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Weakly Operator Harmonizable Processes in Complete Correlated Actions

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***Abstract*— We prove that a stochastic process in a correlated action is weakly operator harmonizable (w.o.h.) if and only if it has a stationary dilation. We identify, by analogy with the stationary case, the shift operator of a w.o.h. process which is, in our context, a linear contraction. Other conditions which are equivalent to the notion of weak operator harmonizability follow the solution of an operator moment problem proposed by Z, Sebestyén and solved by Z. Sebestyén and D. Popovici. The main result of the paper extends, to the case of w.o.h. processes in complete correlated actions, the classical decomposition of H. Cramér.**

I. PRELIMINARIES

Extending the classical prediction theory developed by N. Wiener and P. Masani, in the late sixties, I. Suciu and I. Valuşescu proposed in [17] a mathematical model for the infinite multivariate case. Using as *parameter space* a complex Hilbert space $\mathfrak{E}$, they considered the *action* of operators in $\mathcal{L}(\mathfrak{E})$ on a right $\mathcal{L}(\mathfrak{E})$-module H, called the *state space*. The action of $\mathcal{L}(\mathfrak{E})$ is *correlated* if there exists a $\mathcal{L}(\mathfrak{E})$-valued inner product Γ on H, i.e., $\Gamma : H\times H \to \mathcal{L}(\mathfrak{E})$ satisfies the following conditions:

(a) $\Gamma(f,f) \geq 0,\ f \in H;\ \Gamma(f,f) = 0$ iff $f = 0$;
(b) $\Gamma(f,g)^* = \Gamma(g,f),\ f,g \in H$;
(c) $\Gamma(f,gA) = \Gamma(f,g)A,\ f,g \in H, A \in \mathcal{L}(\mathfrak{E})$;
(d) $\Gamma(f,g+h) = \Gamma(f,g) + \Gamma(f,h),\ f,g,h \in H$.

To any given complex Hilbert spaces $\mathfrak{E}$ and $\mathfrak{K}$ we can associate an *operator model* correlated action in which the correlation of the action of $\mathcal{L}(\mathfrak{E})$ on $H = \mathcal{L}(\mathfrak{E},\mathfrak{K})$ is the inner product

$$\mathcal{L}(\mathfrak{E},\mathfrak{K}) \times \mathcal{L}(\mathfrak{E},\mathfrak{K}) \ni (S,T) \mapsto \Gamma_m(S,T) := S^*T \in \mathcal{L}(\mathfrak{E}).$$

As proved in [17] any correlated action $\{\mathfrak{E},H,\Gamma\}$ can be embedded in an operator model. More precisely, there exist a complex Hilbert space $\mathfrak{K}$ and an embedding $\varphi : H \to \mathcal{L}(\mathfrak{E},\mathfrak{K})$ such that

$$\Gamma(f,g) = \varphi(f)^*\varphi(g),\ f,g \in H.$$

The minimality condition $\mathfrak{K} = \bigvee_{f\in H}\varphi(f)\mathfrak{E}$ (the notation "$\bigvee$" stands for the closed linear span) ensures the uniqueness (up to a unitary equivalence) of φ and $\mathfrak{K}$. $\mathfrak{K}$ is said to be, in this case, the *measuring space* of the correlated action. If φ is onto the correlated action $\{\mathfrak{E},H,\Gamma\}$ is said to be *complete*.

A discrete *stochastic process* is just a sequence $(f_n)_{n\in I}$ ($I = \mathbb{Z}_+$ or $I = \mathbb{Z}$) of elements in H. We identify several remarkable subspaces of $\mathfrak{K}$ attached to f :

- $\mathfrak{K}^f := \bigvee_{n\in I}\varphi(f_n)\mathfrak{E}$ is the *measuring space of* f;
- $\mathfrak{K}^f_n := \bigvee_{m\leq n}\varphi(f_m)\mathfrak{E}$ is the *past up to the moment* $n \in I = \mathbb{Z}$

and

- $\mathfrak{K}^f_{-\infty} := \bigcap_{n\in I=\mathbb{Z}}\mathfrak{K}^f_n$ is the *distant past*.

A process $f = (f_n)_{n\in I}$ is called *stationary* if

$$\Gamma(f_{m+1},f_{n+1}) = \Gamma(f_m,f_n)$$

for every $m,n \geq 0$ (the case $I = \mathbb{Z}_+$), respectively

$$\Gamma(f_{m+n},f_m) = \Gamma(f_n,f_0)$$

for every $m,n \in \mathbb{Z}$ (the case $I = \mathbb{Z}$). Equivalently, the associated *shift operator*

$$\mathfrak{K}^f \ni \varphi(f_n)e \mapsto U_f\varphi(f_n)e := \varphi(f_{n+1})e \in \mathfrak{K}^f \qquad (n \in I, e \in \mathfrak{E})$$

is well defined and isometric, resp. unitary on $\mathfrak{K}^f$. The process $f = (f_n)_{n\in\mathbb{Z}}$ is called *deterministic* if $\mathfrak{K}^f = \mathfrak{K}^f_{-\infty}$.

In the following section we prove that, in our generalized context, a process is weakly operator harmonizable (w.o.h.) if and only if it has a stationary dilation. We identify, by analogy with the stationary case, the shift operator of a weakly operator harmonizable process f and introduce the unilateral and bilateral extensions of f. The main result of the paper, which is included in the last section, extends, to the case of weakly operator harmonizable processes in complete correlated actions, the well known decomposition of H. Cramér [1].

II. WEAKLY OPERATOR HARMONIZABLE PROCESSES AND STATIONARY DILATIONS

In many applications stationarity seemed too restrictive and the need to introduce larger classes of stochastic processes became obvious. In this direction two different notions of harmonizability have been proposed by M. Loève [6] and, respectively, J.A. Rozanov [14]. These notions have been distinguished several years later in strong, respectively weak harmonizability by M.M. Rao [13].

Definition 1: A discrete stochastic process $f = (f_n)_{n\geq 0}$ in a correlated action $\{\mathfrak{E},H,\Gamma\}$ is said to be *weakly operator harmonizable* if there exists a semispectral measure F on the unit circle $\mathbb{T}$ into $\mathcal{L}(\mathfrak{K}^f)$ such that

$$\int_{\mathbb{T}}\bar{\lambda}^n dF(\lambda) = \left(\int_{\mathbb{T}}\bar{\lambda}dF(\lambda)\right)^n,\ n \geq 0$$

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and

$$\varphi(f_n)e = \int_{\mathbb{T}} \bar{\lambda}^n dF(\lambda)\varphi(f_0)e,\ n \geq 0, e \in \mathfrak{E}.$$

A similar definition has been given in [16] (for the bidimensional case we refer to [10], [11]).

Definition 2: Let $\{\mathfrak{E}, H, \Gamma\}$ and $\{\mathfrak{E}, \widetilde{H}, \widetilde{\Gamma}\}$ be correlated actions such that $H \subseteq \widetilde{H}$ and $\widetilde{\Gamma}|_{H\times H} = \Gamma$ (we assume, without any loss of generality, that φ is extended by $\widetilde{\varphi}$ and, consequently, $\mathfrak{K} \subseteq \widetilde{\mathfrak{K}}$).

- A process $g = (g_n)_{n\in\mathbb{Z}}$ in $\widetilde{H}$ is said to be a *stationary dilation of* $f = (f_n)_{n\geq 0}$ in H
  - (a) g is a stationary process;
  - (b) $\mathfrak{K}^f$ is a closed subspace of $\mathfrak{K}^g$;
  - (c) $P^{\widetilde{\mathfrak{K}}}_{\mathfrak{K}^f} U_g^m \varphi(f_n)e = \varphi(f_{m+n})e,\ m, n \geq, e \in \mathfrak{E}$

  ($P^{\widetilde{\mathfrak{K}}}_{\mathfrak{K}^f}$ denotes the orthogonal projection of $\widetilde{\mathfrak{K}}$ onto $\mathfrak{K}^f$).
- The stationary dilation g of f is called *minimal* if

$$\mathfrak{K}^g = \bigvee_{m\in\mathbb{Z}} U_g^m \mathfrak{K}^f.$$

Remark 3: If g is a stationary dilation of f then a simple computation shows that

$$\Big(\sum_{m\in\mathbb{Z},\ n\geq 0}{}' U_g^m\varphi(f_n)\Big)^*\Big(\sum_{m\in\mathbb{Z},\ n\geq 0}{}' U_g^m\varphi(f_n)\Big) = \sum_{m,n\in\mathbb{Z},\ p,q\geq 0} \Gamma(f_{(m-n)^-+q}, f_{(m-n)^++p})$$

(the symbol $\sum'$ denotes a finite sum; for $m \in \mathbb{Z}$ we used the notations $m^+ = \max\{m, 0\}$ and $m^- = -\min\{m, 0\}$). Consequently, if g and g' are minimal stationary dilations of f then the map

$$\mathfrak{K}^g \ni U_g^m\varphi(f_n)e \overset{Z}{\mapsto} U_{g'}^m\varphi(f_n)e \in \mathfrak{K}^{g'}$$

can be extended to a unitary operator Z on $\mathfrak{K}^g$ onto $\mathfrak{K}^{g'}$ which has the properties:

(a) $Z\varphi(f_n) = \varphi(f_n),\ n \geq 0$;
(b) $U_g = Z^* U_{g'} Z$.

In other words, a *minimal stationary dilation is uniquely determined up to a unitary equivalence which leaves invariant* $\varphi(f_n),\ n \geq 0$. ■

A *semispectral measure* on $\mathbb{T}$ into $\mathcal{L}(\mathfrak{H})$ ($\mathfrak{H}$ is a given complex Hilbert space) is a $\mathcal{L}(\mathfrak{H})$-valued function $\sigma \mapsto F(\sigma)$ defined on the Borel subsets of $\mathbb{T}$ such that, for any $h \in \mathfrak{H}$, the map $\sigma \mapsto \langle F(\sigma)h, h\rangle$ is a positive Radon measure on $\mathbb{T}$.

The following classical results in dilation theory are needed in our approach:

Theorem 4 ([7], [8]): A map $\Phi : \mathbb{Z} \to \mathcal{L}(\mathfrak{H})$ is positive definite (i.e., $\sum_{m,n\in\mathbb{Z}}\langle\Phi(m-n)h_m, h_n\rangle \geq 0$ for every sequence $\{h_n\}_{n\in\mathbb{Z}}$ with finite support) if and only if there exists a semispectral measure F on $\mathbb{T}$ into $\mathcal{L}(\mathfrak{H})$ such that

$$\Phi(n) = \int_{\mathbb{T}} \bar{\lambda}^n dF(\lambda),\ n \in \mathbb{Z}.$$

Theorem 5 ([18, § I.8.1]): Let $T \in \mathcal{L}(\mathfrak{H})$. The map

$$n \mapsto T(n) := \begin{cases} T^n, & \text{if } n \geq 0 \\ T^{*|n|}, & \text{if } n < 0 \end{cases}$$

is positive definite on $\mathbb{Z}$ if and only if T is a contraction.

Theorem 6 ([19]): Let T be a contraction on $\mathfrak{H}$. Then there exists a Hilbert space $\mathfrak{K}$ which contains $\mathfrak{H}$ (as a closed subspace) and a unitary operator U (acting on $\mathfrak{K}$) such that

$$P^{\mathfrak{K}}_{\mathfrak{H}} U^n h = T^n h, \quad n \geq 0, h \in \mathfrak{H}$$

(U is said to be a unitary dilation *of T).*

We prove that a process is weakly operator harmonizable if and only if it has a stationary dilation. More precisely it holds:

Theorem 7: Let $f = (f_n)_{n\geq 0}$ be a discrete stochastic process in a correlated action $\{\mathfrak{E}, H, \Gamma\}$. The following conditions are equivalent:

(i) *f is weakly operator harmonizable;*
(ii) *f can be dilated to a stationary process;*
(iii) *There exists a contraction $T_f \in \mathcal{L}(\mathfrak{K}^f)$ such that*

$$T_f\varphi(f_n)e = \varphi(f_{n+1})e$$

for every $n \geq 0$ and $e \in \mathfrak{E}$;

(iv)

$$\sum_{\substack{m,m',\\ n,n'}} \langle\Gamma(f_{(m-n)^-+n'}, f_{(m-n)^++m'})e_{m,m'}, e_{n,n'}\rangle \geq 0$$

for every finite double sequence $\{e_{n,n'}\}_{n,n'\geq 0}$ of vectors in $\mathfrak{E}$;

(v)

$$\sum_{\substack{m,m',\\ n,n'}} \langle\Gamma(f_{(m-n)^-+n'}, f_{(m-n)^++m'})e_{m,m'}, e_{n,n'}\rangle \geq \sum_{\substack{m,m',\\ n,n'}} \langle\Gamma(f_{n+n'}, f_{m+m'})e_{m,m'}, e_{n,n'}\rangle$$

for every finite double sequence $\{e_{n,n'}\}_{n,n'\geq 0}$ of vectors in $\mathfrak{E}$;

(vi)

$$\sum_{m,n}\langle\Gamma(f_{n+1}, f_{m+1})e_m, e_n\rangle \leq \sum_{m,n}\langle\Gamma(f_n, f_m)e_m, e_n\rangle$$

for every finite sequence $\{e_n\}_{n\geq 0}$ of vectors in $\mathfrak{E}$.

Proof: The proof follows some arguments used for the solution of an operator moment problem proposed by Z. Sebestyén in [15] and completely solved in [16]. For the multidimensional case similar moment problems have been considered in [2], [3], [4], [9], [10], [12].

$(i) \Rightarrow (iii)$. Let us define

$$T_f := \int_{\mathbb{T}} \bar{\lambda} dF(\lambda).$$

We observe, in view of (i) and by Theorem 4, that the map $n \mapsto T_f(n)$ is positive definite on $\mathbb{Z}$. Hence, by Theorem 5, T_f is a contraction on $\mathfrak{K}^f$. In addition, since

$$T_f^n = \Big(\int_{\mathbb{T}} \bar{\lambda} dF(\lambda)\Big)^n = \int_{\mathbb{T}} \bar{\lambda}^n dF(\lambda),\ n \geq 0,$$

we deduce that

$$T_f^n \varphi(f_0)e = \varphi(f_n)e, \quad n \geq 0, e \in \mathfrak{E}$$

or, equivalently,

$$T_f \varphi(f_n)e = \varphi(f_{n+1})e, \quad n \geq 0, e \in \mathfrak{E}.$$

$(iii) \Rightarrow (i)$. By Theorem 5 the map $n \mapsto T_f(n)$ is positive definite on $\mathbb{Z}$. Therefore, by Theorem 4, there exists a semispectral measure F on $\mathbb{T}$ into $\mathcal{L}(\mathfrak{K}^f)$ such that

$$T_f(n) = \int_{\mathbb{T}} \bar{\lambda}^n dF(\lambda), \ n \in \mathbb{Z}.$$

Obviously

$$T_f(n) = \Big(T_f(1)\Big)^n, \ n \geq 0$$

and

$$\varphi(f_n)e = T_f^n \varphi(f_0)e = \int_{\mathbb{T}} \bar{\lambda}^n dF(\lambda)\varphi(f_0)e, \quad n \geq 0, \ e \in \mathfrak{E}.$$

$(iii) \Rightarrow (ii)$. Let U (acting on $\mathfrak{K}'$) be a unitary dilation of T_f (cf. Theorem 6). We define $\widetilde{\mathfrak{K}} := \mathfrak{K}' \oplus (\mathfrak{K} \ominus \mathfrak{K}^f)$, $\widetilde{H} = \mathcal{L}(\mathfrak{E}, \widetilde{\mathfrak{K}})$ and identify H with the closed submodule $\varphi(H)$ of $\mathcal{L}(\mathfrak{E}, \mathfrak{K}) \subseteq \mathcal{L}(\mathfrak{E}, \widetilde{\mathfrak{K}})$.

Let $g_0 = f_0 \equiv \varphi(f_0)$ and, for $n \in \mathbb{Z}^*$, consider $g_n = U^n g_0$. Then $(g_n)_{n\in\mathbb{Z}}$ is a stationary process in the operator model correlated action $\{\mathfrak{E}, \widetilde{H}, \Gamma_m\}$. In addition,

$$\begin{aligned} P_{\mathfrak{K}^f}^{\widetilde{\mathfrak{K}}} U_g^m \varphi(f_n)e &= P_{\mathfrak{K}^f}^{\widetilde{\mathfrak{K}}} U^m \varphi(f_n)e \\ &= T_f^m \varphi(f_n)e \\ &= \varphi(f_{m+n})e, \quad m, n \geq 0, \ e \in \mathfrak{E}. \end{aligned}$$

Hence g is a stationary dilation of f.

$(ii) \Rightarrow (v)$. Let $g = (g_n)_{n\in\mathbb{Z}}$ be a stationary dilation of f. Then, for every finite double sequence $\{e_{n,n'}\}_{n,n'\geq 0}$ of vectors in $\mathfrak{E}$,

$$\begin{aligned} &\sum_{m,m',n,n'} \langle \Gamma(f_{n+n'}, f_{m+m'})e_{m,m'}, e_{n,n'} \rangle \\ &= \| \sum_{m,m'} \varphi(f_{m+m'})e_{m,m'} \|^2 \\ &= \| P_{\mathfrak{K}^f}^{\widetilde{\mathfrak{K}}} \sum_{m,m'} U_g^m \varphi(f_{m'})e_{m,m'} \|^2 \\ &\leq \| \sum_{m,m'} U_g^m \varphi(f_{m'})e_{m,m'} \|^2 \\ &= \sum_{m,m',n,n'} \langle U_g^m \varphi(f_{m'})e_{m,m'}, U_g^n \varphi(f_{n'})e_{n,n'} \rangle \\ &= \sum_{m\geq n,m',n'} \langle \varphi(f_{m-n+m'})e_{m,m'}, \varphi(f_{n'})e_{n,n'} \rangle \\ &\quad + \sum_{m<n,m',n'} \langle \varphi(f_{m'})e_{m,m'}, \varphi(f_{n-m+n'})e_{n,n'} \rangle \\ &= \sum_{m,m',n,n'} \langle \Gamma(f_{(m-n)^- + n'}, f_{(m-n)^+ + m'})e_{m,m'}, e_{n,n'} \rangle. \end{aligned}$$

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (vi)$. Let $\{e_n\}_{n\geq 0}$ be a finite sequence of vectors in $\mathfrak{E}$. The inequality of (vi) can be obtained if the family $\{e_{n,n'}\}_{n,n'\geq 0}$ in (iv) takes the following form:

$$e_{n,n'} = \begin{cases} -e_{n'-1}, & \text{if } n = 0 \text{ and } n' \geq 1 \\ e_{n'}, & \text{if } n = 1 \\ 0, & \text{for the other cases.} \end{cases}$$

$(vi) \Rightarrow (iii)$. By (vi), the operator

$$\mathfrak{K}^f \ni \sum\nolimits' \varphi(f_n)e_n \mapsto \sum\nolimits' \varphi(f_{n+1})e_n \in \mathfrak{K}^f$$

is well-defined and contractive on a dense subset of $\mathfrak{K}^f$. It can be therefore extended to a contraction T_f of $\mathfrak{K}^f$ which satisfies

$$T_f \varphi(f_n)e = \varphi(f_{n+1})e, \quad n \geq 0, \ e \in \mathfrak{E}.$$

■

Remark 8: (a) If $f = (f_n)_{n\geq 0}$ is weakly operator harmonizable then the contraction T_f mentioned in the condition (iii) of Theorem 7 is uniquely determined. It will be called, by analogy with the stationary case, the *shift operator* of f.

(b) Any stationary process $f = (f_n)_{n\geq 0}$ is weakly operator harmonizable.

■

III. A GENERALIZED CRAMÉR DECOMPOSITION

One of the most important results towards the study of stochastic processes is a structure theorem proposed by H. Cramér in [1]. Separating the deterministic part by the purely non-deterministic one this decomposition became the cornerstone of prediction theory for this kind of processes. It is our aim in the following to extend this decomposition into the framework of complete correlated actions.

Definition 9: Let $\{\mathfrak{E}, H, \Gamma\}$ be a correlated action and $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ weakly operator harmonizable stochastic processes.

- *f contains g* if
  - (a) $\mathfrak{K}^g$ is a closed subspace of $\mathfrak{K}^f$;
  - (b) $\mathfrak{K}^g$ reduces T_f (i.e., $\mathfrak{K}^g$ is invariant under both T_f and T_f^*);
  - (c) $T_f|_{\mathfrak{K}^g} = T_g$.
- The *unilateral extension* of f is the process $\bar{f} = (\bar{f}_n)_{n\in\mathbb{Z}}$ defined as

$$\bar{f}_n := \begin{cases} f_{-n}, & \text{if } n \leq 0 \\ 0, & \text{if } n > 0. \end{cases}$$

- If $\{\mathfrak{E}, H, \Gamma\}$ is complete then the *bilateral extension* of f is the process $\tilde{f} = (\tilde{f}_n)_{n\in\mathbb{Z}}$ defined as

$$\tilde{f}_n := \begin{cases} f_{-n}, & \text{if } n \leq 0 \\ \varphi^{-1}(T_f^{*^n} \varphi(f_0)), & \text{if } n > 0 \end{cases}.$$

We are now in position to extend the decomposition of H. Cramér in our generalized context:

Theorem 10: Let $f = (f_n)_{n\geq 0}$ be a weakly operator harmonizable process in a complete correlated action $\{\mathfrak{E}, H, \Gamma\}$. There exists a decomposition of the form

$$\tilde{f}_n = u_n + v_n, \ n \in \mathbb{Z}$$

with the following properties

(a) $u_- = (u_{-n})_{n\geq 0}$ *is weakly operator harmonizable and stationary, it is contained in* f, $\tilde{u} = (u_n)_{n\in\mathbb{Z}}$ *is stationary and deterministic and* $\bar{u_-}$ *is deterministic;*

(b) $v_- = (v_{-n})_{n\geq 0}$ *is weakly operator harmonizable, it is contained in* f *and does not contain any process* w *such that* w *is stationary and* $\bar{w}$ *is deterministic;*

(c) $\Gamma(u_m, v_n) = 0$, *for every* $m, n \in \mathbb{Z}$.

Proof: H. Langer proved in [5] that, for any contraction T acting on a complex Hilbert space $\mathfrak{H}$, the set

$$\mathfrak{H}_T := \{h \in \mathfrak{H} : \|T^n h\| = \|h\| = \|T^{*^n} h\|,\ n \geq 0\}$$

is a closed subspace of $\mathfrak{H}$, reduces T to a unitary operator and it is maximal with this last property.

We define

$$u_n := \varphi^{-1}(P^{\mathfrak{K}^f}_{\mathfrak{K}^f_{T_f}} \varphi(\tilde{f}_n))$$

and

$$v_n := \tilde{f}_n - u_n, \quad n \in \mathbb{Z}.$$

Let us firstly compute the measuring space $\mathfrak{K}^{\tilde{u}}$ of $\tilde{u} = (u_n)_{n\in\mathbb{Z}}$:

$$\begin{aligned} \mathfrak{K}^{\tilde{u}} &= \bigvee_{n\in\mathbb{Z}} \varphi(u_n)\mathfrak{E} \\ &= \bigvee_{n\in\mathbb{Z}} P^{\mathfrak{K}^f}_{\mathfrak{K}^f_{T_f}} \varphi(\tilde{f}_n)\mathfrak{E} \\ &= P^{\mathfrak{K}^f}_{\mathfrak{K}^f_{T_f}} \mathfrak{K}^{\tilde{f}} \\ &= P_{\mathfrak{K}^f_{T_f}} \mathfrak{K}^f \\ &= \mathfrak{K}^f_{T_f}. \end{aligned}$$

Similarly,

$$\mathfrak{K}^{\tilde{u}}_0 = \bigvee_{n\geq 0} P^{\mathfrak{K}^f}_{\mathfrak{K}^f_{T_f}} \varphi(f_n)\mathfrak{E} = \mathfrak{K}^f_{T_f}.$$

It follows that $\mathfrak{K}^{\tilde{u}} = \mathfrak{K}^{\tilde{u}}_0 = \mathfrak{K}^f_{T_f}$. Consequently, $\mathfrak{K}^{\tilde{u}}$ reduces T_f, u_- is weakly operator harmonizable, it is contained in f and $T_{u_-} = T_f|_{\mathfrak{K}^{\tilde{u}}}$ is a unitary operator. Hence u_- is also stationary.

We observe that, for every $n \in \mathbb{Z}$,

$$T_{u_-}\varphi(u_n) = \varphi(u_{n-1}). \tag{1}$$

Indeed, the case $n > 0$ follows by the equalities:

$$\begin{aligned} T_{u_-}\varphi(u_n) &= T_{u_-} P^{\mathfrak{K}^f}_{\mathfrak{K}^f_{T_f}} \varphi(\tilde{f}_n) \\ &= T_{u_-} P^{\mathfrak{K}^f}_{\mathfrak{K}^f_{T_f}} T_f^* \varphi(\tilde{f}_{n-1}) \\ &= T_{u_-} T_f^* P^{\mathfrak{K}^f}_{\mathfrak{K}^f_{T_f}} \varphi(\tilde{f}_{n-1}) \\ &= T_{u_-} T_{u_-}^* \varphi(u_{n-1}) \\ &= \varphi(u_{n-1}). \end{aligned}$$

As noted before the case $n \leq 0$ is a direct consequence of the definition of T_f.

We use (1) and proceed inductively to show that

$$T^m_{u_-}\varphi(u_n) = \varphi(u_{n-m}), \quad m, n \in \mathbb{Z}.$$

We deduce that, for every $m, n \in \mathbb{Z}$,

$$\begin{aligned} \Gamma(u_{m+n}, u_n) &= \varphi(u_{m+n})^* \varphi(u_m) \\ &= \varphi(u_n)^* T^m_{u_-} T^{-m}_{u_-} \varphi(u_0) \\ &= \Gamma(u_n, u_0), \end{aligned}$$

that is $\tilde{u}$ is stationary. In addition, for every $n \in \mathbb{Z}$,

$$\mathfrak{K}^{\tilde{u}}_n = T^{-n}_{u_-} \mathfrak{K}^{\tilde{u}}_0 = T^{-n}_{u_-} \mathfrak{K}^{\tilde{u}} = \mathfrak{K}^{\tilde{u}},$$

hence $\tilde{u}$ is also deterministic. It follows immediately that $\bar{u_-}$ is deterministic.

We can proceed analogously to prove that the measuring space of $\tilde{v} := (v_n)_{n\in\mathbb{Z}}$ is $\mathfrak{K}^{\tilde{v}} = \mathfrak{K}^f \ominus \mathfrak{K}^f_{T_f}$ and reduces T_f, v_- is weakly operator harmonizable, it is contained in f and $T_{v_-} = T_f|_{\mathfrak{K}^{\tilde{v}}}$.

Let us suppose that v_- contains a weakly operator harmonizable process $w = (w_n)_{n\geq 0}$ such that w is stationary and $\bar{w}$ is deterministic. Then T_w is an isometric operator. In addition, since $\varphi(w_0)\xi \subseteq \bigvee_{n\geq 1} \varphi(w_n)\xi$ ($\bar{w}$ is deterministic), we obtain that T_w is also surjective, hence a unitary operator on $\mathfrak{K}^w$. This means that $\mathfrak{K}^v$ has a closed subspace reducing T_f to a unitary operator, contradicting the maximality of $\mathfrak{K}^f_{T_f}$.

Finally, for every $m, n \in \mathbb{Z}$,

$$\begin{aligned} \Gamma(u_m, v_n) &= \varphi(u_m)^* \varphi(v_n) \\ &= \varphi(\tilde{f}_m)^* P^{\mathfrak{K}^f}_{\mathfrak{K}^{\tilde{u}}} P^{\mathfrak{K}^f}_{\mathfrak{K}^{\tilde{v}}} \varphi(f_n) \\ &= 0, \end{aligned}$$

since $\mathfrak{K}^{\tilde{u}} = \mathfrak{K}^f_{T_f} \perp \mathfrak{K}^f \ominus \mathfrak{K}^f_{T_f} = \mathfrak{K}^{\tilde{v}}$. ■

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