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Design of optimal deterministic output estimators for distributed parameter systems

Jochem Vissers and Siep Weiland

***Abstract*— This paper considers the optimal H_2 estimation problem for infinite dimensional systems with finite dimensional outputs. It is shown that this problem is equivalent to a dual problem that allows an interpretation as a standard Linear Quadratic optimization problem for an infinite dimensional system. A solution to the latter problem is derived which, in turn solves the optimal estimator problem.**

I. INTRODUCTION

Distributed parameter systems occur in numerous engineering applications. Estimation of non-measured outputs of distributed systems based on measurements or observed outputs is of key importance to infer information of system variables from partial information. One typically distinguishes estimation from filtering problems. Estimation problems are concerned with the (optimal) approximation of non-observed variables from measurements, filtering problems deal with the estimation of state variables. Estimators and filters infer estimates of variables in a causal manner from observed data. The estimation problem is depicted in Fig. 1 and typically involves a given dynamical system that is effected by noise and that produces noise-corrupted measurements y, which are subsequently used to estimate a non-observed signal z. The estimator to be designed is a causal system that processes measurements y to estimates $\hat{z}$.

For distributed parameter systems, the solution of the optimal state estimation problem has been solved in a stochastic setting, for instance in [3]. In this paper we present a complete solution to the design of a deterministic optimal H_2 output estimator for linear distributed parameter systems. The approach avoids stochastic interpretations of variables and is based on methods from functional analysis. For the finite dimensional H_2-estimator design problem it is known that the problem is dual to the linear quadratic optimal control problem. We will generalize this result to the H_2-estimator design problem in an infinite dimensional setting. The result is a generalization of a result for finite dimensional systems presented in [1] and is based on [2]. In the first section we give a precise formulation of the H_2 estimator design problem. Subsequently, we introduce an equivalent dual linear quadratic (LQ) control problem and show that this problem can be solved by using a completion of the squares argument. Finally the solution of the LQ-control problem is used to derive the optimal H_2-estimator.

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J. Vissers is PhD-student and S. Weiland is associate professor with the Faculty of Electrical Engineering, Control Systems, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands `j.a.w.vissers@tue.nl, s.weiland@tue.nl`

II. PROBLEM FORMULATION

Consider a system Σ_p given by the abstract evolution equations:

$$\Sigma_p : \begin{cases} \dot{x} = Ax + Gd_1, \\ y = Cx + Sd_2, \\ z = Hx, \end{cases}$$

with state $x(t) \in \mathcal{X}$, measurement $y(t) \in \mathcal{Y}$, a to-be-estimated output $z(t) \in \mathcal{Z}$, and disturbances $d_1(t) \in \mathcal{D}_1$, $d_2(t) \in \mathcal{D}_2$. Here, $\mathcal{X}$ is a Hilbert space and $\mathcal{Y} = \mathbb{R}^m$, $\mathcal{Z} = \mathbb{R}^n$, $\mathcal{D}_1 = \mathbb{R}^{d_1}$ and $\mathcal{D}_2 = \mathcal{Y}$ are Euclidean spaces of finite dimension. We assume that $A : \mathcal{X} \to \mathcal{X}$ is a linear (possibly unbounded) operator and $G : \mathcal{D} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$ and $H : \mathcal{X} \to \mathcal{Z}$ are bounded linear operators. We study the problem on the time horizon $\mathbb{T} = [0, t_e]$ with $t_e \in \mathbb{R}^+$. Assume that the operator A is the infinitesimal generator of a strongly continuous semigroup operator $T(t) : \mathcal{X} \to \mathcal{X}$ and that is $T(t)$ exponentially stable i.e. there exist positive constants α such that for all $x_0 \in \mathcal{X}$, the exists an M such that $||T(t)x_0|| \leq Me^{-\alpha t}$ for $t \in \mathbb{T}$. Moreover, assume that the pair (A, C) is $\mathbb{T}$-observable, which means that $x_0 = 0$ whenever $CT(t)x_0 = 0$ for all $t \in \mathbb{T}$. Let $d_1 \in L_2(\mathbb{T}; \mathcal{D}_1)$ and $d_2 \in L_2(\mathbb{T}; \mathcal{D}_2)$.

We consider the problem of designing an estimator Σ_e for the outputs z of Σ_p as illustrated in Figure 1. We demand

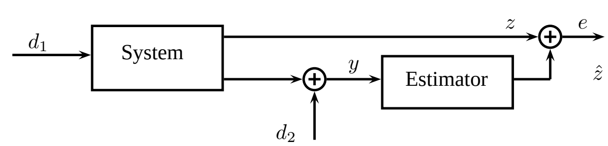

Fig. 1. Estimation scheme

that Σ_e is a linear mapping $L_2(\mathbb{T}, \mathcal{Y}) \to L_2(\mathbb{T}, \mathcal{Z})$ which is causal and which can be represented by the input/output-map:

$$\hat{z}(t_e) = (M(\cdot, t_e) * y(\cdot))(t_e) := \int_0^{t_e} M(\tau, t_e) y(\tau) \mathrm{d}\tau \quad (1)$$

Throughout the paper, the possibly time-dependent integration kernel $M(\cdot, t_e) \in L_2(\mathbb{T}; \mathcal{Z})$ and $M(t, t_e) \in \mathcal{L}(\mathcal{Y}; \mathcal{Z})$ characterizes the estimator. The problem to find the optimal deterministic H_2-estimator amounts to the problem of finding $M(\cdot, t_e) \in L_2(\mathbb{T}; \mathcal{Z})$ such that a given cost criterion is minimal at time instance t_e.

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As design criterion we consider the functional $J : L_2(\mathbb{T}, \mathcal{Z}) \to \mathbb{R}$ for estimation of the output $z(t)$ at $t = t_e$:

$$J(M) = \int_0^{t_e} ||K_M(\tau, t_e)||_2^2 + ||L_M(\tau, t_e)||_2^2 \, \mathrm{d}\tau. \quad (2)$$

Here, operator $K_M(\cdot, t_e)$ is the integration kernel of the mapping $d_1 \mapsto z - \hat{z}$ and $L_M(\cdot, t_e)$ is the integration kernel of the mapping $d_2 \mapsto z - \hat{z}$. It follows that

$$\begin{aligned} K_M(t, t_e) =& HT(t)G - (M(\cdot, t_e) * CT(\cdot)G)(t), \\ L_M(t, t_e) =& M(t, t_e)S. \end{aligned}$$

Suppose that $\dim(\mathcal{Z}) = n$. Since H is a bounded linear operator, it admits the representation $z(t) = Hx(t) = \sum_{i=1}^n \langle h_i, x(t) \rangle e_i$ with $h_i \in \mathcal{X}$ fixed and $\{e_i\}_{i=1}^n$ the canonical basis in $\mathbb{R}^n$. To simplify the analysis we will (first) assume that $n = 1$. This means that the to-be-estimated signal is scalar valued and given by $z(t)Hx(t) = \langle h, x(t) \rangle$ for some element $h \in \mathcal{X}$. We derive the optimal output estimator for this case and will use the result to derive a solution for the more general case where $n > 1$.

III. Existence of a dual problem

In this section it is shown that the optimal H_2 estimation problem is equivalent to a (dual) optimal control problem with a cost criterion that equals the criterion of the estimator design problem given by (2). For the optimal control problem to be derived in this section, the signal $M(t, t_e)$ does serve as control input to be designed.

Consider the adjoint operator of $K_M(t, t_e)$, which (for fixed $t, t_e \in \mathbb{T}$) is given by:

$$K_M^*(t, t_e) = G^* T^*(t) h - \int_0^t G^* T^*(t - \tau) C^* M^*(\tau, t_e) \mathrm{d}\tau$$

In the following theorem 3.1 it is shown that $K_M^*(t, t_e)$ coincides with the output of an artificial system Σ' represented by an abstract evolution equation.

Theorem 3.1: Consider the system:

$$\Sigma^{'}(t_e) : \begin{cases} \dot{\xi}(t) = +A^* \xi(t) - C^* M^{'}(t, t_e) \\ \kappa(t) = G^* \xi(t) \\ \lambda(t) = S M^{'}(t, t_e) \\ \xi(0) = h \in \mathcal{X}, \end{cases} \quad (3)$$

where $t \in \mathbb{T}$.

Then for all $M^{'}(\cdot, t_e) \in L_2(\mathbb{T})$, the outputs $\kappa(t)$ and $\lambda(t)_{M'}$ are uniquely defined for $t \in \mathbb{T}$. Moreover $K_M^*(t, t_e) = \kappa(t)$ and $L_M^*(t, t_e) = \lambda(t)$ for all $t \in \mathbb{T}$ whenever $M^{'}(t, t_e) = M^*(t, t_e)$.

Proof: For the proof first use lemma 5.1. Given that A generates the semi-group $T(t)$, it follows that A^* generates the semigroup $T^*(t)$. Then it is a standard result (see for instance Theorem 3.1.3 in [2]) that

$$\kappa(t) = G^* T^*(t) h - \int_0^t T(t - \tau) C^* M^{'}(\tau, t_e) \mathrm{d}\tau$$

is the unique strong solution of (3). Whenever $M^{'}(t, t_e) = M^*(t, t_e)$ we have that

$$\kappa(t) = G^* T^*(t) h - \int_0^t T(t - \tau) C^* M^*(\tau, t_e) \mathrm{d}\tau = K_M^*(t, t_e).$$

In a similar fashion one shows that $L_M^*(t, t_e) = \lambda(t)$ for all $t \in \mathbb{T}$. ■

Moreover $K_M^*(t, t_e)$ and $L_M^*(t, t_e)$ are also the outputs of an anti-causal differential equation, as is shown in theorem 4:

Theorem 3.2: Consider the system:

$$\begin{cases} \dot{\zeta}(t) = -A^* \zeta(t) + C^* M^{'}(t_e - t, t_e), \\ \bar{\kappa}(t) = G^* \zeta(t), \\ \bar{\lambda}(t) = S M^{'}(t_e - t, t_e), \\ \zeta(t_e) = h \in \mathcal{X}, \end{cases} \quad (4)$$

where $t \in \mathbb{T}$. Then for all $M^{'}(t, t_e) \in L_2(\mathbb{T})$, the outputs $\bar{\kappa}(t)$ and $\bar{\lambda}(t)$ are uniquely defined for $t \in \mathbb{T}$. Moreover $K_M^*(t_e - t, t_e) = \bar{\kappa}(t)$ and $L_M^*(t_e - t, t_e) = \bar{\lambda}(t)$ for all $t \in \mathbb{T}$ whenever $M^{'}(t, t_e) = M^*(t, t_e)$.

Proof: The proof is a direct result of Lemma 5.2 in the appendix. ■

For the dynamical system $\Sigma^{'}(t_e)$ given by (3), we will be interested in finding the input $M'(t, t_e)$, $t \in \mathbb{T}$ that minimizes the criterion:

$$J^{'}(M^{'}) := \int_0^{t_e} ||\kappa(t)||_2^2 + ||\lambda(t)||_2^2 \, \mathrm{d}t. \quad (5)$$

That is, we aim to minimize $J^{'}(M^{'})$ over all $M^{'}(t, t_e)$ subject to the equations (3). Note that this is an linear quadratic regulator problem in the input $M^{'}(\cdot, t_e) \in L_2(\mathbb{T})$. The relation between the estimation problem and the latter optimization problem is given by the following theorem.

Theorem 3.3: Suppose $M^{'}(t, t_e) = M^*(t, t_e)$ for all $t \in \mathbb{T}$. Then

$$J(M) = J^{'}(M^{'}).$$

Proof: By theorem 3.1 it follows that if $M^{'}(t, t_e) = M^*(t, t_e)$ for all $t \in \mathbb{T}$ the following holds:

$$\begin{aligned} J(M) &= \int_0^{t_e} ||K_M(\tau, t_e)||_2^2 + ||L_M(\tau, t_e)||_2^2 \, \mathrm{d}\tau \\ &= \int_0^{t_e} ||\kappa(\tau)||_2^2 + ||\lambda(\tau)||_2^2 \, \mathrm{d}\tau = J^{'}(M^{'}). \end{aligned}$$

■

In particular, it follows that the minimization of the estimator cost J over all estimator integration kernels M is equivalent to the minimization of the control cost J' over all input variables M'. What is more, if M_{opt} is an optimal integration kernel that solves the H_2 estimation problem, then $M'_{\text{opt}} = M^*_{\text{opt}}$ is an optimal control that minimizes J subject to (3). Conversely, if M'_{opt} is an optimal input that minimizes J subject to (3), then $M_{\text{opt}} := (M'_{\text{opt}})^*$ is the integration kernel of the H_2 optimal estimator.

From the theorem above it therefore follows that the estimator design problem to estimate $z(t_e)$ is equivalent to an linear quadratic regulator design problem for the artificial system $\Sigma'(t_e)$ where the optimal input $M^*(t,t_e)$ is to be designed. Specifically, the integration kernel of the optimal estimator is designed according to:

$$M_{opt}(t,t_e) = (M^*_{opt}(t,t_e))^* = (M'_{opt})^*(t,t_e) \text{ for all } t \in \mathbb{T}, \tag{6}$$

where M'_{opt} is the optimal input that minimizes J' subject to (3).

In the next section we will derive the solution to the latter optimization problem and use its solution to derive the H_2 optimal estimator.

A. solution LQ-control problem

The solution of the LQ control problem for the infinite dimensional system (3) is widely known and can be found in, for instance [2]. In this section we present a derivation of the solution to this problem, in which we exploit the quadratic structure of the cost criterion with respect to the optimization variable $M'(t,t_e)$ by use of a completion of the square argument.

In theorem 3.4 it is shown that the cost criterion can be expressed as a combination of a constant term and a quadratic term that is parametrized in $M'(t,t_e)$.

Theorem 3.4: Let the operator $P(t,t_e) : \mathcal{X} \to \mathcal{X}$ with $(t,t_e) \in \mathbb{T}$ be defined as the unique self-adjoint solution to the Ricatti differential equation:

$$-\frac{\mathrm{d}\langle \xi_n, P\xi_m\rangle}{\mathrm{d}t} = \langle \xi_n, PA^*\xi_m\rangle + \langle PA^*\xi_n, \xi_m\rangle + \langle G^*\xi_n, G^*\xi_m\rangle - \left\langle S^{-1}CP\xi_m, S^{-1}CP\xi_n\right\rangle,$$

with end-point condition at $t = t_e$ given by $P(t_e,t_e) = P_e$ where P_e is self-adjoint and positive semi-definite. Moreover, assume that $\{\xi, \kappa, \lambda_{M'}\}$ satisfy (3) for all $t \in \mathbb{T}$. Then

$$J'(M') = \langle \xi(0), P(0)\xi(0)\rangle - \langle \xi(t_e), P(t_e)\xi(t_e)\rangle + \int_0^{t_e} \left|\left|SM'(\tau,t_e) - S^{-1}CP(\tau)\xi(\tau)\right|\right|_2^2 \mathrm{d}\tau.$$

Proof: We introduce the following identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(t), P(t)\xi(t)\rangle = 2\left\langle \xi, P(t)\dot{\xi}(t)\right\rangle + \left\langle \xi(t), \dot{P}(t)\xi(t)\right\rangle$$

which follows from differentiation of $\langle \xi(t), P(t)\xi(t)\rangle$ with respect to t and the fact that P is self-adjoint.

Substitute the system dynamics (3) in this equation and use that $P(t)$ solves the Ricatti differential equation for $\xi_m(t) = \xi_n(t) = \xi(t)$ to derive that, along solutions of the dynamical system, the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(t), P(t)\xi(t)\rangle = \left\langle S^{-1}CP(t)\xi(t), S^{-1}CP(t)\xi(t)\right\rangle + 2\langle P(t)\xi(t), A^*\xi(t)\rangle - 2\langle P(t)A^*\xi(t), \xi(t)\rangle - 2\left\langle P(t)\xi(t), C^*M'(t)\right\rangle - \langle G^*\xi(t), G^*\xi(t)\rangle \tag{7}$$

Using of a ‘completing of the squares’ argument one can split off the expressions $||\kappa||_2^2$ and $||\lambda||_2^2$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(t), P(t)\xi(t)\rangle = -\langle G^*\xi(t), G^*\xi(t)\rangle - \left\langle SM'(t), SM'(t)\right\rangle + \left\langle SM'(t) - S^{-1}CP\xi(t), SM'(t) - S^{-1}CP(t)\xi(t)\right\rangle \tag{8}$$

Since we assume that ξ is a solution to the system given by (3) we have that $||\kappa(t)||_2^2 = \langle G^*\xi(t), G^*\xi(t)\rangle$ and $||\lambda(t)||_2^2 = \left\langle SM'(t), SM'(t)\right\rangle$. Rearrangement and integration of the left hand side as well as the right hand side of the latter from $t = 0$ to $t = t_e$ completes the proof. By equation (5) it follows that $J'(M')$ equals:

$$\int_0^{t_e} ||\kappa(\tau)||_2^2 + ||\lambda(\tau)||_2^2 \,\mathrm{d}\tau = \int_0^{t_e} -\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(t), P(t)\xi(t)\rangle(\tau) + \left|\left|SM'(\tau) - S^{-1}CP\xi(\tau)\right|\right|_2^2 \mathrm{d}\tau, \tag{9}$$

which shows the proof. ∎

The solution to the linear quadratic optimal control problem for can now be deduced from the quadratic structure of $J'(M')$. This is shown in the following theorem.

Theorem 3.5: The cost $J'(M')$ is minimized subject to the system equations (3) if and only if

$$M'(\tau) = M'_{\text{opt}}(\tau) := S^{-2}CP(\tau)\xi(\tau)$$

Proof: The term $\langle \xi(0), P(0)\xi(0)\rangle - \langle \xi(t_e), P(t_e)\xi(t_e)\rangle$ is independent of M'. Therefore $J(M')$ is minimized if and only if

$$\int_0^{t_e} \left|\left|SM'(\tau) - S^{-1}CP(\tau)\xi(\tau)\right|\right|_2^2 \mathrm{d}\tau$$

is minimized. The integrand is a norm and therefore non-negative. The integral is minimized if and only if

$$\left|\left|SM'(\tau) - S^{-1}CP(\tau)\xi(\tau)\right|\right|_2 = 0$$

for all $\tau \in \mathbb{T}$. From this it follows that $M'_{opt}(t) = S^{-2}CP(t)\xi(t)$ is the unique minimizer of $J'(M')$. ∎

From the structure of $M'_{opt}(t)$ it follows that the optimal input is given by a linear time dependent state feedback law. The optimally controlled closed loop system is found by substituting the optimal state feedback law in the system equations (3). The closed loop system dynamics of the optimally controlled system are given by:

$$\dot{\xi}(t) = (A^* - C^*S^{-2}CP(t,t_e))\xi(t) \tag{10}$$

$$\kappa(t) = G^*\xi(t) \tag{11}$$

Let us assume that $V(t,t_e) : \mathcal{X} \to \mathcal{X}$ be the mild evolution operator generated by the time dependent operator $A^* - C^*S^{-2}CP(t,t_e)$ defined for $t \in \mathbb{T}$. That is,

$\lim_{h\to 0} \frac{(V(t+h)-V(t))\zeta}{h} = (A^* - C^*S^{-2}CP(t))\zeta$ for any $\zeta \in \mathcal{X}$. The operator $(A^* - C^*S^{-2}CP(t,t_e))$ can be seen as perturbed version of A^* where the perturbation is time dependent. From theorem 3.2.5 in [2] it follows that $A^* - C^*S^{-2}CP(t,t_e)$ indeed is the generator of mild evolution operator $V(t,t_e)$, which is basically due to the fact that operator P is well behaved.

From the closed loop system it follows that the state trajectory of the optimally controlled system is given by:

$$\xi(t) = V(t,t_e)h$$

When this trajectory is substituted in to feedback form of $M_{opt}^{'}(t,t_e)$ it follows that $M_{opt}^{'}(t,t_e)$ also has a feed forward representation, which is given by $M_{opt}^{'}(t,t_e) = S^{-2}CPV(t,t_e)h$.

B. Solution of estimation problem by dualization

Now the solution of the optimal LQ problem with equivalent cost criterion is known this solution can be used to solve the optimal estimator design problem. Using the relation between the optimal input $M_{opt}^{'}(t,t_e)$ for the control problem and the integration kernel $M_{opt}(t,t_e)$ of the optimal estimator given by (6) it follows that convolution kernel of the optimal estimator is given by:

$$M_{opt}(t,t_e) = (M_{opt}^{'}(t,t_e))^* = HU(t,t_e)P(t,t_e)C^*S^{-2}, \tag{12}$$

where $U(t,t_e) : \mathcal{X} \to \mathcal{X}$ is the mild evolution operator with the infinitesimal generator $A - P(t,t_e)CS^{-2}C^*$. The convolution kernel of the optimal estimator can be substituted in the input/ouput map of the estimator given by equation (1). It follows that the optimal estimator has an input/output map that is given by:

$$\hat{z}(t_e) = \int_0^{t_e} HU(t,t_e)P(t,t_e)C^*S^{-2}y(\tau)\mathrm{d}\tau. \tag{13}$$

C. Realization of estimator

Given convolution kernel of the optimal estimator and the structure of causal input output map (1) of the estimator, the input output map of the optimal estimator is known.

The optimal estimator is realized by the following state space system:

Theorem 3.6:

$$\dot{\alpha} = (A - P(t,t_e)C^*S^{-2}C)\alpha + PC^*S^{-2}y, \tag{14}$$
$$\hat{z} = H\alpha, \tag{15}$$

With initial condition $\alpha = 0$.

Proof: Assume that $(A - P(t,t_e)C^*S^{-2}C)$ is the generator of $U(t,t_e)$. Then the unique solution of this system is given by:

$$\hat{z}(t_e) = \int_0^{t_e} HU(t,t_e)P(t,t_e)CS^{-2}y\mathrm{d}t$$

Now it follows from comparison of the latter with (13) that the realization given by (14) implements a system with an input/output map that is equal to $\int_0^{t_e} M_{opt}(t,t_e)y\mathrm{d}t$. ■

It follows from the structure of realization of the optimal estimator, equation (14) specifically, that state evolution of the optimal estimator does not depend on the operator H. Moreover the readout map of the optimal estimator is equal the to readout map of the system under study. Therefore generalization of the optimal estimator for one estimated output to an optimal estimator for system with a n-dimensional output is straight forward. One can solve the optimal estimation problem for each output individually and combine these estimators. This results into an estimator for n-dimensional with a readout map that equals the operator H of the original system Σ_p.

IV. Conclusions

In this paper it has been shown that the design of a optimal H_2 output estimator for distributed parameters systems with finite dimensional outputs can be done by optimization of the design criterion over the convolution kernel of the input output map of the estimator to be designed. It has been shown that the estimator design problem is dual to the linear quadratic optimal control problem of an artificial system. Due to the specific structure of the optimal control problem, this problem can be solved elegantly by use of a "'completion of the square"'-argument applied on the cost criterion that is used for optimization in the optimal control problem. It has been shown that the solution of the dual control problem can be used to calculate the convolution kernel of the input/output map of the optimal output estimator. The solution to the optimal estimator has been presented in terms of the its convolution kernel as well as its state space realization.

V. Appendix

Lemma 5.1: Suppose the operator A with domain $D(A)$ is the infinitesimal generator of the semigroup $T(t)$. Let the operator A^* be the adjoint operator of A and let for every t the operator $T^*(t)$ be the adjoint operator of $T(t)$. Then A^* is the infinitesimal generator of $T^*(t)$.

Proof: Given that A generates $T(t)$ and suppose $\bar{A}$ generates $T^*(t)$. Then it follows from the definitions of generator of the semigroup operator that:

$$Ax = \lim_{h\to 0} \frac{T(t+h)x - T(t)x}{h} \text{for :} x \in D(A)$$
$$\bar{A}x = \lim_{h\to 0} \frac{T(t+h)^*x - T^*(t)x}{h} \text{for :} x \in D(A^*)$$

Then from the definition of the adjoint operator of A it follows that

$$\langle Ax_1, x_2 \rangle = \langle x_1, A^*x_2 \rangle \quad \text{for all } x_1 \in D(A) \text{ and } x_2 \in D(A^*)$$

Then:

$$\begin{aligned}\langle Ax_1, x_2 \rangle &= \lim_{h\to 0} \frac{1}{h} \langle T(t+h)x_1 - T(t)x_1, x_2 \rangle \\ &= \lim_{h\to 0} \frac{1}{h} (\langle x_1, T^*(t+h)x_2 - T^*(t)x_2 \rangle) \\ &= \langle x_1, \bar{A}x_2 \rangle .\end{aligned}$$

Which shows $A^* = \bar{A}$ for $x \in D(A) \cap D(A^*)$ ■

Lemma 5.2 (Time reversal): Let $x(t)$ satisfy

$$\dot{x}(t) = Ax(t) + f(t) \quad \text{with: } x(0) = x_0$$

for $0 \leq t \leq t_e$ then $\xi(t) = x(t_e - t)$ satisfies:

$$\dot{\xi}(t) = -A\xi(t) - \phi(t) \quad \text{with: } \xi(t_e) = x_0 \quad \text{and } \phi(t) = f(t_e - t)$$

Proof: Given $\xi(t) = x(t_e - t)$, it follows that

$$\dot{\xi}(t) = -\dot{x}(t_e - t).$$

From this it follows for $\xi(t)$ the following hold:

$$\begin{aligned}\dot{\xi}(t) &= -Ax(t_e - t) - f(t_e - t) \\ &= -A\xi(t) - \phi(t)\end{aligned}$$

which shows the proof. ■

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