Existence of Strict Optimal Controls for Long-term Average Stochastic Control Problems

Francois Dufour and Richard H. Stockbridge

Abstract— **Convexity conditions are identified under which optimal controls in the class of strict controls exist for a large class of stochastic processes under a long-term average criterion in the presence of hard and/or soft constraints. The result adapts a similar result obtained by Haussmann and Lepeltier (1990) for a controlled diffusion under a mixed optimal-stopping/finite-horizon/first-exit criterion. The approach taken in this paper is to utilize an equivalent linear programming formulation of the control problem. These results apply to controlled processes such as diffusions, Markov chains, simple Markov jump processes, diffusions with jumps, regime-switching diffusions and solutions to Lévy stochastic differential equations.**

I. Introduction

This paper considers a long-term average stochastic control problem with the goal of establishing sufficient conditions for the existence of strict (or pure) optimal controls. Let E denote the state space and U denote the space of available controls and let $c : E \times U \to \mathbb{R}$ be a function that captures the cost rate per unit of time. The goal is to minimize the expected long-term average cost

$$\overline{\lim_{t\to\infty}}\, t^{-1}\mathbb{E}\left[\int_0^t e^{-\alpha s} c(X(s), u(s))\, ds\right] \tag{1}$$

over processes satisfying some controlled dynamics. In (1), $X(s)$ denotes the state of the process and $u(s)$ is the control applied at time s. We assume that X has controlled generator A, which means that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s), u(s))\, ds, \quad t \geq 0, \tag{2}$$

is a martingale for every function f in the domain of A. An optimal solution of the stochastic control problem consists of an admissible control process u^* and the corresponding state process X^* satisfying (2) which minimizes (1).

In many situations the controls must be chosen so that the state process X and control process u satisfy some budget or resource constraints that may be either hard (a.s.) or soft (in mean). Such constraints have the effect of reducing the number of actions available to the decision maker at any time. We merely mention the possibility of additional constraints at this point and delay their specification to the formal description of the problem in Section I-A.

F. Dufour is with the Institut de Mathématiques de Bordeaux, Université de Bordeaux, INRIA Bordeaux Sud Ouest, Team: CQFD, 351 Cours de Libération, 33405 Talence Cedex, France, Francois.Dufour@math.u-bordeaux1.fr

R.H. Stockbridge is with the Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201-0413, USA, stockbri@csd.uwm.edu

Often the characteristics of the problem are such that it is not evident any control u will achieve the infimal value of the cost (1). One helpful approach, therefore, is to relax the definition of the control processes in a manner that embeds the original problem in a related control problem such that the infimal costs of the two problems are equal and existence of an optimizing relaxed control can be proved. By considering $\mathscr{P}(U)$-valued processes Λ, one convexifies the space of admissible controls so that (under some conditions) the space is compact; here $\mathscr{P}(U)$ denotes the space of probability measures on U. The relaxed control problem is one of minimizing

$$\overline{\lim_{t\to\infty}}\, t^{-1}\mathbb{E}\left[\int_0^t \int_U e^{-\alpha s} c(X(s), u)\, \Lambda_s(du)\, ds\right] \tag{3}$$

subject to

$$f(X(t)) - f(X(0)) - \int_0^t \int_U Af(X(s), u)\, \Lambda_s(du)\, ds,\ t \geq 0, \tag{4}$$

being a martingale for every function $f \in \mathscr{D}(A)$. Under mild conditions, it then follows that an optimal solution (X^*, Λ^*) exists, in which Λ^* is a relaxed control that at every time instant randomly selects the control u according to its distribution.

The purpose of this paper is to provide conditions on the model which ensure the existence of an optimal strict control u^*. These results adapt to more general processes the conditions provided by Haussmann and Lepeltier [3] for diffusion processes on $\mathbb{R}^d$ for a mixed optimal-stopping/finite-horizon/first-exit problem. This paper announces the results for a long-term average control problems; the paper [2] provides full details for discounted, first passage, finite horizon and optimal stopping problems. Furthermore, we employ a very general model that includes controlled diffusions, Markov chains, Markov jump processes, diffusions with jumps and Lévy processes as special cases.

The paper is organized as follows. Section I-A establishes the formulation for the controlled dynamics in terms of a controlled martingale problem and specifies the additional constraints. The key to our approach is to analyze an equivalent linear programming (LP) formulation for the stochastic control problem; this reformulation is given in Section I-B. The equivalence between the stochastic and LP problem has essentially been established by Kurtz and Stockbridge [4] though Condition 1.1 of this paper is more general than the conditions in [4]. We therefore also indicate how to extend the equivalence. Section II then presents the convexity condition and proves the existence of an optimal strict control.

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We illustrate the result in Section III by examining control problems for Markov chains, simple Markov jump processes, jump-diffusion processes, regime-switching diffusions and solutions to Lévy stochastic differential equations.

A. Formulation

For a complete, separable, metric space S, we define $M(S)$ to be the space of Borel measurable functions on S, $B(S)$ to be the space of bounded, measurable functions on S, $C(S)$ to be the space of continuous functions on S, $\overline{C}(S)$ to be the space of bounded, continuous functions on S, $\widehat{C}(S)$ to be the space of continuous functions vanishing at ∞, $\mathscr{M}_F(S)$ to be the space of finite Borel measures on S, $\mathscr{M}_{\sigma F}(S)$ to be the space of σ-finite measures on S and $\mathscr{P}(S)$ to be the space of probability measures on S. $\mathscr{M}_F(S)$, $\mathscr{M}_{\sigma F}(S)$ and $\mathscr{P}(S)$ are topologized by weak convergence.

Dynamics. Let E and U be complete, separable metric spaces. Let A be a linear operator satisfing the following condition.

Condition 1.1: (i) $A : \mathscr{D} \subset \overline{C}(E) \to C(E \times U)$, $1 \in \mathscr{D}$, and $A1 = 0$.

(ii) There exist $\psi_A \in C(E \times U)$, $\psi_A \geq 1$, and constants a_f, $f \in \mathscr{D}$, such that

$$|Af(x,u)| \leq a_f \psi_A(x,u), \quad \forall (x,u) \in \mathscr{U}.$$

(iii) Defining $A_0 = \{(f, \psi_A^{-1} Af) : f \in \mathscr{D}\}$, A_0 is separable in the sense that there exists a countable collection $\{f_k : k \in \mathbb{N}\} \subset \mathscr{D}$ such that A_0 is contained in the bounded, pointwise closure of the linear span of $\{(f_k, A_0 f_k) = (f_k, \psi_A^{-1} A f_k) : k \in \mathbb{N}\}$.

(iv) For each $u \in U$, the operator A_u defined by $A_u f(x) = Af(x,u)$ is a pre-generator (see [5] for the definition and examples of pregenerators; we note that a generator of a Markov process is also a pregenerator).

(v) $\mathscr{D}$ is closed under multiplication and separates points.

The model dynamics are specified as pairs of processes (X, Λ) that are solutions to the controlled martingale problem for A; that is, there exists a probability space $(\Omega, \mathscr{F}, P)$ such that X is a stochastic process taking values in E, Λ is a $\mathscr{P}(U)$-valued process, there exists some filtration $\{\mathscr{F}_t\}$ with respect to which (X, Λ) is progressively measurable and (4) is an $\{\mathscr{F}_t\}$-martingale for every $f \in \mathscr{D}$.

We remark that the set of functions $\mathscr{D}$ does not have to contain *all* of the functions in the domain of the operator A. It simply needs to be sufficiently large so that the martingale property will be satisfied for all f in the domain of A.

Remark 1.2: Condition 1.1 is obtained from Condition 1.2 of [5] by specifying the operator $B \equiv 0$. The paper [5] provides a relaxed framework for stochastic processes having singular behavior (with respect to Lebesgue measure in time). This behavior is captured by the operator B so by taking $B \equiv 0$ in this paper, we are assuming that the evolution of the process X, the effect of the control decisions on X and the accrual of costs are absolutely continuous in time. The only discontinuous instantaneous behavior allowed in this paper is that for which the compensator is absolutely continuous in time (e.g. Markov jump processes).

The extension of the equivalence of the linear programs and stochastic control problems in [4] to include singular stochastic processes is non-trivial. The paper [5] establishes existence of a stationary solution to a singular controlled martingale problem (a key step in the proof of equivalence), but not the equivalence of the stochastic problem with the linear program.

Cost Criteria. The cost rate function $c : E \times U \to \overline{\mathbb{R}}$ is assumed time-homogeneous, lower semi-continuous and bounded below. Further assume

Condition 1.3: (i) for each $a > 0$, $\{(x,u) : c(x,u) \leq a\}$ is compact, and

(ii) there exist constants a, b and β, with $0 < \beta < 1$ such that

$$\psi_A(x,u) \leq a + b\, c(x,u)^{\beta}. \tag{5}$$

The long-term average criterion is given by (3). We seek a solution (X, Λ) of the controlled martingale problem for A with unspecified initial distribution, which allows optimization over stationary processes.

Constraints. Haussmann and Lepeltier [3] discuss how to express hard constraints as soft constraints by suitably allowing the function c to take value ∞. We therefore express all additional constraints as soft constraints.

Let $m, n \in \mathbb{N}$ be fixed. For $i \leq m$, let $F_i : E \times U \to \overline{\mathbb{R}}_+$ denote the running constraint rate and let $\lambda_i \in \mathbb{R}_+$ denote the constraint limits. We assume that each F_i is lower semi-continuous in (x,u). The additional budget constraints for the long-term average criterion are

$$\overline{\lim_{t \to \infty}}\, t^{-1} \mathbb{E}\left[\int_0^t \int_U F_i(X(s),u)\, \Lambda_s(du)\, ds\right] \leq \lambda_i, \quad i \in \{1, \ldots, m\}. \tag{6}$$

B. Equivalent Linear Programming Formulation

This section states the equivalent linear programming formulation; it is important to point out that Condition 1.1 is more general than the conditions on A assumed in [4].

Theorem 1.4: Suppose there exists a feasible pair (X, Λ) for which the long-term average cost (3) is finite. Then the long-term average stochastic control problem of minimizing (3) over solutions of the martingale problem for A satisfying the additional constraints (6) is equivalent to the linear program

$$\begin{cases} \text{Min.} & \int c(x,u)\, \mu_0(dx \times du) \\ \text{S.t.} & \int Af(x,u)\, \mu_0(dx \times du) = 0, \quad f \in \mathscr{D}, \\ & \int F_i(x,u)\, \mu_0(dx \times du) \leq \lambda_i, \ i \in \{1, \ldots, m\}, \\ & \mu_0 \in \mathscr{P}(E \times U). \end{cases} \tag{7}$$

Moreover, an optimal measure μ_0^* and an optimal stationary relaxed solution (X^*, Λ^*) exist with the property that $X^*(s)$ has distribution μ_E^* and $\Lambda_s^*(\cdot) = \eta^*(X^*(s), \cdot)$ for $s \geq 0$, where μ_E^* is the state marginal of μ_0^* and η^* is the regular conditional distribution of μ_0^* on U given x.

Remark 1.5: Observe that the stationarity of X^* implies that for each $h \in M(E \times U)$ that is bounded below,

$$\mathbb{E}\left[\int_U h(X^*(s),u)\,\eta^*(X^*(s),du)\right] = \int_{E\times U} h(x,u)\mu_0^*(dx\times du). \quad (8)$$

Proof: Proceed exactly as in the proof of Theorem 6.1 of [4] but invoke Theorem 3.4 of [5] in place of Theorem 2.2 of [4] to obtain the existence of the stationary relaxed solution $(X^*,\eta^*(X^*,\cdot))$ of the martingale problem for A satisfying (8). The feasibility of $(X^*,\eta^*(X^*,\cdot))$ relative to the additional constraints follows from the observation in Remark 1.5 and the feasibility of μ_0^* for (7). ■

II. Existence of a Strict Optimal Control

We now turn to the issue of establishing conditions under which the existence of an optimal control in the class of strict controls can be proven.
We first need an extension of Theorem A.9 by Haussmann and Lepeltier [3]. For $z,w \in \mathbb{R}^{n+1}$, the notation $z \geq w$ means $z_i \geq w_i$ for $i = 0,\ldots n$.

Lemma 2.1: Let $\psi : E\times U \mapsto \overline{\mathbb{R}}^{n+1}$ and $\phi : E\times U \to \mathbb{R}^{\mathbb{N}}$ be measurable functions with ψ bounded below, $\psi(x,\cdot)$ lower semi-continuous and $\phi(x,\cdot)$ continuous for each x. Assume also that ψ_{j_0} satisfies Condition 1.3(i) for some $0 \leq j_0 \leq n$. First define the set $k(x) = \{(z,u) \in \mathbb{R}^{n+1}\times U : z_i \geq \psi_i(x,u), i = 0,\ldots,n\}$ and then define

$$K(x) = \left\{(z,\phi(x,u)) \in \mathbb{R}^{n+1}\times\mathbb{R}^{\mathbb{N}} : (z,u) \in k(x)\right\}.$$

Let $h_1 : E \mapsto \mathbb{R}^{n+1}$ and $h_2 : E \mapsto \mathbb{R}^{\mathbb{N}}$ be measurable functions such that for all $x \in E$, $(h_1(x),h_2(x)) \in K(x)$. Then there exists a measurable function $\widehat{u} : E \mapsto U$ such that for all $x \in E$, $h_1(x) \geq \psi(x,\widehat{u}(x))$ and $h_2(x) = \phi(x,\widehat{u}(x))$.

Proof: With one small change, the proof follows exactly the arguments developed in Theorem A.9 of [3]. Their proof uses the fact that $U \subset \mathbb{R}^d$ for some integer d so that $U_n = \{u \in \mathbb{R}^d : ||u|| \leq n\}$ is compact. We replace this compact subset by $U_n = \{u \in U : \psi_{j_0}(x,u) \leq n \text{ for some } x \in E\}$ which is clearly compact by hypothesis. ■

Now to be precise, a strict feedback control process is defined to be (X,u) where X is an E-valued process and $u : E \mapsto U$ is a measurable function such that $(X,u(X))$ is a solution of the martingale problem for A; that is, $f(X(t)) - f(X(0)) - \int_0^t Af(X(s),u(X(s)))\,ds$ is a martingale for each $f \in \mathscr{D}$.

We now introduce the key convexity condition under which the existence of an optimal strict control can be selected. For each $x \in E$, define the sets

$$\kappa(x) = \{(z,u) \in \mathbb{R}^{m+1}\times U : z_0 \geq c(x,u), z_i \geq F_i(x,u), i = 1,\ldots,m\}$$

and

$$\mathscr{K}(x) = \left\{\left(z,(Af_k(x,u))_{k\in\mathbb{N}}\right) \in \mathbb{R}^{m+1}\times\mathbb{R}^{\mathbb{N}} : (z,u) \in \kappa(x)\right\}.$$

Observe that the z_0-coordinate gives the epi-graph of $c(x,\cdot)$ and, for $i \in \{1,\ldots,m\}$, the z_i-coordinate produces the epi-graph of $F_i(x,\cdot)$.

Condition 2.2: For each x, the set $\mathscr{K}(x)$ is closed and convex.

Theorem 2.3: Let A satisfy Condition 1.1, c satisfy Condition 1.3 and assume Condition 2.2. Assume there exists a feasible measure μ_0 for the linear program (7) for which $\int c\,d\mu_0 < \infty$. Then there exists an optimal strict feedback control u^*.

Proof: We refer the reader to the proof of Theorem 4.3 of [2]. ■

III. Examples of controlled processes

Condition 2.2 for the convexity of $\mathscr{K}(x)$ is expressed in part using the infinite collection $\{(Af_k(x,u))_{k\in\mathbb{N}}\}$ since this sequence suffices to characterize the solutions to the martingale problem for A. Verification of the condition can be obtained by the particulars of the models. This section analyzes several different models and determines sufficient conditions for Condition 2.2 to hold. We denote the set of symmetric, positive semi-definite $d\times d$ matrices by $\mathbb{S}^d$ and by $\mathbb{M}$ the set of rate matrices in which $q_{ij} \geq 0$ for all $j \neq i$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$.

Example 3.1: Diffusions
Let X be an $\mathbb{R}^d$-valued, almost surely continuous process satisfying the stochastic differential equation

$$X(t) = x + \int_0^t b(X(s),u(s))\,ds + \int_0^t \sigma(X(s),u(s))\,dW(s) \quad (9)$$

where W is a d'-dimensional standard Brownian motion, $\sigma(x,u)$ is a $(d\times d')$-dimensional matrix with $\sigma(x,u)\sigma(x,u)' = a(x,u)$, and $(a,b) : E\times U \to \mathbb{S}^d\times\mathbb{R}^d$ is continuous and such that there exist nonnegative constants k,β,γ,ν,p with $0 \leq \beta \leq 2$, $\nu \leq p$, $\gamma\bar{\beta} \leq p$ where $\bar{\beta} = 1\vee\beta$, for which

$$|a(x,u)| \leq k(1+|x|^\beta+|u|^\nu),$$
$$|b(x,u)| \leq K(1+|x|+|u|^\gamma).$$

The generator A of X is

$$Af(x,u) = \tfrac{1}{2}\sum_{i,j} a_{ij}(x,u)f_{x_ix_j}(x) + \sum_i b_i(x,u)f_{x_i}(x),$$

defined for $f \in C_b^2(\mathbb{R}^d)$, where $C_b^2(\mathbb{R}^d)$ denotes the set of bounded functions on $\mathbb{R}^d$ having two bounded derivatives. The time-homogeneous version of the convexity condition assumed by Haussmann and Lepeltier [3, p. 859] is

Condition 3.1: For each $x \in \mathbb{R}^d$, define the set $K(x) = \{(a(x,u),b(x,u),z) : u \in U, z \in \mathbb{R}^{m+1}, z_0 \geq c(x,u), z_i \geq F_i(x,u), i = 0,1,\ldots,m\}$. Assume $K(x)$ is closed and convex for almost all $x \in E$.

Observe that the closedness and convexity of $(a(x,u),b(x,u),z) \in K(x)$ implies closedness and convexity of $\{(z,(Af_k(x,u))_{k\in\mathbb{N}}) : z_i \geq F_i(x,u), i \in \{1,\ldots,m\}$ as well.

EXAMPLE 3.2: MARKOV CHAINS
Let $\mathscr{I}$ denote $\{1,\ldots,n\}$ for some $n<\infty$ and let $E=\{x_i : i\in\mathscr{I}\}$ denote the state space for a continuous-time controlled Markov chain X. Let U denote the space of controls and let $Q(u)=((q_{ij}(u)))_{i,j\in\mathscr{I}}$ be the control-dependent transition rate matrix for X. We remind the reader that $q_{ii}(u)=-\sum_{j\in\mathscr{I},j\neq i} q_{ij}(u)$. Define the generator A of X by

$$Af(x_i,u)=\sum_{j\in\mathscr{I}}(f(x_j)-f(x_i))\,q_{ij}(u)=\sum_{j\in\mathscr{I}}f(x_j)\,q_{ij}(u)$$

with $\mathscr{D}=B(E)$. In order to show that Condition 2.2 is satisfied in this model, it is sufficient to impose the following condition on the rate matrix $Q(u)$.

Condition 3.2: For each $x_l\in E$, define the set $k(x_l)=\{(z,u)\in\mathbb{R}^{m+1}\times U : z_0\geq c(x_l,u), z_i\geq F_i(x_l,u), i=1,\ldots,m\}$ and then define the set $K(x_l)=\{(z,Q(u))\in\mathbb{R}^{m+1}\times\mathbb{M} : (z,u)\in k(x_l)\}$. Assume $K(x_l)$ is closed and convex for every $x_l\in E$.

EXAMPLE 3.3: SIMPLE MARKOV JUMP PROCESSES
We use the description of simple Markov jump processes given by Ethier and Kurtz in Section IV.2 of [1] but formulated to include control inputs to the process. Let E and U be complete, separable metric spaces. Let $\widehat{\lambda}: E\times U\to\mathbb{R}_+$ be bounded. Define the generator A by

$$Af(x,u)=\widehat{\lambda}(x,u)\int_E[f(y)-f(x)]\,\mu(x,u,dy),\quad f\in B(E).$$

Observe that the dependence of A on the control variable u occurs in the product $\widehat{\lambda}(x,u)\mu(x,u,\cdot)$. Since $\widehat{\lambda}$ gives the rate at which jumps occur and μ provides the distribution of the new jump location y, this product can be interpreted to be the rate of change from x to y using control u, similarly to the meaning of $q_{ij}(u)$ in the previous example. It will therefore be sufficient to impose the following convexity condition on this product.

Condition 3.3: For each $x\in E$, define the set $k(x)=\{(z,u)\in\mathbb{R}^{m+1}\times U : z_0\geq c(x,u), z_i\geq F_i(x,u), i=1,\ldots,m\}$ and then define the set $K(x)=\{(z,\widehat{\lambda}(x,u)\mu(x,u,\cdot))\in\mathbb{R}^{m+1}\times\mathscr{M}_F(E) : (z,u)\in k(x)\}$. Assume $K(x)$ is closed and convex for every $x\in E$.

EXAMPLE 3.4: DIFFUSIONS WITH JUMPS
This example looks at a common, though simple, controlled jump diffusion process. We assume that X satisfies (9) between jumps and that jumps occur according to the jump model of the previous example. Thus the generator A is

$$\begin{aligned}Af(x.u)=\tfrac{1}{2}\sum_{i,j}a_{ij}(x,u)f_{x_ix_j}(x)+\sum_i b_i(x,u)f_{x_i}(x)\\ +\widehat{\lambda}(x,u)\int_{\mathbb{R}^d}[f(y)-f(x)]\,\mu(x,u,dy)\end{aligned}\qquad(10)$$

acting on functions $f\in C_b^2(\mathbb{R}^d)$. The admissible strict control processes consist of those mappings $u:\mathbb{R}^d\mapsto U$ such that there exists some filtration $\{\mathscr{F}_t\}$ and $\{\mathscr{F}_t\}$-adapted process X for which for each $f\in\mathscr{D}\subset C_b^2(\mathbb{R}^d)$,

$$f(X(t))-f(X(0))-\int_0^t Af(X(s),u(X(s)))\,ds$$

is a mean 0, $\{\mathscr{F}_t\}$-martingale. Relaxed solutions (X,Λ) of the controlled martingale problem replace the strict control u with an $\{\mathscr{F}_t\}$-adapted, $\mathscr{P}(U)$-valued process Λ.

The convexity condition which ensures existence of an optimal strict control for this stochastic control problem combines Conditions 3.1 and 3.3.

Condition 3.4: For each $x\in E$, define the set $k(x)=\{(z,u)\in\mathbb{R}^{m+1}\times U : z_0\geq c(x,u), z_i\geq F_i(x,u), i=1,\ldots,m\}$ and then define the set $K(x)=\{(z,a(x,u), b(x,u),\widehat{\lambda}(x,u)\mu(x,u,\cdot))\in\mathbb{R}^{m+1}\times\mathbb{S}^d\times\mathbb{R}^d\times\mathscr{M}_F(U) : (z,u)\in k(x)\}$. Assume $K(x)$ is closed and convex for every $x\in E$.

EXAMPLE 3.5: REGIME-SWITCHING DIFFUSION
A common model used in mathematical finance has the process X satisfying the stochastic differential equation

$$\begin{aligned}X(t)=x+\int_0^t b(X(s),Y(s),u(s))\,ds\\ +\int_0^t\sigma(X(s),Y(s),u(s))\,dW(s)\end{aligned}\qquad(11)$$

in which Y is a continuous-time finite-state Markov chain, independent of the Brownian motion W, the process X and the control input u. X might, for example, represent the wealth invested in a portfolio of risky assets in which the price processes of the assets experience changes in the underlying dynamics based on whether there is a "bull" or "bear" market, with Y indicating the current type of market. Let $\mathscr{I}=\{1,\ldots,n\}$ denote the state space for Y and $Q=(q_{kl})_{k,l\in\mathscr{I}}$ its rate matrix. The generator A of the paired process (X,Y) is

$$\begin{aligned}Af(x,y_k,u)=\tfrac{1}{2}\sum_{i,j}a_{ij}(x,y_k,u)f_{x_i,x_i}(x,y_k)\\ +\sum_i b_i(x,y_k,u)f_{x_i}(x,y_k)\\ +\sum_{l\in\mathscr{I}}[f(x,y_l)-f(x,y_k)]q_{kl}.\end{aligned}$$

In this setting, the constraints (6) might represent requirements on the portfolios, such as a diversification requirement. Thus only those solutions (X,Y,u) satisfying (6) would be admissible.

Typically in finance, one is interested in maximizing a reward rather than minimizing a cost and often the goal is to be achieved within a specified time. To simplify the presentation of this example, however, we stay with our current framework by considering the problem of minimizing the long-term average criterion (3). The convexity condition guaranteeing the existence of an optimal strict control is:

Condition 3.5: For each $x\in E$, define the set $k(x)=\{(z,u)\in\mathbb{R}^{m+1}\times U : z_0\geq c(x,u), z_i\geq F_i(x,u), i=1,\ldots,m\}$ and then define the set $K(x)=\{(z,a(x,u), b(x,u))\in\mathbb{R}^{m+1}\times\mathbb{S}^d\times\mathbb{R}^d : (z,u)\in k(x)\}$. Assume $K(x)$ is closed and convex for every $x\in E$.

Observe that the convexity requirements for this model only involve the drift and diffusion coefficients since the control has no impact on the regime-switching mechanism. It would also be possible to include control dependence in Q as in Example 3.2 with Condition 3.5 so modified. This case would therefore allow the modelling of a large investor whose purchases and sales of the asset might trigger a response in the market and therefore a change in the regime change structure.

EXAMPLE 3.6: LÉVY STOCHASTIC DIFFERENTIAL EQUATIONS

We consider a more general Lévy process X specified as the solution to a Lévy stochastic differential equation. We adopt the formulation in Chapter 3 of Øksendal and Sulem [6] and refer the reader to [6] for the growth and Lipschitz conditions assumed on the coefficients b, σ and ϕ (see Theorem 1.19, p. 10). To simplify the notation, we assume X is one-dimensional so $E = \mathbb{R}$ and $U \subset \mathbb{R}$ is also one-dimensional; the more general case only requires minor modifications. Let W be a standard Brownian motion process, N be a marked jump process (as well as the induced measure on $\mathbb{R}_+ \times \mathbb{R}$), ν be the associated Lévy measure and $\tilde{N}$ be the compensated measure $\tilde{N}(dt,dy) = N(dt,dy) - \nu(dy)\,dt$. We assume X satisfies

$$\begin{aligned} X(t) = X(0) + \int_0^t b(X(s),u(s))\,ds + \int_0^t \sigma(X(s),u(s))\,dW(s) \\ + \int_0^t \int_{\mathbb{R}} \phi(X(s-),u(s-),y)\,\tilde{N}(ds \times dy). \end{aligned}$$

We note that Theorem 1.19 on page 10 of [6] implies $E[X(t)] < \infty$ for all $t \geq 0$ and hence one may take $R = \infty$ in the formulation of Chapter 3 of [6], in which case $\bar{N} = \tilde{N}$ (see p. 5 for the definition of $\bar{N}$). The generator A of X is therefore

$$\begin{aligned} Af(x,u) = b(x,u)f'(x) + \tfrac{1}{2}a(x,u)f''(x) \\ + \int_{\mathbb{R}} [f(x+\phi(x,u,y)) - f(x) \\ - f'(x)\cdot\phi(x,u,y)]\,\nu(dy) \end{aligned}$$

in which $a(x,u) = \sigma^2(x,u)$ and $f \in C_c^2(\mathbb{R})$, the space of twice-continuously differentiable functions having compact support.

To identify the convexity condition under which existence of an optimal strict control is proven, we need to address the term of Af involving the Lévy measure ν. For each $x \in E$ and $u \in U$, define a kernel Q on $\mathbb{R}$ given $\mathbb{R} \times U$ by

$$Q(x,u,G) = \int_{\mathbb{R}} I_G(\phi(x,u,y))\,\nu(dy).$$

For $f \in C_c^2(\mathbb{R})$, define the function $\bar{f}(x,z) = f(x+z) - f(x) - f'(x)\cdot z$. It then follows that

$$\begin{aligned} \int [f(x+\phi(x,u,y)) - f(x) - f'(x)\cdot\phi(x,u,y)]\,\nu(dy) \\ = \int \bar{f}(x,z)\,Q(x,u,dz) \end{aligned}$$

and this integral inherits any convexity in u of the kernel Q.

Condition 3.6: For each $x \in E$, define the set $k(x) = \{(z,u) \in \mathbb{R}^{m+1} \times U : z_0 \geq c(x,u), z_i \geq F_i(x,u), i = 1,\ldots,m\}$ and then define the set $K(x) = \{(z,a(x,u), b(x,u), Q(x,u,\cdot)) \in \mathbb{R}^{m+1} \times \mathbb{R}_+ \times \mathbb{R} \times \mathscr{M}_{\sigma F}(\mathbb{R}) : (z,u) \in k(x)\}$. Assume $K(x)$ is closed and convex for every $x \in E$.

IV. CONCLUDING REMARKS

This paper has extended the convexity conditions of Haussmann and Lepeltier [3] for diffusions in $\mathbb{R}^d$ to more general controlled processes evolving in a complete, separable metric space E. Furthermore, by taking advantage of an equivalent linear programming formulation of stochastic control problems, these results address the long-term average cost criterion. The processes are characterized by a countable collection of test functions $\{f_k : k \in \mathbb{N}\}$ so the convexity conditions are stated in terms of the sequence $\{(Af_k) : k \in \mathbb{N}\}$. For many processes of interest, however, simpler conditions are derived which imply the general convexity condition. The paper [2] addresses the existence of strict optimal controls for stochastic control problems having infinite-horizon discounted, finite-horizon, first passage and optimal stopping cost criteria as well as the long-term average cost criterion.

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