Stochastic input-output realization of bilinear systems

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Abstract

A bilinear stochastic system given in state space form is studied when both the input and the output are measured. The Hankel matrix of the system is built up in terms of the Fliess-series representation of the process. The Fliess-coefficients are calculated by the cumulants between the output and the polynomials of the input.

1 Introduction

Identification of nonlinear systems is an important problem in engineering since most of the systems exhibit nonlinearities. Bilinear systems show up in practical problems when the linear representation is not sufficiently significant, the realization of that systems is well studied, it started in the seventies [Forn76]. A possible identification of the parameters for a stochastic bilinear system with measured input and output is based on the cross-cumulants between the output and higher order products of the input [Tso2001].

We suppose that a bilinear stochastic system is given in state space representation form and both the input and the output of the system are measured. When the bilinear process is stationary and the input is Gaussian white noise, it is possible to build up the multiple Wiener-Ito stochastic spectral representation of the process [Ter99]. It turns out that the Fourier series of the transfer functions provide the coefficients of the Hermite polynomials in the Fliess-series representation of the process. The object of this paper is to reconstruct the state and the output equations from measured data in terms of the Hankle matrix. The Hankel matrix of the system, similarly to the deterministic case [Isi95], is built up in terms of the Fliess-series representation of the process. The Fliess-coefficients are calculated by the cumulants between the output and the Hermite polynomials of the input.

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2 Bilinear model with scalar input

An observation Y_t with state process $\{\underline{X}_t\,,\, t \in \mathbb{Z}\}$ is called bilinear if it satisfies the bilinear state-space equation

$$\begin{aligned} \underline{X}_t &= \mathbf{A}\underline{X}_{t-1} + \mathbf{D}\underline{X}_{t-1}\mathrm{w}_{t-1} + \underline{b}\mathrm{w}_{t-1}, \\ Y_t &= \underline{c}^{\mathsf{T}}\underline{X}_t + \mathrm{w}_t, \end{aligned} \tag{1}$$

where $\underline{X}_t$ is m–dimensional random variable, $\mathbf{A}$, $\mathbf{D} \in \mathbf{R}^{m\times m}$, $\underline{b}$, $\underline{c} \in \mathbf{R}^m$. We assume that there is no eigenvalue of $\mathbf{A}$ with modulus 1, w_t is a Gaussian white noise scalar process with $\mathsf{E}w_t = 0$, $\mathsf{E}w_t^2 = \sigma^2$, and $\{\underline{Y}_t\,,\, t \in \mathbb{Z}\}$ is physically realizable and subordinated to $\{\mathrm{w}_t\,,\, t \in \mathbb{Z}\}$. It is seen that $\underline{X}_t$ is independent of w_t.and $\mathsf{E}\underline{X}_t = 0$.

We have the Wiener–Itô representation of the stochastic process $\{\underline{X}_t\,,\, t \in \mathbb{Z}\}$ from the Wiener–Itô representation of its entries $\{(\underline{X}_t)_i\,,\, t \in \mathbb{Z}\}$, $i = 1,\ldots,m$, i.e.,

$$\underline{X}_t = \sum_{r=1}^{\infty} \int_{\mathcal{D}^r} \exp\left(i2\pi t \sum_{j=1}^{r} \omega_j\right) \underline{f}_r(\omega_{1:r}) W(d\omega_{1:r}), \qquad t \in \mathbb{Z},$$

where $(f_r)_i \in \overline{L_\Phi^r}$, $i = 1,\ldots,m$ are the transfer functions of $\{(\underline{X}_t)_i\,,\, t \in \mathbb{Z}\}$, $i = 1,\ldots,m$, respectively. The first transfer function $\underline{f}_1(\omega_1)$ corresponds to the linear part of (1), the second one contains the contribution of all possible second order products of the input and so on. The following recursive formula for the transfer functions can be derived easily:

$$\begin{aligned} \underline{f}_1(\omega_1) &= \left(\exp\left(i2\pi\omega_1\right) I - \mathbf{A}\right)^{-1} \underline{b} \\ \underline{f}_2(\omega_{(1:2)}) &= \left(\exp(i2\pi(\omega_1+\omega_2) I - \mathbf{A}\right)^{-1} \mathbf{D}\underline{f}_1(\omega_1), \end{aligned}$$

and in general $r \geq 2$,

$$\underline{f}_r(\omega_{1:r}) = (\exp\left(i2\pi \sum_{j=1}^{r} \omega_j\right) I - \mathbf{A})^{-1}\mathbf{D}\underline{f}_{r-1}(\omega_{(1:r-1)}).$$

There is no question of the existence of inverses because of the physical realizability of the process all the eigenvalues of $\mathbf{A}$ must be inside the unit circle. It has been pointed out, see [Ter99], [Ter85], [LB88], that the above Wiener–Itô representation exists if and only if all the eigenvalues of the matrix $\mathbf{A}^{\otimes 2} + \sigma^2\mathbf{D}^{\otimes 2}$ are less than one in modulus, which is the necessary and sufficient condition for the existence of a second order stationary physically realizable solution of equation (1). .

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3 Fliess functional expansion

Now we are interested in series expansion of $\underline{X}_t$ in terms of Hermite polynomials $H_k(\mathrm{w}_{t-s_1}, \mathrm{w}_{t-s_2}, \ldots, \mathrm{w}_{t-s_k}) = H_k(z^{-s_1}\mathrm{w}_t, z^{-s_2}\mathrm{w}_t, \ldots, z^{-s_k}\mathrm{w}_t)$. The state variable $\underline{X}_t$ of the bilinear equation

$$\underline{X}_t = \mathbf{A}\underline{X}_{t-1} + \mathbf{D}\underline{X}_{t-1}\mathrm{w}_{t-1} + \underline{b}\mathrm{w}_{t-1},$$

is given in the form of multiple Wiener Ito integrals

$$\underline{X}_t = \sum_{r=1}^{\infty}\int_{\mathcal{D}^r}\exp\left(i2\pi t\sum_{j=1}^{r}\omega_j\right)\underline{f}_r(\omega_{1:r})W(d\omega_{1:r}), \quad t\in\mathbb{Z},$$

where the transfer functions are given above. The first transfer function is clearly the linear part of the system and corresponds to the series

$$\begin{aligned}&\int_{\mathcal{D}}\exp(i2\pi t\omega_1)\,\underline{f}_r(\omega_1)\,W(d\omega_1)\\&=\int_{\mathcal{D}}\exp(i2\pi t\omega_1)\left(\exp(i2\pi\omega_1)I-\mathbf{A}\right)^{-1}W(d\omega_1)\,\underline{b}\\&=\sum_{k=0}^{\infty}\mathbf{A}^k\mathrm{w}_{t-1-k}\underline{b}.\end{aligned}$$

The first order Hermite polynomial is $H_1(\mathrm{w}_{t-1-k}) = \mathrm{w}_{t-1-k}$, hence the linear part is written as

$$\sum_{k=0}^{\infty}\mathbf{A}^k\underline{b}H_1(\mathrm{w}_{t-1-k}),$$

where the coefficients $\mathbf{A}^k\underline{b}$ will be denoted as $c_k(\underbrace{0,0,\ldots,0}_{k})$, in particular $c_0 = \underline{b}$, $c_1(0) = \mathbf{A}\underline{b}$, and so on. The second order transfer function

$$\begin{aligned}\underline{f}_2(\omega_{1:2}) = &\left(\exp\left(i2\pi(\omega_1+\omega_2)\right)I-\mathbf{A}\right)^{-1}\mathbf{D}\\&\times\left(\exp(i2\pi\omega_1)I-\mathbf{A}\right)^{-1}\underline{b},\end{aligned}$$

has the series expansion

$$\begin{aligned}\underline{f}_2(\omega_{1:2}) &= \exp\left(-i2\pi(2\omega_1+\omega_2)\right)\\&\times\sum_{n,m=0}^{\infty}\mathbf{A}^n\mathbf{D}\mathbf{A}^m\underline{b}\exp\left(-i2\pi\left((n+m)\omega_1+m\omega_2\right)\right),\end{aligned}$$

hence the coefficient $\mathbf{A}^n\mathbf{D}\mathbf{A}^m\underline{b}$ corresponds to the Hermite polynomial $H_2(\mathrm{w}_{t-1-m}, \mathrm{w}_{t-2-n-m}) = H_2(z^{-m}\mathrm{w}_{t-1}, z^{-m-n}\mathrm{w}_{t-2})$. We introduce the notation for $\mathbf{A}^n\mathbf{D}\mathbf{A}^m\underline{b}$ as $c_{n+m+1}(\underbrace{0,0,\ldots,0}_{m},1,\underbrace{0,0,\ldots,0}_{n})$. If $n=m=0$ then the coefficient of $H_2(\mathrm{w}_{t-1}, \mathrm{w}_{t-2})$ is $c_1(1) = \mathbf{D}\underline{b}$.

Now set

$$\begin{aligned}\underline{f}_3(\omega_{1:2}) &= \left(\exp\left(i2\pi\sum_{j=1}^{3}\omega_j\right)I-\mathbf{A}\right)^{-1}\mathbf{D}\underline{f}_2(\omega_{1:2})\\&= e^{-i2\pi(3\omega_1+2\omega_2+\omega_3)}\\&\times\sum_{n,m,k=0}^{\infty}\mathbf{A}^n\mathbf{D}\mathbf{A}^m\mathbf{D}\mathbf{A}^k\underline{b}e^{-i2\pi((n+m+k)\omega_1+(m+n)\omega_2+k\omega_3)},\end{aligned}$$

here $c_{n+m+k+2}(\underbrace{0,0,\ldots,0}_{n},1,\underbrace{0,0,\ldots,0}_{m},1,\underbrace{0,0,\ldots,0}_{k})$ is the coefficient $\mathbf{A}^n\mathbf{D}\mathbf{A}^m\mathbf{D}\mathbf{A}^k$, in particular $c_2(1,1) = \mathbf{D}^2\underline{b}$, it corresponds to the $H_3(\mathrm{w}_{t-1}, \mathrm{w}_{t-2}, \mathrm{w}_{t-3})$. It is seen that the degree of the Hermite polynomial is the number of ones plus one.

In general we introduce a set of indices $i_{1:k} = (i_1, i_2, \ldots, i_k)$ where i_n is either 0,or 1. assume that there are exactly $K \geqq 0$ ones among i_n and $n_\ell \geqq 0$ denotes the number of consecutive zeros, these blocks of zeros are either at the beginning of the series $i_{1:k}$, or they are between ones, or closing $i_{1:k}$. If $K=0$, then $c_k(\underbrace{0,0,\ldots,0}_{k})$ has been defined above. If $K>0$, and $n_\ell = 0$, $\ell = 1,2,\ldots,K+1$, then $c_K(\underbrace{1,1,\ldots,1}_{K}) = \mathbf{D}^K\underline{b}$ is the coefficient of $H_{K+1}\left(\mathrm{w}_{t-1}, z^{-1}\mathrm{w}_{t-1}, \ldots, z^{-K}\mathrm{w}_{t-1}\right)$. We list n_ℓ, $\ell = 1,2,\ldots,K+1$, starting with the last block of zeros in $i_{1:k}$. Namely $i_{k-n_1} = 1$, $i_{k-n_1-n_2-1} = 1$, and so on. Now we define $c_k(i_{1:k}) = \mathbf{A}^{n_{K+1}}\mathbf{D}\ldots\mathbf{A}^{n_2}\mathbf{D}\mathbf{A}^{n_1}\underline{b}$ it is going to be the coefficient of $H_{K+1}\left(z^{-n_1}\mathrm{w}_{t-1}, z^{-n_1-n_2}\mathrm{w}_{t-2}, \ldots, z^{-\Sigma n_\ell}\mathrm{w}_{t-1-K},\right)$ in the series expansion of $\underline{X}_t$.

Building the Hankel matrix the columns and rows indices are in the alphabetical order of the zero-one series by the convention that a shorter series precedes a longer one, then a matrix element defined by the concatenation of the row-index and column-index, Table 1.

In general we introduce the coefficients of the Hermite polynomials of higher order and use them for the definition of the Hankel matrix, see [Isi95] for the deterministic case. The Hankel matrix for the state process $\underline{X}_t$ is in Table 2.

The estimation of the entries $c_k(i_{1:k})$ of the Hankel matrix follows from the cross covariances between the output and the Hermite polynomials of the input

$$\begin{aligned}&EY_tH_{K+1}\left(z^{-n_1}\mathrm{w}_{t-1}, z^{-n_1-n_2-1}\mathrm{w}_{t-1}, \ldots, z^{-\Sigma n_\ell-K}\mathrm{w}_{t-1}\right)\\&=E\left(\underline{c}^\intercal\underline{X}_t+\mathrm{w}_t\right)\\&\times H_{K+1}\left(z^{-n_1}\mathrm{w}_{t-1}, z^{-n_1-n_2}\mathrm{w}_{t-2}, \ldots, z^{-\Sigma n_\ell}\mathrm{w}_{t-1-K}\right)\\&=\sigma^{2(K+1)}\underline{c}^\intercal\mathbf{A}^{n_{K+1}}\mathbf{D}\ldots\mathbf{A}^{n_2}\mathbf{D}\mathbf{A}^{n_1}\underline{b}.\end{aligned}$$

	$\emptyset$	(0)	(1)	$(0,0)$	$(0,1)$	$(1,0)$	$\cdots$
$\emptyset$	c_0	$c_1(0)$	$c_1(1)$	$c_2(0,0)$	$c_2(0,1)$	$c_2(1,0)$	$\cdots$
(0)	$c_1(0)$	$c_2(0,0)$	$c_2(0,1)$	$c_3(0,0,0)$	$c_3(0,0,1)$	$c_3(0,1,0)$	$\cdots$
(1)	$c_1(1)$	$c_2(1,0)$	$c_2(1,1)$	$c_3(1,0,0)$	$c_3(1,0,1)$	$c_3(1,1,0)$	$\cdots$
$(0,0)$	$c_2(0,0)$	$c_3(0,0,0)$	$c_3(0,0,1)$	$c_4(0,0,0,0)$	$c_4(0,0,0,1)$	$c_4(0,0,1,0)$	$\cdots$
$(0,1)$	$c_2(0,1)$	$c_3(0,1,0)$	$c_3(0,1,1)$	$c_4(0,1,0,0)$	$c_4(0,1,0,1)$	$c_4(0,1,1,0)$	$\cdots$
$(1,0)$	$c_2(1,0)$	$c_3(1,0,0)$	$c_3(1,0,1)$	$c_4(1,0,0,0)$	$c_4(1,0,0,1)$	$c_4(1,0,1,0)$	$\cdots$
$(1,1)$	$c_2(1,1)$	$c_3(1,1,0)$	$c_3(1,1,1)$	$c_4(1,1,0,0)$	$c_4(1,1,0,1)$	$c_4(1,1,1,0)$	$\cdots$
$(0,0,0)$	$c_3(0,0,0)$	$c_4(0,0,0,0)$	$c_4(0,0,0,1)$	$c_5(0,0,0,0,0)$	$c_5(0,0,0,0,1)$	$c_5(0,0,0,1,0)$	$\cdots$
$(0,0,1)$	$c_3(0,0,1)$	$c_4(0,0,1,0)$	$c_4(0,0,1,1)$	$c_5(0,0,1,0,0)$	$c_5(0,0,1,0,1)$	$c_5(0,0,1,1,0)$	$\cdots$
$(0,1,0)$	$c_3(0,1,0)$	$c_4(0,1,0,0)$	$c_4(0,1,0,1)$	$c_5(0,1,0,0,0)$	$c_5(0,1,0,0,1)$	$c_5(0,1,0,1,0)$	$\cdots$
$(0,1,1)$	$c_3(0,1,1)$	$c_4(0,1,1,0)$	$c_4(0,1,1,1)$	$c_5(0,1,1,0,0)$	$c_5(0,1,1,0,1)$	$c_5(0,1,1,1,0)$	$\cdots$
$(1,0,0)$	$c_3(1,0,0)$	$c_4(1,0,0,0)$	$c_4(1,0,0,1)$	$c_5(1,0,0,0,0)$	$c_5(1,0,0,0,1)$	$c_5(1,0,0,1,0)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1: Hankel matrix of the observation.

	$\emptyset$	(0)	(1)	$(0,0)$	$(0,1)$	$(1,0)$	$\cdots$
$\emptyset$	$\underline{c}^\mathsf{T}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{AD}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DA}\underline{b}$	$\cdots$
(0)	$\underline{c}^\mathsf{T}\mathbf{A}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{AD}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^3\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^2\mathbf{D}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{ADA}\underline{b}$	$\cdots$
(1)	$\underline{c}^\mathsf{T}\mathbf{D}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DA}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{D}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DA}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DAD}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{D}^2\mathbf{A}\underline{b}$	$\cdots$
$(0,0)$	$\underline{c}^\mathsf{T}\mathbf{A}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^3\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^2\mathbf{D}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^4\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^3\mathbf{D}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^2\mathbf{DA}\underline{b}$	$\cdots$
$(0,1)$	$\underline{c}^\mathsf{T}\mathbf{AD}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{ADA}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{AD}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{ADA}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{ADAD}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{AD}^2\mathbf{A}\underline{b}$	$\cdots$
$(1,0)$	$\underline{c}^\mathsf{T}\mathbf{DA}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DA}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DAD}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DA}^3\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DA}^2\mathbf{D}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{DADA}\underline{b}$	$\cdots$
$(1,1)$	$\underline{c}^\mathsf{T}\mathbf{D}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{D}^2\mathbf{A}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{D}^3\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{D}^2\mathbf{A}^2\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{D}^2\mathbf{AD}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{D}^3\mathbf{A}\underline{b}$	$\cdots$
$(0,0,0)$	$\underline{c}^\mathsf{T}\mathbf{A}^3\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^4\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^3\mathbf{D}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^5\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^4\mathbf{D}\underline{b}$	$\underline{c}^\mathsf{T}\mathbf{A}^3\mathbf{DA}\underline{b}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2: Hankel matrix of the state.

3.1 Simulation

In this section we consider the state representation form

$$\mathbf{X}_t = A\mathbf{X}_{t-1} + D\mathbf{X}_{t-1}\mathrm{w}_{t-1} + \mathbf{b}\mathrm{w}_{t-1},$$
$$Y_t = \mathbf{c}'\mathbf{X}_t. \quad (2)$$

Now, if we put a state space form for lower triangular bilinear models as

$$\begin{bmatrix} X \\ V_1 \\ Z_1 \end{bmatrix}_t = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ V_1 \\ Z_1 \end{bmatrix}_{t-1} + \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} \mathrm{w}_{t-1}$$
$$+ \begin{bmatrix} d_{11} & 0 & d_{12} \\ d_{22} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ V_1 \\ Z_1 \end{bmatrix}_{t-1} \mathrm{w}_{t-1},$$

and

$$Y_t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ V_1 \\ Z_1 \end{bmatrix}_t + \mathrm{w}_t$$
$$= X_t + \mathrm{w}_t,$$

then we see that X_t satisfies equation (3) because

$$X_t = -a_1 X_{t-1} + V_{1,t-1} + b_1 \mathrm{w}_{t-1}$$
$$+ d_{11} X_{t-1}\mathrm{w}_{t-1} + d_{12}.Z_{1,t-1}\mathrm{w}_{t-1},$$
$$V_{1,t} = -a_2 X_{t-1} + b_2 \mathrm{w}_{t-1} + d_{22} X_{t-1}\mathrm{w}_{t-1},$$
$$Z_{1,t} = X_{t-1},$$

provides the lower triangular bilinear model

$$\sum_{m=0}^{2} a_m Y_{t-m} = \sum_{m=0}^{2} (b_m + a_m)\,\mathrm{w}_{t-m} \quad (3)$$
$$+ \sum_{m=1}^{2}\sum_{n=0}^{2-m} d_{m,m+n} Y_{t-m-n}\mathrm{w}_{t-m} \quad (4)$$
$$- \sum_{m=1}^{2}\sum_{n=0}^{2-m} d_{m,m+n} \mathrm{w}_{t-m-n}\mathrm{w}_{t-m},$$

where $a_0 = b_0 = 1$. The general treatment follows easily from this example, see also in [Mohl88], [Ter99]

For the simulation we put $.\sigma^2 = 2$, sample size 2^{17}, $A = \begin{bmatrix} -0.15 & 1 & 0 \\ 0.045 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ with eigenvalues 0.15, -0.3 and 0.

The bilinear parameters are $D = \begin{bmatrix} .3 & 0 & -.2 \\ .5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with eigenvalues 0.3 and 0, 0, finally $\mathbf{b} = \begin{bmatrix} .6 \\ 0 \\ 0 \end{bmatrix}$.

The Hankel matrix is given in Table 3.it is calculated from the parameters. The Hankel rank follows from the number of the nonzero singular values, Table 4., being 3.

The estimated cross-covariances $EY_t H_{K+1}\left(z^{-n_1}\mathrm{w}_{t-1}, z^{-n_1-n_2-1}\mathrm{w}_{t-1}, \ldots, z^{-\Sigma n_\ell - K}\mathrm{w}_{t-1}\right)$ give an estimate for the above Hankel matrix, Table 5.

The singular values of this Hankel matrix give a possible estimation for the Hankel rank of the system, Table 6.

One may conclude that the Hankel rank is 3.

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	$\emptyset$	(0)	(1)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(0,0,0)$	$(0,0,1)$
$\emptyset$	0.6	-0.09	0.18	0.0405	0.093	0.273	0.054	-0.0101	-0.0058
(0)	-0.09	0.0405	0.093	-0.0101	-0.0058	-0.059	0.0279	0.0033	0.0051
(1)	0.18	0.273	0.054	-0.0329	0.1179	0.0819	0.0162	0.0172	0.0447
$(0,0)$	0.0405	-0.0101	-0.0058	0.0033	0.0051	0.0211	-0.0018	-0.001	-0.001
$(0,1)$	0.093	-0.059	0.0279	0.013	0.0009	0.0423	0.0084	-0.0046	-0.0079
$(1,0)$	0.273	-0.0329	0.1179	0.0172	0.0447	0.1188	0.0354	-0.0041	-0.0014
$(1,1)$	0.054	0.0819	0.0162	-0.0099	0.0354	0.0246	0.0049	0.0052	0.0134
$(0,0,0)$	-0.0101	0.0033	0.0051	-0.001	-0.001	-0.0058	0.0015	0.0003	0.0004
$(0,0,1)$	-0.0058	0.0211	-0.0018	-0.0034	0.0052	-0.0027	-0.0005	0.0015	0.0032
$(0,1,0)$	-0.059	0.013	0.0009	-0.0046	-0.0079	-0.0296	0.0003	0.0013	0.0012
$(0,1,1)$	0.0279	0.0423	0.0084	-0.0051	0.0183	0.0127	0.0025	0.0027	0.0069

Table 3: Hankel matrix of the model.

0.8152	0.3636	0.1314	0	0	0	0	0	0	0	0

Table 4: Singular values.

	$\emptyset$	(0)	(1)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(0,0,0)$	$(0,0,1)$
$\emptyset$	0.5921	-0.099	0.1831	0.0374	0.0852	0.2756	0.0505	-0.0194	-0.007
(0)	-0.099	0.0374	0.0852	-0.0194	-0.007	-0.0671	0.0294	0.0008	0.0014
(1)	0.1831	0.2756	0.0505	-0.0381	0.1181	0.0808	0.0147	0.0109	0.0426
$(0,0)$	0.0374	-0.0194	-0.007	0.0008	0.0014	0.0168	0.0045	0.0018	0.0043
$(0,1)$	0.0852	-0.0671	0.0294	0.0082	0	0.0408	0.0105	-0.0053	-0.0104
$(1,0)$	0.2756	-0.0381	0.1181	0.0109	0.0426	0.1276	0.0424	0.0017	-0.0104
$(1,1)$	0.0505	0.0808	0.0147	-0.0113	0.0337	0.022	0.0027	0.0043	0.0056
$(0,0,0)$	-0.0194	0.0008	0.0014	0.0018	0.0043	-0.0095	0.0024	0.0074	0.0011
$(0,0,1)$	-0.007	0.0168	0.0045	-0.0093	0.0027	-0.0039	0.0031	0.0053	0.0012
$(0,1,0)$	-0.0671	0.0082	0	-0.0053	-0.0104	-0.0247	0.004	0.0086	0.003
$(0,1,1)$	0.0294	0.0408	0.0105	-0.0015	0.0168	0.0171	0.0041	0.0072	0.0079

Table 5: Estimated Hankel matrix.

0.8137	0.3718	0.1299	0.0203	0.0142	0.0123	0.0118	0.0091	0.0068	0.0032	0.0017

Table 6: Estimated singular values.

[Tso2001] Tsoulkas, V., Koukoulas, P., and Kalouptsidis, N., Identification of Input Output Bilinear Systems using Cumulants, IEEE Transactions on Signal Processing, 49(11), 2753–2761, 2001.