Linear-Quadratic Differential Games Revisited

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***Abstract*— This paper revisits the pioneering work of P. Bernhard [2] on two-person zero-sum linear quadratic differential games and generalize it to utility functions without positivity assumptions on the matrices acting on the state variable and to linear dynamics with bounded measurable data matrices. The paper specializes to *state feedback* via Lebesgue measurable *affine closed loop strategies* with possible non L^2-integrable singularities. It first deals with L^2-integrable closed loop strategies and then with the larger family of strategies that may have non L^2-integrable singularities.**

I. INTRODUCTION

Two-person zero-sum games with a quadratic utility function and linear dynamics have been studied in the pioneering work of P. Bernhard [2] in 1979. He gave necessary and sufficient conditions for the existence of a closed loop saddle point for smooth feedback strategies with possible isolated singularities at a finite number of times. He restricted his attention to utility functions where the matrices F and $Q(t)$ in front of the state are positve semidefinite that implies that the utility function is convex with respect to the control of the minimizing player.

Next to the concept of closed loop saddle point, we have the open loop concepts of lower value, upper value, and value of the game. In 2005 P. Zhang [8] proved that if the upper and lower values of the game are finite, then they are equal and the game has a finite value. In that case we have an open loop saddle point that coincides with the concept of a Nash equilibrium. This work was done without the positivity assumptions on the matrices F and $Q(t)$ and opened the way to other investigations.

In 2007 M. Delfour [4] gave a necessary and sufficient condition for the existence of a lower value of the game in term of the usual coupled state-adjoint state system and natural concavity-convexity conditions on the utility function without the positivity assumptions. By duality, those conditions extend to the upper value of the game, and, by combining the two, to the value of the game. In a recent paper M. Delfour and O. Dello Sbarba [6] extended the work of [2] to the case of bounded measurable coefficients, symmetrical matrices F and $Q(t)$ that are not necessarily positive, and affine feedback strategies that are not necessarily L^2-integrable. Connections were made between the fundamental notions from the Calculus of Variations of *normality* and *normalizability* and *Invariant Embedding for all and almost all initial times*. The apparently new and equivalent concept of *structural feedback saddle point* was also introduced.

This work was supported by an NSERC Grant in Canada

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In this paper, we review the above cited literature where detailed proofs can be found. We focus our attention on the finite dimensional case for which a fairly complete theory is available under the most general assumptions. At the same time we single out finite dimensional concepts that do not carry over to evolution equations in infinite dimensional spaces. We give equivalent notions and concepts. One of them is the invariant embedding for almost all initial times that turns out to have been observed for the Helmholtz equation of waveguides (cf. I. Champagne [3]). We give complete classifications in terms of open loop values of the game and compare results for L^2-integrable and possibly non L^2-integrable closed loop saddle points.

II. SYSTEM, UTILITY FUNCTION, AND ASSUMPTIONS

Given the Euclidean space $\mathbf{R}^d$ of dimension $d \geq 1$, the *norm* and *inner product* will be denoted by $|x|$ and $x \cdot y$, respectively and irrespective of the dimension d of the space.

Consider the following two-player zero-sum game over the finite time interval $[0, T]$ with quadratic *utility function*

$$C_{x_0}(u, v) \overset{\text{def}}{=} Fx(T) \cdot x(T) + \int_0^T Q(t)x(t) \cdot x(t) + |u(t)|^2 - |v(t)|^2 \, dt, \tag{II.1}$$

where x is the solution of the linear differential system

$$\begin{aligned} &x'(t) = A(t)x(t) + B_1(t)u(t) + B_2(t)v(t) \quad \text{a.e. in } [0, T] \\ &x(0) = x_0, \end{aligned} \tag{II.2}$$

x_0 is the *initial state* at time $t = 0$, $u \in L^2(0, T; \mathbf{R}^m)$, $m \geq 1$, is the strategy of the first player and $v \in L^2(0, T; \mathbf{R}^k)$, $k \geq 1$, is the strategy of the second player. Assume that F is an $n \times n$-matrix and that A, B_1, B_2, and Q are matrix-functions of appropriate order that are measurable and bounded almost everywhere in $[0, T]$. Moreover $Q(t)$ and F are symmetrical. Define $R = B_1 B_1^\top - B_2 B_2^\top$. The above assumptions on F, A, B_1, B_2, and Q will be used throughout this paper. We shall also use the compact notation

$$x' = Ax + B_1 u + B_2 v \quad \text{in } [0, T], \quad x(0) = x_0.$$

III. OPEN LOOP STRATEGIES

Definition III.1. Let x_0 be an initial state in $\mathbf{R}^n$ at $t = 0$.

(i) The game is said to achieve its *open loop lower value* (resp. *upper value*) if

$$v^-(x_0) \overset{\text{def}}{=} \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) \tag{III.1}$$

$$\left(\text{resp. } v^+(x_0) \overset{\text{def}}{=} \inf_{u \in L^2(0,T;\mathbf{R}^m)} \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u, v)\right) \tag{III.2}$$

is finite. By definition $v^-(x_0) \leq v^+(x_0)$.

(ii) The game is said to achieve its *open loop value* if its open loop lower value $v^-(x_0)$ and upper value $v^+(x_0)$ are achieved and $v^-(x_0) = v^+(x_0)$. The *open loop value* of the game will be denoted by $v(x_0)$.

(iii) A pair $(\bar{u}, \bar{v})$ in $L^2(0,T;\mathbf{R}^m) \times L^2(0,T;\mathbf{R}^k)$ is an *open loop saddle point* of $C_{x_0}(u,v)$ if for all u in $L^2(0,T;\mathbf{R}^m)$ and all v in $L^2(0,T;\mathbf{R}^k)$

$$C_{x_0}(\bar{u}, v) \le C_{x_0}(\bar{u}, \bar{v}) \le C_{x_0}(u, \bar{v}). \tag{III.3}$$

We recall and sharpen the results of [4, Thms 2.2, 2.3, and 2.4] when the open loop lower or upper value of the game is finite for a given initial state x_0. In all cases the existence of a solution to the coupled system (III.5) is a necessary condition and the difference is in the *feasibility conditions* (III.6), (III.9), and (III.10) for the null initial state. Moreover, in each case, the global assumption of finiteness for *all* initial state $x_0 \in \mathbf{R}^n$ yields the *uniqueness* of solution (x,p) of the coupled system (III.5) (cf. [4, Thms 2.6, 2.7, and 2.8]).

Theorem III.1. *The following conditions are equivalent.*

(i) $\exists \hat{u} \in L^2(0,T;\mathbf{R}^m)$ *and* $\hat{v} \in L^2(0,T;\mathbf{R}^k)$ *such that*

$$\begin{aligned} C_{x_0}(\hat{u}, \hat{v}) &= \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) \\ &= \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v). \end{aligned} \tag{III.4}$$

(ii) *The open loop lower value* $v^-(x_0)$ *of the game is finite.*

(iii) $\exists$ *an* $H^1(0,T;\mathbf{R}^n)^2$ *solution of the* coupled system

$$\begin{cases} x' = Ax - Rp, & x(0) = x_0 \\ p' + A^\top p + Qx = 0, & p(T) = Fx(T), \end{cases} \tag{III.5}$$

and the following identities are verified

$$\begin{aligned} &\sup_{v \in V(0)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u,v) \\ &\quad = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u,0) = C_0(0,0). \end{aligned} \tag{III.6}$$

Under such conditions, the optimal controls and the open loop lower value are given by the following expressions

$$\hat{u} = -B_1^\top p, \quad \hat{v} = B_2^\top p, \quad C_{x_0}(\hat{u}, \hat{v}) = p(0) \cdot x_0. \tag{III.7}$$

Remark III.1. Condition $B_2^\top p \in V(x_0)$ in [4, Thms 2.2, 2.6],

$$V(x_0) \overset{\text{def}}{=} \left\{ v \in L^2(0,T;\mathbf{R}^k) : \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v) > -\infty \right\}$$

was redundant. The last part of identity (III.6) is equivalent to the convexity of $u \mapsto C_{x_0}(u,v)$. By [4, Thm 3.1] the convexity plus a solution of the coupled system (III.5) yields

$$\inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, B_2^\top p) = C_{x_0}(-B_1^\top p, B_2^\top p) > -\infty$$

and hence $B_2^\top p \in V(x_0)$.

Similarly, the additional condition $-B_1^\top p \in U(x_0)$ in [4, Thms 2.3 and 2.7] is also redundant.

Theorem III.2. *The following conditions are equivalent.*

(i) $\exists$ $\hat{u}$ *in* $L^2(0,T;\mathbf{R}^m)$ *and* $\hat{v}$ *in* $L^2(0,T;\mathbf{R}^k)$ *such that*

$$\begin{aligned} C_{x_0}(\hat{u}, \hat{v}) &= \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u}, v) \\ &= \inf_{u \in L^2(0,T;\mathbf{R}^m)} \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v). \end{aligned} \tag{III.8}$$

(ii) *The open loop upper value* $v^+(x_0)$ *of the game is finite.*

(iii) *There exists a solution* (x,p) *of the* coupled system (III.5) *and the following identities are verified*

$$\begin{aligned} &\inf_{u \in U(0)} \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_0(u,v) \\ &\quad = \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_0(0,v) = C_0(0,0). \end{aligned} \tag{III.9}$$

Under such conditions, the optimal controls and the open loop upper value are given by expressions (III.7)*.*

Finally, by combining the previous two theorems and a result of P. Zhang [8], we get

Theorem III.3. *The following conditions are equivalent.*

(i) *There exists an open loop* saddle point *of* $C_{x_0}(u,v)$.

(ii) *The open loop value* $v(x_0)$ *of the game is finite.*

(iii) *There exists a pair* $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ *solution of the* coupled system (III.5) *and the identities*

$$\sup_{v \in L^2(0,T;\mathbf{R}^k)} C_0(0,v) = C_0(0,0) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u,0). \tag{III.10}$$

are verified (convexity-concavity condition).

Under such conditions, the optimal controls and the value are given by expressions (III.7) *and* $v(x_0) = v^-(x_0) = v^+(x_0)$.

There are six possible cases that can all occur. The first three are (a) $v^-(x_0)$ finite and $v^+(x_0)$ finite; (b) $v^-(x_0)$ finite and $v^+(x_0) = +\infty$; (c) $v^-(x_0) = -\infty$ and $v^+(x_0)$ finite. The last three are *degenerate ones*: (d) $v^-(x_0) = -\infty$ and $v^+(x_0) = +\infty$; (e) $v^-(x_0) = v^+(x_0) = +\infty$; (f) $v^-(x_0) = v^+(x_0) = -\infty$.

IV. L^2-INTEGRABLE CLOSED LOOP STRATEGIES

Definition IV.1. L^2*-integrable affine closed loop strategies*.

$$\Phi \overset{\text{def}}{=} \left\{ \phi : [0,T] \times \mathbf{R}^n \to \mathbf{R}^m \;\middle|\; \begin{array}{l} \text{such that } x \mapsto \phi(t,x) \text{ is} \\ \text{affine and } t \mapsto \phi(t,x) \\ \text{belongs to } L^2(0,T;\mathbf{R}^m) \end{array} \right\}$$

$$\Psi \overset{\text{def}}{=} \left\{ \psi : [0,T] \times \mathbf{R}^n \to \mathbf{R}^k \;\middle|\; \begin{array}{l} \text{such that } x \mapsto \psi(t,x) \text{ is} \\ \text{affine and } t \mapsto \psi(t,x) \\ \text{belongs to } L^2(0,T;\mathbf{R}^k) \end{array} \right\}.$$

To each $\phi \in \Phi$ (resp. $\psi \in \Psi$) we can associate an $L^2(0,T;\mathbf{R}^m)$-vector function u and an $m \times n$ matrix L^2-function U such that $\phi(t,x) = u(t) + U(t)x$ (resp. an $L^2(0,T;\mathbf{R}^k)$-vector function v and a $k \times n$ matrix L^2-function V such that $\psi(t,x) = v(t) + V(t)x$). The matrix functions U and V may have singularities, but they are globally L^2-integrable. As a result the fundamental matrix associated with the L^2-matrix $A + B_1 U + B_2 V$ is invertible everywhere in $[0,T]$. Therefore, the closed loop system

$$x' = (A + B_1 U + B_2 V)x + B_1 u + B_2 v, \quad x(0) = x_0 \tag{IV.1}$$

has a unique solution in $H^1(0,T;\mathbf{R}^n)$ for all $x_0 \in \mathbf{R}^n$ and all pairs $(\phi,\psi) \in \Phi \times \Psi$ are *admissible*.

Definition IV.2 (L^2*-integrable closed loop saddle point*).

(i) Given $x_0 \in \mathbf{R}^n$, we say that $(\phi^*,\psi^*) \in \Phi \times \Psi$ is *an* L^2*-integrable closed loop saddle point* of $C_{x_0}(\phi,\psi)$ if for all $\phi \in \Phi$ and $\psi \in \Psi$

$$C_{x_0}(\phi^*,\psi) \le C_{x_0}(\phi^*,\psi^*) \le C_{x_0}(\phi,\psi^*). \quad \text{(IV.2)}$$

(ii) We say that $(\phi^*,\psi^*) \in \Phi \times \Psi$ is *a global* L^2*-integrable closed loop saddle point* of $C_{x_0}(\phi,\psi)$ in $\Phi \times \Psi$ if for all $x_0 \in \mathbf{R}^n$, $\phi \in \Phi$, and $\psi \in \Psi$ the inequalities (IV.2) are verified.

Theorem IV.1. *The following statements are equivalent.*

(i) $(\phi^*,\psi^*) \in \Phi \times \Psi$ *is* a global L^2-integrable closed loop saddle point *of* $C_{x_0}(\phi,\psi)$ *in* $\Phi \times \Psi$.

(ii) *(*L^2*-linear feedback) For all* $x_0 \in \mathbf{R}^n$ *there exist a unique solution* $(\hat{x},\hat{p})$ *in* $H^1(0,T;\mathbf{R}^n)^2$ *to the system*

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0 \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T) \end{cases} \quad \text{(IV.3)}$$

and L^2*-matrices* U_* *and* V_* *of appropriate orders such that for all* $x_0 \in \mathbf{R}^n$

$$\hat{u} = -B_1^\top \hat{p} = U_*\hat{x}, \quad \hat{v} = B_2^\top \hat{p} = V_*\hat{x}. \quad \text{(IV.4)}$$

(iii) *(Invariant embedding) For all* $s \in [0,T[$*, there exists a unique* $H^1(s,T)$ *solution of the coupled system*

$$\begin{cases} \widehat{X}_s' = A\widehat{X}_s - R\widehat{\Lambda}_s, & \widehat{X}_s(s) = I \\ \widehat{\Lambda}_s' + A^\top \widehat{\Lambda}_s + Q\widehat{X}_s = 0, & \widehat{\Lambda}_s(T) = F\widehat{X}_s(T). \end{cases} \quad \text{(IV.5)}$$

By convention, set $\widehat{X}_T(T) = I$ *and* $\widehat{\Lambda}_T(T) = F$.

(iv) *(Normality)* $\det X(t) \neq 0$ *everywhere in* $[0,T]$*, where* (X,Λ) *is the* $H^1(0,T)$ *solution of the system*

$$\begin{cases} X' = AX - R\Lambda, & X(T) = I \\ \Lambda' + A^\top \Lambda + QX = 0, & \Lambda(T) = F. \end{cases} \quad \text{(IV.6)}$$

(v) *There exists a symmetrical solution* P *with elements in* $H^1(0,T)$ *to the matrix RDE*

$$P' + PA + A^*P - PRP + Q = 0, \quad P(T) = F. \quad \text{(IV.7)}$$

(vi) *(Structural* L^2*-integrable closed loop saddle point) There exists* $(\phi^*,\psi^*) \in \Phi \times \Psi$ *such that for all* $x_0 \in \mathbf{R}^n$, $u \in L^2(0,T;\mathbf{R}^m)$*, and* $v \in L^2(0,T;\mathbf{R}^k)$,

$$C_{x_0}(\phi^*,\psi^*+v) \le C_{x_0}(\phi^*,\psi^*) \le C_{x_0}(\phi^*+u,\psi^*).$$

In particular $C_{x_0}(\phi^*,\psi^*) = P(0)x_0 \cdot x_0$ *and the closed loop strategies are given by* $\phi^*(t,x) = -B_1^\top(t)P(t)x = U_*(t)x$ *and* $\psi^*(t,x) = B_2^\top(t)P(t)x = V_*(t)x$.

So a necessary condition for the existence of a closed loop saddle point is the existence of a solution to the coupled system. This condition was also necessary for the finiteness of the open loop lower value, upper value, or value of the game. This leads to the following natural classification.

Theorem IV.2. *Assume that* $(\phi^*,\psi^*) \in \Phi \times \Psi$ *is a closed loop saddle point of* $C_{x_0}(\phi,\psi)$.

(a) $v(x_0)$ *is finite if and only if* $C_{x_0}(u,v)$ *is convex in* u *and concave in* v.
(b) $v^-(x_0)$ *is finite and* $v^+(x_0) = +\infty$ *if and only if* $C_{x_0}(u,v)$ *is convex in* u *and not concave in* v.
(c) $v^+(x_0)$ *is finite and* $v^-(x_0) = -\infty$ *if and only if* $C_{x_0}(u,v)$ *is concave in* v *and not convex in* u.
(d) $v^-(x_0) = -\infty$ *and* $v^+(x_0) = +\infty$ *if and only if* $C_{x_0}(u,v)$ *is not convex in* u *and not concave in* v.
(e) $v^-(x_0) = v^+(x_0) = +\infty$ *cannot occur.*
(f) $v^-(x_0) = v^+(x_0) = -\infty$ *cannot occur.*

In the first three cases $C_{x_0}(\phi^*,\psi^*)$ *is equal to* $v(x_0)$, $v^-(x_0)$*, and* $v^+(x_0)$*, respectively.*

Remark IV.1. Case (d) can occur. An example can be constructed by using a first system of the type (b) and a second system of the type (c) without interconnection with utility function equal to the sum of the two utility functions.

V. Closed loop strategies with non L^2-integrable singularities

We now extend the definitions and results of the previous section to Lebesgue measurable feedback strategies with singularities that are not necessarily L^2-integrable in any of their neighborhood. We shall consider the families $\widetilde{\Phi}$ and $\widetilde{\Psi}$ of affine feedback strategies that are measurable in time.

Definition V.1. *Measurable affine closed loop strategies.*

$$\widetilde{\Phi} \stackrel{\text{def}}{=} \left\{ \phi : [0,T] \times \mathbf{R}^n \to \mathbf{R}^m \,\middle|\, \begin{array}{l} \text{such that } x \mapsto \phi(t,x) \text{ is} \\ \text{is affine, } t \mapsto \phi(t,x) \text{ is} \\ \text{Lebesgue measurable, and} \\ t \mapsto \phi(t,0) \text{ belongs to} \\ L^2(0,T;\mathbf{R}^m) \end{array} \right\}$$

$$\widetilde{\Psi} \stackrel{\text{def}}{=} \left\{ \psi : [0,T] \times \mathbf{R}^n \to \mathbf{R}^k \,\middle|\, \begin{array}{l} \text{such that } x \mapsto \psi(t,x) \text{ is} \\ \text{is affine, } t \mapsto \psi(t,x) \text{ is} \\ \text{Lebesgue measurable, and} \\ t \mapsto \psi(t,0) \text{ belongs to} \\ L^2(0,T;\mathbf{R}^k) \end{array} \right\}.$$

We say that ϕ (resp. ψ) is a *linear closed loop strategy* if ϕ (resp. ψ) is linear in x. To any $(\phi,\psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ are associated measurable matrix functions $U(t)$ and $V(t)$ and L^2 vector functions u and v such that $\phi(t,x) = U(t)x + u(t)$ and $\psi(t,x) = V(t)x + v(t)$.

At that level of generality, an *admissibility condition* on the pair $(\phi,\psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ is required to make sense of a solution of the state equation. The set of admissible strategies is no longer $\Phi \times \Psi$ but a subspace S of $\widetilde{\Phi} \times \widetilde{\Psi}$ containing $\Phi \times \Psi$.

A. P. Bernhard [2]'s conditions in the free end case

In the free end case with $F \ge 0$ and $Q(t) \ge 0$, the necessary and sufficient condition of P. Bernhard [2, Thm 3.1] for the existence of a non-degenerate closed loop saddle point in the sense of [2, Definition 2.3 and Remark 5.1] reduces to the following three properties:

(ii) $X(t)$ is invertible except possibly at isolated points in $[0,T]$, where (X,Λ) is the $H^1(0,T)$ solution of

$$\begin{cases} X' = AX - R\Lambda, & X(T) = I \\ \Lambda' + A^\top \Lambda + QX = 0, & \Lambda(T) = F \end{cases} \tag{V.1}$$

(iii) $x_0 \in \operatorname{Im} X(0)$
(iv) for all $t \in [0,T]$, $P(t) \geq 0$,

where P is defined in terms the pseudo inverse $X(t)^\dagger$ of X

$$P(t) = \Lambda(t)\, X(t)^\dagger. \tag{V.2}$$

Condition (ii) defines the matrix function $P(t)$ a.e. in $[0,T]$ and gives a meaning to a solution of the RDE via the solution (Λ, X) of system (V.1). The positivity of F and $Q(t)$ makes the utility function $C_{x_0}(u,v)$ convex in u and this leads to the positivity of $P(t)$. Hence only cases (a), (b) and (e) can occur. Finally, (iii) and (iv) are redundant.

The relaxation of the positivity assumptions generates the two dual cases (c) and (f) of (a) and (e), and a new case (d) that can occur in the presence of an L^2-integrable closed loop saddle point. The main difficulty is to make sense of the definition of a closed loop saddle point since some of the competitive terms in the utility function may simultaneously blow up making it difficult to set the utility function equal to $\pm\infty$ (cf. [2, p. 68 and Remark 5.1]).

B. Normalizability and its consequences

Given the matrices A, B_1, B_2, Q, and F verifying the conditions of § II, system (V.1) always has a unique solution (X,Λ) with elements in $H^1(0,T)$. Introduce the notation

$$Z \overset{\text{def}}{=} \{s \in [0,T] : \det X(s) = 0\}. \tag{V.3}$$

We now relax the definition of [2] (property (ii) in § V-A).

Definition V.2. The problem (II.1)–(II.2) is *normalizable* if $\det X(t) \neq 0$ a.e. in $[0,T]$.

Lemma V.1. (i) *If problem* (II.1)–(II.2) *is normalizable in the sense of Definition* V.2, *then Z contains at most a countable number of instants.*
(ii) *If problem* (II.1)–(II.2) *is* normalizable *in the sense of [2], then Z contains a finite number of instants.*

Here Z is possibly infinite with accumulation points that are not isolated as can be seen from the following example.

Example V.1. Consider an extension of the example from [2, Example 5.1, p. 67]:

$$\begin{aligned} &x'(t) = B_1(t)\, u(t) + B_2(t)\, v(t) \text{ a.e. in } [0,2], \\ &x(0) = x_0 \end{aligned} \tag{V.4}$$

$$C_{x_0}(u,v) = \frac{1}{2}|x(2)|^2 + \int_0^2 |u(t)|^2 - |v(t)|^2\, dt, \tag{V.5}$$

where

$$B_1(t) \overset{\text{def}}{=} \begin{cases} 2-t, & 1 < t \leq 2, \\ 2^{\frac{n}{2}+1}\left(\dfrac{1}{2^n} - t\right), & \dfrac{1}{2^{n+1}} < t \leq \dfrac{1}{2^n}, \\ & n \geq 0, \end{cases} \tag{V.6}$$

$$B_2(t) \overset{\text{def}}{=} \begin{cases} t, & 1 < t \leq 2, \\ 2^{\frac{n}{2}+1}\left(t - \dfrac{1}{2^{n+1}}\right), & \dfrac{1}{2^{n+1}} < t \leq \dfrac{1}{2^n}, \\ & n \geq 0, \end{cases} \tag{V.7}$$

It is readily seen that both B_1 and B_2 are measurable and bounded. Here $A = 0$, $F = 1/2$, $Q = 0$, and $R = B_1 B_1^* - B_2 B_2^*$

$$R(t) = \begin{cases} 4(1-t), & 1 < t \leq 2, \\ \left(\dfrac{3}{2^n} - 4t\right), & \dfrac{1}{2^{n+1}} < t \leq \dfrac{1}{2^n}, \quad n \geq 0. \end{cases} \tag{V.8}$$

The solution of system (V.1) is given by the following expressions

$$X(t) = \begin{cases} (1-t)^2, & 1 < t \leq 2, \\ \left(t - \dfrac{1}{2^n}\right)\left(t - \dfrac{1}{2^{n+1}}\right), & \dfrac{1}{2^{n+1}} < t \leq \dfrac{1}{2^n}, \\ & n \geq 0, \\ 0, & t = 0, \end{cases}$$
$$\Lambda(t) = \frac{1}{2}. \tag{V.9}$$

Here $Z = \{1/2^n : n \geq 0\} \cup \{0\}$ has an infinite number of isolated points plus the accumulation point 0 that is not isolated since $1/2^n \to 0$.

The normalizability property relies on the fact that the state equation can be solved backward in finite dimension. In general, this would not be true for infinite dimensional evolution systems. Yet, we have the following equivalent property that would be more natural in infinite dimension. Denote by Z' the set of all initial times $s \in [0,T[$ such that the matrix differential system

$$\begin{cases} \widehat{X}_s' = A\widehat{X}_s - R\widehat{\Lambda}_s, & \widehat{X}_s(s) = I, \\ \widehat{\Lambda}_s' + A^\top \widehat{\Lambda}_s + Q\widehat{X}_s = 0, & \widehat{\Lambda}_s(T) = F\widehat{X}_s(T), \end{cases} \tag{V.10}$$

has a solution $(\widehat{X}_s, \widehat{\Lambda}_s)$ with elements in $H^1(s,T)$.

Lemma V.2. (i) $Z = Z'$.
(ii) *For $s \in [0,T[\,\backslash Z'$, the matrix differential system* (V.10) *has a unique solution $(\widehat{X}_s, \widehat{\Lambda}_s)$ with elements in $H^1(s,T)$, for all $t \in [s,T[\,\backslash Z'$, $\det \widehat{X}_s(t) \neq 0$, and $P(s) = \Lambda(s) X(s)^{-1} = \widehat{\Lambda}_s(s)$.*
By convention we set $\widehat{X}_T(T) = I$ and $\widehat{\Lambda}_T(T) = F$.

Remark V.1. Thus, normalizabilty is equivalent to invariant embedding with respect to almost all initial times as in [4]. This equivalence is analogous to Theorem IV.1 (iii). It says that the decoupling operator $P(s)$ can be defined a.e. as

$\widehat{\Lambda}_s(s)$. This property was observed for the RDE associated with Helmholtz equation of waveguides. Due to a resonance phenomenon, the invariant embedding cannot not be done at an at most countable set of initial times (cf. I. Champagne [3] where a sup-inf formulation is introduced).

Starting with the normalizability property, we now proceed in a constructive way to identify the appropriate definition of a closed loop saddle point in the presence of non L^2-integrable singularities in the closed loop strategies.

Lemma V.3. *Assume that the problem* (II.1)–(II.2) *is normalizable. Then*

(i) $P(s) = \Lambda(s)X(s)^{-1}$ *is uniquely defined and symmetrical for all* $s \in [0,T]\backslash Z$*,* P *verifies the matrix RDE*

$$P' + PA + A^\top P - PRP + Q = 0, \quad P(T) = F$$

in $[0,T]\backslash Z$*, and* PX *is the unique* $H^1(0,T)$ *solution of the matrix system*

$$(PX)' + A^\top(PX) + QX = 0, \quad (PX)(T) = F.$$

(ii) X *is an* $H^1(0,T)$ *solution of the closed loop equation*

$$X' = (A - RP)X, \quad X(T) = I, \tag{V.11}$$

and $\det X(t) \neq 0$ *in* $[0,T]\backslash Z$*.*

(iii) *For almost all* $s \in [0,T]\backslash Z$*,* $\widehat{X}_s$ *is an* $H^1(s,T)$ *solution of the closed loop matrix differential equation*

$$\widehat{X}_s' = (A - RP)\widehat{X}_s, \quad \widehat{X}_s(s) = I, \tag{V.12}$$

and $\det \widehat{X}_s(t) \neq 0$ *in* $[s,T]\backslash Z$*.*

C. Admissible closed loop affine strategies pairs

Definition V.3. We say that the pair $(\phi,\psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ belongs to S or simply that (ϕ,ψ) is an *admissible pair* if the associated matrix differential equation

$$X' = (A + B_1U + B_2V)X, \quad X(T) = I \tag{V.13}$$

has an $H^1(0,T)$ solution such that $\det X(t) \neq 0$ a.e. in $[0,T]$, UX and VX are $L^2(0,T)$ matrices, $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$, and $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$.

Remark V.2. As in Lemma V.1, it can be proved that the condition $\det X(t) \neq 0$ a.e. in $[0,T]$ implies that $\det X(t) \neq 0$ except at an at most countable number of instants in $[0,T]$. Therefore, the matrix $A + B_1U + B_2V = X'X^{-1}$ is the product of an L^2-matrix function and a continuous matrix function with possible non L^2-integrable singularities at an at most countable number of instants. Since A, B_1, and B_2 are L^∞-matrix functions, it implies that the feedback matrix functions U and V will have properties similar to $X'X^{-1}$ and hence possible non L^2-integrable singularities at an at most countable number of instants.

As a consequence, given $(\phi,\psi) \in S$ and $y_0 \in \mathbf{R}^n$, $x(t) = X(t)y_0$ is a solution in $H^1(0,T;\mathbf{R}^n)$ of

$$x' = [A + B_1U + B_2V]x, \quad x(0) = X(0)y_0, \tag{V.14}$$

(or $x(T) = y_0$) such that

$$u = Ux = UXy_0 \in L^2(0,T;\mathbf{R}^m) \text{ and}$$
$$v = Vx = VXy_0 \in L^2(0,T;\mathbf{R}^k).$$

However this solution of system (V.14) is not unique. The admissibility condition amounts to a change in the state variable via the associated transformation $X(t)$.

Lemma V.4. *Assume that* $(\phi,\psi) \in S$*,* $\phi(t,x) = U(t)x + u_0(t)$*, and* $\psi(t,x) = V(t)x + v_0(t)$*. Let* X *be a solution of system* (V.13) *such that* $\det X(t) \neq 0$ *a.e. in* $[0,T]$*.*

(i) *For all* $y_0 \in \mathbf{R}^n$*,* $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$*, and* $v \in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$*, the system*

$$y' = X^{-1}(B_1u + B_2v), \quad y(0) = y_0, \tag{V.15}$$

has a unique solution in $H^1(0,T;\mathbf{R}^n)$*,* $x \stackrel{\text{def}}{=} Xy$ *is the unique solution in*

$$H^1_X(0,T;\mathbf{R}^n) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} x \in H^1(0,T;\mathbf{R}^n) : \\ \exists y \in H^1(0,T;\mathbf{R}^n) \text{ s. t. } x = Xy \end{array} \right\} \tag{V.16}$$

of the system

$$\begin{array}{l} x' = [A + B_1U + B_2V]x + B_1u + B_2v, \\ x(0) = X(0)y_0, \end{array} \tag{V.17}$$

up to a function of the form $X(t)z_0$ *for some* $z_0 \in \ker X(0)$*, and all the solutions of* (V.17) *in* $H^1_X(0,T;\mathbf{R}^n)$ *are given by the expression*

$$x(t) = X(t)\left[y_0 + z_0 + \int_0^t X^{-1}(B_1u + B_2v)\,ds\right] \tag{V.18}$$

for all $z_0 \in \ker X(0)$*.*

(ii) *The subspaces* $\mathcal{U} = \{u \in L^2(0,T;\mathbf{R}^m) : |X^{-1}|u \in L^2(0,T;\mathbf{R}^m)\}$ *and* $\mathcal{V} = \{v \in L^2(0,T;\mathbf{R}^k) : |X^{-1}|v \in L^2(0,T;\mathbf{R}^k)\}$ *are dense in* $L^2(0,T;\mathbf{R}^m)$ *and* $L^2(0,T;\mathbf{R}^k)$*, respectively.*

As for normalizability, Definition V.3 is equivalent to

Definition V.4. We say that the pair $(\phi,\psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ belongs to S or simply that (ϕ,ψ) is an *admissible pair* if, for almost all $s \in [0,T[$, the matrix differential equation

$$X_s' = (A + B_1U + B_2V)X_s, \quad X_s(s) = I, \tag{V.19}$$

has an $H^1(s,T)$ solution, UX_s and VX_s are $L^2(s,T)$ matrices, $|X_s^{-1}|u \in L^2(s,T;\mathbf{R}^m)$, and $|X_s^{-1}|v \in L^2(s,T;\mathbf{R}^k)$.

D. Necessary and sufficient conditions for normalizability

Lemma V.5. *Assume that problem* (II.1)–(II.2) *is normalizable. Let* P *be defined by* (V.2)*.*

(i) X *is a solution of the matrix equation* (V.11)

$$X' = (A - RP)X, \quad X(T) = I,$$

in $H^1(0,T)$ *such that* $\det X(t) \neq 0$ *in* $[0,T]\backslash Z$*,*

$$\begin{array}{l} U_*X = -B_1^\top PX = -B_1^\top \Lambda \in L^2(0,T;\mathbf{R}^n)^n \\ V_*X = B_2^\top PX = B_2^\top \Lambda \in L^2(0,T;\mathbf{R}^n)^n \end{array} \tag{V.20}$$

for the matrices

$$U_*(t) \overset{\text{def}}{=} -B_1^\top(t)P(t) \text{ and } V_*(t) \overset{\text{def}}{=} B_2^\top(t)P(t), \quad \text{(V.21)}$$

and the linear pair $(\phi^*, \psi^*) \in S$, *where* $(\phi^*(t,x), \psi^*(t,x)) = (U_*(t)x, V_*(t)x)$.

(ii) *Given* $x_0 \in \operatorname{Im} X(0)$, $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$, *and* $v \in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$, *the system*

$$x' = [A - RP]x + B_1 u + B_2 v, \quad x(0) = x_0, \quad \text{(V.22)}$$

has a solution $x \in H^1_X(0,T;\mathbf{R}^n)$ *unique up to an element* $X(t)z_0$ *for some* $z_0 \in \ker X(0)$. *Then, for any solution* $x \in H^1_X(0,T;\mathbf{R}^n)^n$ *of* (V.22)

$$\begin{aligned} &C_{x_0}(\phi^*(x)+u, \psi^*(x)+v) \\ &= \Lambda(0)y_0 \cdot x_0 + \int_0^T |u|^2 - |v|^2\, dt, \end{aligned} \quad \text{(V.23)}$$

$C_{x_0}(\phi^*,\psi^*) = \Lambda(0)y_0 \cdot x_0$ *is independent of the choice of* y_0 *such that* $X(0)y_0 = x_0$, *and* $C_{x_0}(\phi^*(x)+u, \psi^*(x)+v)$ *is independent of the choice of the solution* x *in* $H^1_X(0,T;\mathbf{R}^n)$ *to system* (V.22).

(iii) *For all* $x_0 \in \operatorname{Im} X(0)$, $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$, *and* $v \in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$

$$C_{x_0}(\phi^*, \psi^* + v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi^* + u, \psi^*). \quad \text{(V.24)}$$

Remark V.3. In view of Lemma V.4 (ii), inequalities (V.24) are verified on dense subsets of $L^2(0,T;\mathbf{R}^m)$ and $L^2(0,T;\mathbf{R}^k)$. They characterize an open loop saddle point in $(0,0) \in \mathcal{U} \times \mathcal{V}$.

We now give the main result that sheds light on the choice of a definition of a closed loop saddle point in the presence of non L^2-integrable closed loop strategies.

Theorem V.1. *The following statements are equivalent.*

(i) *Problem* (II.1)–(II.2) *is* normalizable.

(ii) *There exists a pair of closed loop strategies* $(\phi^*, \psi^*) \in S$ *that for all* $x_0 \in \operatorname{Im} X(0)$, *all* $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$, *and all* $v \in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$

$$C_{x_0}(\phi^*, \psi^* + v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi^* + u, \psi^*). \quad \text{(V.25)}$$

(iii) *There exists a linear pair* $(\phi^*, \psi^*) \in S$ *(that is,* $\psi^*(t,x) = V_*(t)x$ *and* $\phi^*(t,x) = U_*(t)x$*) such that for all* $x_0 \in \operatorname{Im} X(0)$, *there exist a solution* $(\hat{x}, \hat{p}) \in H^1_X(0,T;\mathbf{R}^n) \times H^1(0,T;\mathbf{R}^n)$ *of*

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0 \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T) \end{cases} \quad \text{(V.26)}$$

and

$$-B_1^\top \hat{p} = U_*\hat{x}, \quad B_2^\top \hat{p} = V_*\hat{x}. \quad \text{(V.27)}$$

For all $x_0 \in \operatorname{Im} X(0)$ *the feedback strategies associated with* C_{x_0} *are given by*

$$\phi^*(t,x) = -B_1^\top(t)P(t)x, \; \psi^*(t,x) = B_2^\top(t)P(t)x, \quad \text{(V.28)}$$

where P *is defined by* (V.2), *and the value of the closed loop saddle point by* $C_{x_0}(\phi^*,\psi^*) = \Lambda(0)y_0 \cdot x_0$ *for some* $y_0 \in \mathbf{R}^n$ *such that* $x_0 = X(0)y_0$ *and this value is independent of the choice of* y_0 *such that* $x_0 = X(0)y_0$.

E. Definition of a closed loop saddle point

In view of Theorem V.1, we adopt the following definition of a *closed loop saddle point*.

Definition V.5. (i) Given $x_0 \in \mathbf{R}^n$, $(\phi^*, \psi^*) \in S$ is *a closed loop saddle point* of C_{x_0} if there exists a solution in $H^1_X(0,T;\mathbf{R}^n)$ to the state equation

$$\begin{aligned} &\hat{x}' = (A + B_1 U_* + B_2 V_*)\hat{x} + B_1 u_* + B_2 v_*, \\ &\hat{x}(0) = x_0, \end{aligned} \quad \text{(V.29)}$$

and for all solutions $\hat{x} \in H^1_X(0,T;\mathbf{R}^n)$ of (V.29), all $u \in L^2(0,T;\mathbf{R}^m)$ such that $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$, all $v \in L^2(0,T;\mathbf{R}^k)$ such that $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$, and all solutions $x_u, x_v \in H^1_X(0,T;\mathbf{R}^n)$ of the state equations

$$x_u' = (A + B_1 U_* + B_2 V_*)x_u + B_1(u_* + u) + B_2 v_*, \quad \text{(V.30)}$$
$$x_v' = (A + B_1 U_* + B_2 V_*)x_v + B_1 u_* + B_2(v_* + v), \quad \text{(V.31)}$$

with $x_u(0) = x_v(0) = x_0$, the following inequalities are verified

$$\begin{aligned} &C_{x_0}(\phi^*(x_v), \psi^*(x_v) + v) \\ &\le C_{x_0}(\phi^*(\hat{x}), \psi^*(\hat{x})) \le C_{x_0}(\phi^*(x_u) + u, \psi^*(x_u)). \end{aligned} \quad \text{(V.32)}$$

(ii) We say that $(\phi^*, \psi^*) \in S$ is an $X(0)$*-global closed loop saddle point* of C_{x_0} if for all $x_0 \in \operatorname{Im} X(0)$ the inequalities (V.32) are verified.

F. Classification: $v(x_0)$ *is finite*

Theorem V.2. (Case (a)) *Given* $x_0 \in \mathbf{R}^n$, *the following statements are equivalent.*

(i) C_{x_0} *has a closed loop saddle point* $(\phi^*, \psi^*) \in S$ *and* $C_{x_0}(u,v)$ *is convex in* u *and concave in* v.

(ii) C_{x_0} *has a closed loop saddle point* $(\phi^*, \psi^*) \in \Phi \times \Psi$ *and* $C_{x_0}(u,v)$ *is convex in* u *and concave in* v (case (a) of Theorem IV.2).

(iii) C_{x_0} *has an open loop saddle point.*

G. Classification: $v^-(x_0)$ *or* $v^+(x_0)$ *not finite*

From Theorem V.2, the conclusion is that *closed loop strategies with non* L^2*-integrable singularities will only occur when either* $v^-(x_0)$ *or* $v^+(x_0)$ *is not finite*. We complete the classification along the lines of Theorem IV.2.

Theorem V.3. *Assume that* C_{x_0} *has a closed loop saddle point* $(\phi^*, \psi^*) \in S$ *and that* $C_{x_0}(u,v)$ *is convex in* u *and not concave in* v. *Denote by* $(\hat{u}, \hat{v}) = (\phi^*, \psi^*)$ *the associated controls. Then two possibilities can occur:*

(i) (case (b) $v^-(x_0)$ *is finite and* $v^+(x_0) = +\infty$ *and*

$$v^-(x_0) = \inf_{u \in L^2(0,T;\mathbf{R}^n)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}) = C_{x_0}(\phi^*, \psi^*).$$

(ii) (case (e) $v^-(x_0) = v^+(x_0) = +\infty$ *and*

$$\sup_{\psi\in\Psi} \inf_{\phi\in\Phi} C_{x_0}(\phi,\psi) = \inf_{\phi\in\Phi} \sup_{\psi\in\Psi} C_{x_0}(\phi,\psi) = +\infty.$$

Since the cases (c) and (f) are dual of cases (b) and (e), we have the dual result.

Theorem V.4. *Assume that C_{x_0} has a closed loop saddle point $(\phi^*,\psi^*) \in S$ and that $C_{x_0}(u,v)$ is concave in v and not convex in u. Denote by $(\hat{u},\hat{v}) = (\phi^*,\psi^*)$ the associated controls. Then two possibilities can occur:*

(i) (case (c) $v^+(x_0)$ *is finite and* $v^-(x_0) = -\infty$ *and*

$$v^+(x_0) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u},v) = C_{x_0}(\hat{u},\hat{v}) = C_{x_0}(\phi^*,\psi^*).$$

(ii) (case (f) $v^-(x_0) = v^+(x_0) = -\infty$ *and*

$$\sup_{\psi\in\Psi} \inf_{\phi\in\Phi} C_{x_0}(\phi,\psi) = \inf_{\phi\in\Phi} \sup_{\psi\in\Psi} C_{x_0}(\phi,\psi) = -\infty.$$

If the problem is normalizable, then the conclusions of Theorems V.3 and V.4 hold for all $x_0 \in \operatorname{Im} X(0)$ (cf. Theorem V.1). Finally, we have the last case (d) of Theorem IV.2.

Theorem V.5. Case (d). *Assume that C_{x_0} has a closed loop saddle point $(\phi^*,\psi^*) \in S$ and that it is not convex in u and not concave in v. Then $v^-(x_0) = -\infty$ and $v^+(x_0) = +\infty$.*

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