requirement that

$$\lim_{s \in \mathbb{R}, s \rightarrow +\infty} G(s) = D \text{ exists.}$$

There is now a rich literature on the abstract theory for regular well-posed linear systems and from practical point of view, the construction of specific examples of distributed parameter systems which belong to this class is of considerable importance. In recent years, a limited number of PDEs with boundary control and observation are proved to be well-posed and regular (see [1], [2], [3], [4], [5], [6], [7], [8], [9]).

In this paper, we shall study the system of transmission of the Schrödinger equation with Dirichlet control and collocated observation. Using the multiplier method, we show that the system is well-posed with input and output space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$. The regularity of the system is also established and the feedthrough operator is found to be zero. Finally, the exact controllability of the open-loop system is obtained by proving the observability inequality of the dual system.

II. SYSTEM DESCRIPTION AND MAIN RESULTS

Let Ω be an open bounded domain of \mathbb{R}^n ($n \geq 2$) with smooth boundary Γ , Ω_1 be a bounded domain contained inside Ω : $\Omega_1 \subset \Omega$ with smooth boundary Γ_1 , Ω_2 be the

domain $\Omega \setminus \Omega_1$ and ν be the unit normal of Γ or Γ_1 pointing towards the exterior of Ω_2 .

Let a time $T > 0$ and two distinct constants $a_1, a_2 > 0$ be given.

In this paper, we shall be concerned with the following system of transmission of the Schrödinger equation with Dirichlet control and collocated observation.

$$y'(x, t) = \mathbf{i}a(x)\Delta y(x, t), \quad (x, t) \in \Omega \times (0, T) \quad (1)$$

$$y(x, 0) = y^0(x), \quad x \in \Omega \quad (2)$$

$$y_2(x, t) = u(x, t), \quad (x, t) \in \Gamma \times (0, T) \quad (3)$$

$$y_1(x, t) = y_2(x, t), \quad (x, t) \in \Gamma_1 \times (0, T) \quad (4)$$

$$a_1 \frac{\partial y_1(x, t)}{\partial \nu} = a_2 \frac{\partial y_2(x, t)}{\partial \nu}, \quad (x, t) \in \Gamma_1 \times (0, T) \quad (5)$$

$$z(x, t) = \mathbf{i} \frac{\partial}{\partial \nu} (A^{-1}y_2(x, t)), \quad (x, t) \in \Gamma \times (0, T) \quad (6)$$

where

$$- y'(x, t) = \frac{\partial y(x, t)}{\partial t}$$

$$- a(x) = \begin{cases} a_1, & x \in \Omega_1 \\ a_2, & x \in \Omega_2 \end{cases},$$

$$- y(x, t) = \begin{cases} y_1(x, t), & (x, t) \in \Omega_1 \times (0, T) \\ y_2(x, t), & (x, t) \in \Omega_2 \times (0, T) \end{cases}$$

- $A : H^{-1}(\Omega) \rightarrow H^{-1}(\Omega)$ is a positive selfadjoint operator defined by

$$Af = -\Delta f, \quad D(A) = H_0^1(\Omega)$$

- $u(\cdot, \cdot)$ is the input function.

- $z(\cdot, \cdot)$ is the output function.

When $a_1 = a_2$, Guo and Shao have shown that the system (1) - (6) is well-posed with input and output space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$ and regular with zero as the feedthrough operator. One of the aims of this paper is to investigate the well-posedness and the regularity of the system (1) - (6) in the case where $a_1 \neq a_2$. Indeed, we shall prove the following

Theorem 1: The equations (1) - (6) determine a well-posed linear system with input and output space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$.

Theorem 2: The system (1) - (6) is regular with zero feedthrough operator. This means that if the initial state $y(\cdot, 0) = 0$ and $u(\cdot, t) = u(t) \in U$ is a step input then the corresponding output satisfies

$$\lim_{\sigma \rightarrow 0} \int_{\Gamma} \left| \frac{1}{\sigma} \int_0^\sigma z(x, t) dt \right| d\Gamma = 0 \quad (7)$$

The second aim is to establish an exact controllability result for the open-loop system (1) - (5). To state it, we shall assume the following conditions:

This work was not supported by any organization

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(A1) $a_2 < a_1$.

(A2) There exists a real vector field $h(\cdot) \in (C^1(\overline{\Omega}))^n$ such that

(A2a)

$$Re\left(\int_{\Omega} H(x)v(x).\overline{v(x)}dx\right) \geq \rho \int_{\Omega} \|v(x)\|^2 dx$$

for all $v(\cdot) \in (L^2(\Omega))^n$ for some $\rho > 0$, where

$$H(x) = \left(\frac{\partial h_i(x)}{\partial x_j}\right), i = 1, \dots, n \text{ and } j = 1, \dots, n$$

(A2b)

$$h(x).v(x) \leq 0, \quad x \in \Gamma_1$$

Theorem 3: Let $T > 0$ be arbitrary. Assume hypotheses (A1) and (A2). Then for any initial data $y^0 \in H^{-1}(\Omega)$, there a exists a control $u \in L^2(0, T; L^2(\Gamma))$ such that the corresponding solution of the system (1) - (5) satisfies $y(x, T) = 0$.

Remark 4: 1- Lasiecka and Triggiani [10] established Theorem 3 when $a_1 = a_2$.

As a consequence of Theorem 1, Theorem 3 and Proposition 3.1.1 of [11], we have the following uniform stabilization result for the system (1) - (5) on the space $H^{-1}(\Omega)$.

Corollary 5: Let the hypotheses of Theorem 3 hold true. Then there exist positive constants M, ω such that the solution of (1) - (5) with $u = -kz$ ($k > 0$) satisfies

$$\|y(t)\|_X \leq Me^{-\omega t} \|y^0\|_X$$

III. ABSTRACT FORMULATION

We define the space

$$H^2(\Omega, \Gamma_1) = \{y \in H_0^1(\Omega) : y_i \in H^2(\Omega_i); i = 1, 2; \\ y_1 = y_2, a_1 \frac{\partial y_1}{\partial \nu} = a_2 \frac{\partial y_2}{\partial \nu} \text{ on } \Gamma_1\}$$

with the norm

$$\|y\|_{H^2(\Omega, \Gamma_1)}^2 = \|y_1\|_{H^2(\Omega_1)}^2 + \|y_2\|_{H^2(\Omega_2)}^2$$

It can be shown that $H^2(\Omega, \Gamma_1)$ is dense in $H_0^1(\Omega)$.

Let $A_1 : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ be the extension of $-a(x)\Delta$ to $H_0^1(\Omega)$. This means that $Af = -a(x)\Delta f$ whenever $f \in H^2(\Omega, \Gamma_1)$ and that $A^{-1}g = (-a(x)\Delta)^{-1}g$ for any $g \in L^2(\Omega)$.

Let $A_{-1} : H^{-1}(\Omega) \rightarrow (D(A))'$ be the extension of A_1 to $H^{-1}(\Omega)$. Notice that $(D(A))'$ is the dual of $D(A)$ with respect to the pivot space $H^{-1}(\Omega)$.

Define the Dirichlet map $\gamma \in \mathcal{L}(L^2(\Gamma), L^2(\Omega))$ by

$$\gamma u = v$$

if and only if

$$a(x)\Delta v = 0 \text{ in } \Omega \\ v = u \text{ on } \Gamma \\ v_1 = v_2 \text{ on } \Gamma_1 \\ a_1 \frac{\partial v_1}{\partial \nu} = a_2 \frac{\partial v_2}{\partial \nu} \text{ on } \Gamma_1$$

Using the operators introduced above, we can rewrite (1), (3) - (5) on $(D(A))'$ as

$$y'(t) = -\mathbf{i}A_{-1}y(t) + Bu(t)$$

where $B \in \mathcal{L}(U, (D(A))')$ is given by

$$Bu = \mathbf{i}A_{-1}\gamma u$$

We have via Green's second theorem

$$\gamma^* A\psi = -\frac{\partial \psi}{\partial \nu}, \quad \psi \in D(A)$$

Hence the adjoint C of B is given by

$$C\psi = \mathbf{i}\frac{\partial}{\partial \nu}(A^{-1}\psi)$$

Now, we can reformulate the system (1) - (6) into an abstract form in the state space $H^{-1}(\Omega)$ as follows:

$$y'(t) = -\mathbf{i}A_1y(t) + Bu(t) \tag{8}$$

$$y(0) = y^0 \tag{9}$$

$$z(t) = Cu(t) \tag{10}$$

IV. SKETCH OF THE PROOF OF THEOREM 1

It is well known that the operator $-\mathbf{i}A_1$ is the infinitesimal generator of a C_0 -group of unitary operators $S(t)$ on X .

A. Admissibility of B and C for the group $S(t)$

Since the system (8) - (10) is colocated, the admissibility of B for the group $S(t)$ is equivalent to the admissibility of C for the group $S(t)$. But the latter means that

$$\int_0^T \int_{\Gamma} |CS(t)\psi|^2 d\Gamma dt \leq C_T \|\psi\|_X^2 \tag{11}$$

for all $\psi \in D(A)$ and for some $T > 0$.

The PDE version of the estimate (11) is

$$\int_0^T \int_{\Gamma} \left|\frac{\partial \varphi}{\partial \nu}\right|^2 d\Gamma dt \leq C_T \|\varphi_0\|_{H_0^1(\Omega)}^2 \tag{12}$$

where φ is the solution of

$$\varphi'(x, t) = \mathbf{i}a(x)\Delta\varphi(x, t), (x, t) \in \Omega \times (0, T) \tag{13}$$

$$\varphi(x, 0) = \varphi_0(x), \quad x \in \Omega \tag{14}$$

$$\varphi_2(x, t) = 0, (x, t) \in \Gamma \times (0, T) \tag{15}$$

$$\varphi_1(x, t) = \varphi_2(x, t), (x, t) \in \Gamma_1 \times (0, T) \tag{16}$$

$$a_1 \frac{\partial \varphi_1(x, t)}{\partial \nu} = a_2 \frac{\partial \varphi_2(x, t)}{\partial \nu}, (x, t) \in \Gamma_1 \times (0, T) \tag{17}$$

To prove (12), we apply the identity (40) with a vector field $h \in (C^1(\overline{\Omega}))^n$ such that

$$h = \nu \text{ on } \Gamma \text{ and } h = 0 \text{ in } \Omega_0$$

where the open set Ω_0 satisfies

$$\overline{\Omega}_1 \subset \Omega_0 \subset \overline{\Omega}_0 \subset \Omega$$

B. Boundedness of the input-output map

It suffices to show that the solution of (1) - (5) with $y(x, 0) = 0$ satisfies

$$\int_0^T \int_{\Gamma} \left| \frac{\partial A^{-1}y(x, t)}{\partial v} \right|^2 d\Gamma dt \leq C_T \int_0^T \int_{\Gamma} |u(x, t)|^2 d\Gamma dt \quad (18)$$

for all $u \in L^2(0, T; U)$.

From the admissibility of B , we have $y \in C(0, T; H^{-1}(\Omega))$ for every $y_0 \in H^{-1}(\Omega)$.

Let us introduce a new variable by setting

$$w(t) = A^{-1}y(t) \in C(0, T; H_0^1(\Omega))$$

Thus by (8), we obtain the abstract equation

$$w'(t) = -\mathbf{i}A_1y(t) + \mathbf{i}\gamma u(t)$$

whose corresponding partial differential problem is

$$w'(x, t) = \mathbf{i}a(x)\Delta w(x, t) + \mathbf{i}\gamma u(x, t), \quad (x, t) \in \Omega \times (0, T) \quad (19)$$

$$w(x, 0) = 0, \quad x \in \Omega \quad (20)$$

$$w_2(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T) \quad (21)$$

$$w_1(x, t) = w_2(x, t), \quad (x, t) \in \Gamma_1 \times (0, T) \quad (22)$$

$$a_1 \frac{\partial w_1(x, t)}{\partial v} = a_2 \frac{\partial w_2(x, t)}{\partial v}, \quad (x, t) \in \Gamma_1 \times (0, T) \quad (23)$$

The estimate (18) becomes

$$\int_0^T \int_{\Gamma} \left| \frac{\partial w(x, t)}{\partial v} \right|^2 d\Gamma dt \leq C_T \int_0^T \int_{\Gamma} |u(x, t)|^2 d\Gamma dt \quad (24)$$

From the identity (40) in the appendix with $f = \mathbf{i}\gamma u$ and the vector field h is as in Section 4.1, we obtain

$$\int_0^T \int_{\Gamma} \left| \frac{\partial w(x, t)}{\partial v} \right|^2 d\Gamma dt \leq C_T (\|w\|_{C(0, T; H_0^1(\Omega))}^2 + \|u\|_{L^2(0, T; L^2(\Gamma))}^2)$$

This together with the admissibility of B for the C_0 -group $S(t)$ yield (24).

V. SKETCH OF THE PROOF OF THEOREM 2

Since the system (1) - (6) is well-posed, then its transfer function $G(s)$ is bounded on some right-half plane (see [14]). To continue we need the following results.

Proposition 6: The assertion of Theorem 2 holds true if for any $u \in C_0^\infty(\Gamma)$ the solution y of

$$sy(x) + \mathbf{i}a(x)\Delta y(x) = 0, \quad x \in \Omega \quad (25)$$

$$y_2(x) = u(x), \quad x \in \Gamma \quad (26)$$

$$y_1(x) = y_2(x), \quad x \in \Gamma_1 \quad (27)$$

$$a_1 \frac{\partial y_1(x)}{\partial v} = a_2 \frac{\partial y_2(x)}{\partial v}, \quad x \in \Gamma_1 \quad (28)$$

satisfies

$$\lim_{s \in \mathbb{R}, s \rightarrow +\infty} \int_{\Gamma} \left| \frac{1}{s} \frac{\partial y(x)}{\partial v} \right|^2 d\Gamma = 0 \quad (29)$$

Proof: We know from [14] that in the frequency domain, (7) is equivalent to

$$\lim_{s \in \mathbb{R}, s \rightarrow +\infty} G(s)u = 0 \quad (30)$$

in strong topology of U for any $u \in U$. Due to the boundedness of $G(s)$ and the density of $L^2(\Gamma)$ in $C_0^\infty(\Gamma)$, it suffices to establish (30) for all $u \in C_0^\infty(\Gamma)$. Now for $u \in C_0^\infty(\Gamma)$ and $s > 0$, let

$$y = (sI + \mathbf{i}A_1)^{-1}Bu$$

Then y satisfies (25) - (28) and

$$(G(s)y)(x) = \mathbf{i} \frac{\partial(A^{-1}y)}{\partial v}(x), \quad x \in \Gamma$$

Define a function $v \in H^2(\Omega, \Gamma_1)$ by

$$a(x)\Delta v(x) = 0, \quad x \in \Omega$$

$$v_2(x) = u(x), \quad x \in \Gamma$$

$$v_1(x) = v_2(x), \quad x \in \Gamma_1$$

$$a_1 \frac{\partial v_1(x)}{\partial v} = a_2 \frac{\partial v_2(x)}{\partial v}, \quad x \in \Gamma_1$$

Then (26) - (28) can be written as

$$sy(x) + \mathbf{i}a(x)\Delta(y(x) - v(x)) = 0, \quad x \in \Omega$$

$$y_2(x) - v_2(x) = 0, \quad x \in \Gamma$$

$$y_1(x) - v_1(x) = y_2(x) - v_2(x), \quad x \in \Gamma_1$$

$$a_1 \frac{\partial(y_1(x) - v_1(x))}{\partial v} = a_2 \frac{\partial(y_2(x) - v_2(x))}{\partial v}, \quad x \in \Gamma_1$$

Hence

$$(G(s)y)(x) = \frac{a_2}{s} \frac{\partial y(x)}{\partial v} - \frac{a_2}{s} \frac{\partial v(x)}{\partial v}$$

This gives (29). ■

Lemma 7: Let h be the vector field introduced in Section 4.1. Let $u \in C_0^\infty(\Gamma)$. Then the solution of (25) satisfies

$$a_2 \int_{\Gamma} \left| \frac{\partial y}{\partial v} \right|^2 d\Gamma = 2a_2 \operatorname{Re} \int_{\Omega_2} \nabla y_2 \cdot H \nabla \bar{y}_2 d\Omega - a_2 \int_{\Omega_2} |\nabla y_2|^2 \operatorname{div}(h) d\Omega - s \operatorname{Im} \int_{\Omega} y h \cdot \nabla \bar{y} \quad (31)$$

Proof: We obtain the identity (31) after multiplying both sides of (25) by $h \cdot \nabla \bar{y}$, integrating over Ω and applying Green's theorem. ■

Now, multiply both sides of (25) by \bar{y} and integrate over Ω to obtain

$$s \int_{\Omega} |y|^2 d\Omega + \mathbf{i} \int_{\Omega} a(x) |\nabla y|^2 d\Omega = \mathbf{i} a_2 \int_{\Gamma} \bar{y}_2 \frac{\partial y_2}{\partial v} d\Gamma$$

This implies

$$\|y\|_{L^2(\Omega)} \leq \left(\frac{a_2}{s} \|y_2\|_{L^2(\Gamma)} \left\| \frac{\partial y_2}{\partial v} \right\|_{L^2(\Gamma)} \right)^{1/2} \quad (32)$$

and

$$\|\nabla y\|_{L^2(\Omega)} \leq \left(\|y_2\|_{L^2(\Gamma)} \left\| \frac{\partial y_2}{\partial v} \right\|_{L^2(\Gamma)} \right)^{1/2} \quad (33)$$

The desired result follows from the identity (31) and the estimates (32) and (33).

VI. SKETCH OF THE PROOF OF THEOREM 3

By classical duality theory, to prove Theorem 3 it is enough to establish the associated observability inequality

$$\int_0^T \int_{\Gamma} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt \geq C_T \|\varphi_0\|_{H_0^1(\Omega)}^2 \tag{34}$$

where φ is the solution of the homogeneous system (13) - (17).

To obtain inequality (34) we use the identity (40) in the appendix with a vector field h satisfying assumption (A2) and a compactness/uniqueness argument.

APPENDIX

Lemma 8: Let $f \in L^1(0, T; H^1(\Omega))$ and h be a vector field of class C^1 on $\bar{\Omega}$. Then for every strong solution of the problem

$$y'(x, t) = ia(x)\Delta y(x, t) + f(x, t), \quad (x, t) \in \Omega \times (0, T) \tag{35}$$

$$y(x, 0) = y^0(x), \quad x \in \Omega \tag{36}$$

$$y_2(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T) \tag{37}$$

$$y_1(x, t) = y_2(x, t), \quad (x, t) \in \Gamma_1 \times (0, T) \tag{38}$$

$$a_1 \frac{\partial y_1(x, t)}{\partial \nu} = a_2 \frac{\partial y_2(x, t)}{\partial \nu}, \quad (x, t) \in \Gamma_1 \times (0, T) \tag{39}$$

we have

$$\begin{aligned} & -2a_1 Re \int_0^T \int_{\Gamma_1} \left| \frac{\partial y_1}{\partial \nu} \right|^2 \left(1 - \frac{a_1}{a_2}\right) h \cdot \nu d\Gamma dt + \\ & a_1 \int_0^T \int_{\Gamma_1} |\nabla y_1|^2 h \cdot \nu d\Gamma dt + 2a_2 Re \int_0^T \int_{\Gamma} \frac{\partial y_2}{\partial \nu} h \cdot \nabla \bar{y}_2 d\Gamma dt - \\ & a_2 \int_0^T \int_{\Gamma} |\nabla y_2|^2 h \cdot \nu d\Gamma dt - a_2 \int_0^T \int_{\Gamma_1} |\nabla y_2|^2 h \cdot \nu d\Gamma dt - \\ & 2a_1 Re \int_0^T \int_{\Omega_1} \nabla y_1 \cdot H \nabla \bar{y}_1 d\Omega dt + \\ & a_1 \int_0^T \int_{\Omega_1} |\nabla y_1|^2 \operatorname{div}(h) d\Omega dt - \\ & 2a_2 Re \int_0^T \int_{\Omega_2} \nabla y_2 \cdot H \nabla \bar{y}_2 d\Omega dt + \\ & a_2 \int_0^T \int_{\Omega_2} |\nabla y_2|^2 \operatorname{div}(h) d\Omega dt = Im \left(\int_{\Omega} y h \cdot \nabla \bar{y} d\Omega \right) \Big|_0^T - \\ & Im \int_0^T \int_{\Gamma} y_2 \bar{y}'_2 h \cdot \nu d\Gamma dt + \\ & a_1 Re \int_0^T \int_{\Omega_1} \nabla \bar{y}_1 \cdot \nabla (y_1 \operatorname{div}(h)) d\Omega dt - \\ & a_2 Re \int_0^T \int_{\Gamma} \frac{\partial \bar{y}_2}{\partial \nu} y_2 \operatorname{div}(h) d\Gamma dt + \\ & a_2 Re \int_0^T \int_{\Omega_2} \nabla \bar{y}_2 \cdot \nabla (y_2 \operatorname{div}(h)) d\Omega dt + \\ & Re \int_0^T \int_{\Omega} \bar{f} y \operatorname{div}(h) d\Omega dt - 2Im \int_0^T \int_{\Omega} f h \cdot \nabla \bar{y} d\Omega dt \tag{40} \end{aligned}$$

Proof: To obtain (40), we multiply both sides of (35) by $h \cdot \nabla \bar{y}$ and integrate over $(0, T) \times \Omega$. ■

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