

# Convergence Rates of Markov Chain Approximation Methods for Controlled Regime-switching Diffusions with Stopping

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**Abstract**—This work summarizes our recent work on rates of convergence of Markov chain approximation methods for controlled switching diffusions, in which both continuous dynamics and discrete events coexist. The discrete events are formulated by continuous-time Markov chains to delineate random environment and other random factors that cannot be represented by diffusion processes. The cost function is over an infinite horizon with stopping times and without discount. The paper demonstrates how to use a probabilistic approach for studying rates of convergence. Although there have been significant developments in the literature using PDE (partial differential equation) methods to approximate controlled diffusions, there appear to be yet any PDE results to date for rates of convergence of numerical solutions for controlled switching diffusions to the best of our knowledge. Moreover, in the literature, to prove the convergence using Markov chain approximation methods for control problems involving cost functions with stopping (even for uncontrolled diffusion without switching), an assumption was used to avoid the so-called tangency problem. By modifying the value function, we demonstrate that the anticipated tangency problem will not arise in the sense of convergence in probability and convergence in  $L^1$ .

## I. INTRODUCTION

This paper summarizes our recent work [11], in which rates of convergence of Markov chain approximation methods for controlled switching diffusions were developed. Using an infinite horizon setup with stopping times and without discount, the regime switching is modeled by a continuous-time Markov chain. The motivation for using such models stems from the needs to deal with emerging applications in manufacturing systems, financial engineering, and wireless communications; see [2], [16], [17] and references therein. The reader is referred to [14], [18] for recent development on switching diffusion processes. The added regime-switching component provides more flexibility to formulate the real-world scenario, but causes much difficulties in analysis and numerical treatment of the associated control and optimization problems.

Numerical methods using Markov chain approximation for controlled diffusions have been developed and studied extensively in [6], [8]. Using probability methods, a systematic approach for proving the convergence of the algorithms was given although the associated convergence rates were

not discussed in the references above. Such a method is very powerful and can handle a wide variety of applications. Because of its importance, a great deal of attention has been devoted to the convergence rate problems; see [1], [4], [5], [9], [15] and references therein. As was demonstrated in these references, rates of convergence may be treated by considering convergence rates of finite difference scheme for Hamilton-Jacobi-Bellman (HJB) equations. Most of the work considered only rates of convergence without boundary conditions or with a finite time horizon. Using analytic (nonlinear partial differential equation (PDE)) techniques, the rates of convergence issues have been studied in great generality in the aforementioned references.

In [10], we presented some preliminary results on rates of convergence using Markov chain approximation for controlled diffusions, but the conditions used were strong. This paper presents a more comprehensive study, in which the conditions posed in [10] are replaced by milder conditions.

Compared with the diffusion models, there are added difficulties due to the Markovian coupling and boundary conditions. First, owing to the presence of the switching mechanism, we need to treat a number of cost and value functions. In lieu of treating a single HJB equation as in the case of controlled diffusions, we have to deal with a system of coupled partial differential equations. Although the convergence of Markov chain approximation to the controlled switching diffusion processes was obtained in [12], there appear to be no rate of convergence result for such controlled switched diffusions to date.

We note that numerics of stochastic controls of diffusions with a stopping time is difficult to study due to the added boundary conditions. As presented in [8, p. 278], because of the appearance of the boundary, a “tangency” problem may arise. Thus an added assumption is commonly used for the convergence of the algorithm to avoid the tangency problem. It is difficult to get convergence rates even for expectation of a stopping time (a stochastic control problem with a constant running cost). To overcome the difficulty due to the stopping time, in this work, we generalize the traditional view of cost and value functions by noting the explicit dependence on another parameter, namely the boundary. The essence is that in addition to the dependence on the state, we regard the value functions as functions of the boundary as well. As an immediate consequence, a nice property, namely, continuous dependence on the boundary of the value function follows. This continuity enables us to take a closer scrutiny of the value function and to get a much better understanding of the numerical approximation scheme using Markov chain

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approximation.

Although some of our working conditions such as one dimensional continuous state variable, non-degenerate diffusions, and control only on the drift may be seemingly strong, they are adequate as the starting point for using this new approach to treat the rates of convergence problems. The classical Markov chain approximation methods developed in [6], [7], [8] uses weak convergence methods for proving the convergence. In the weak convergence setup, various processes (the approximation sequences and the original stochastic processes) may live in different probability spaces. To study the convergence rates, comparisons of different processes are needed. To facilitate the study, we use a strong approximation technique and embed all the stochastic processes in the same space. Owing to the Markov chain scheme used, we have to face the problem of “adaptation” of continuous controls to discrete controls. The rate of convergence is ascertained by obtaining upper and lower bounds, respectively. To get lower bounds, adaptation from discrete control to continuous one is needed. Similar techniques were used in [9]. In contrast to the rates of convergence study to date, to obtain upper bounds, we adapt continuous controls to discrete controls by using certain properties of relaxed controls.

While the formulation and main results are presented, we refer the reader to [11] for verbatim proofs. The rest of the paper is arranged as follows. The precise formulation of the problem is presented next together with the notion of weak approximation. Section 3 proceeds with strong approximation using relaxed control representation. Section 4 obtains upper and lower bounds of the approximate value functions. Section 5 contains discussions on several issues. After some specific models are considered, another assumption is proposed leading to the verification of the condition on the cost function. Somewhat surprising, as a by-product, we prove that the anticipated tangency problem will not happen in the sense of convergence in probability and in  $L^1$ .

## II. PROBLEM FORMULATION

Suppose that  $\mathcal{M} = \{1, \dots, m_0\}$  is a finite set,  $\alpha_t$  is a continuous-time Markov chain with state space  $\mathcal{M}$  and generator  $Q = (q_{ij}) \in \mathbb{R}^{m_0 \times m_0}$  satisfying  $q_{ij} \geq 0$  for  $i \neq j$  and  $\sum_{j=1}^{m_0} q_{ij} = 0$  for each  $i \in \mathcal{M}$ . Consider a pair of random processes  $(X_t, \alpha_t)$  in the filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P, W, \cdot, \alpha)$ , which satisfies

$$\begin{cases} X_t = x + \int_0^t b_{\alpha_s}(X_s, u_s) ds + \int_0^t \sigma_{\alpha_s}(X_s) dW_s, \\ \alpha_t \text{ is a continuous-time Markov chain with } \alpha_0 = i, \end{cases} \quad (1)$$

where  $W_t$  is a standard Brownian motion independent of the Markov chain  $\alpha_t$ . For a given  $B > 0$ , define a stopping time as  $\tau_B^{x,i,u} = \inf\{t : X^{x,i,u}(t) \notin (-B, B)\}$ . Our objective is to choose the control  $u$ , so as to minimize the expected cost

function

$$\begin{cases} J_i^B(x, u) = E\left[\int_0^{\tau_B^{x,i,u}} f_{\alpha_s}(X_s, u_s) ds\right], \\ \forall (x, i) \in (-B, B) \times \mathcal{M}, \\ J_i^B(x, u) = 0, \forall x \notin (-B, B), i \in \mathcal{M}, \end{cases} \quad (2)$$

where for each  $i \in \mathcal{M}$ ,  $f_i(\cdot, \cdot)$  is an appropriate function representing the running cost function. For each  $(x, i) \in (-B, B) \times \mathcal{M}$ , the value of the optimization problem is given by

$$V_i^B(x) = \inf_{u \in \mathcal{U}} J_i^B(x, u), \quad (3)$$

where  $\mathcal{U}$  is the space of all  $\mathcal{F}_t$ -adapted controls taking values on a compact set  $U$ , which is referred to as an ordinary control space. Formally, the value functions satisfy a system of  $m_0 = |\mathcal{M}|$  Hamilton-Jacobi-Bellman (HJB) equations,

$$\begin{cases} \inf_{r \in U} \{L^r V_i^B(x) + f_i(x, r)\} = 0, & \forall x \in (-B, B), i \in \mathcal{M}, \\ V_i^B(x) = 0, & \forall x \notin (-B, B), i \in \mathcal{M}, \end{cases} \quad (4)$$

where the operator  $L$  with parameter  $r \in U$  on  $\{\varphi_i \in C^2(\mathbb{R}) : i \in \mathcal{M}\}$  is

$$L^r \varphi_i(x) = \frac{1}{2} \sigma_i^2(x) \frac{d^2 \varphi_i(x)}{dx^2} + b_i(x, r) \frac{d \varphi_i(x)}{dx} + \sum_{j \in \mathcal{M}} q_{ij} \varphi_j(x).$$

For the controlled switching diffusion, in [12], we constructed a locally consistent, discrete-time, controlled Markov chain (see [12] for a definition of local consistency; see also [8] for the diffusion counterpart). We briefly explain the idea and refer the aforementioned references for further reading. Let  $h > 0$  be a discretization parameter. Define  $S_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \dots\}$ . Let  $\{(\xi_n^h, \alpha_n^h), n < \infty\}$  be a controlled discrete-time Markov chain on a discrete state space  $S_h \times \mathcal{M}$  with transition probabilities from a state  $(x, i) \in \mathcal{M}$  to another state  $(y, j) \in \mathcal{M}$ , denoted by  $p^h((x, i), (y, j) | r)$  for  $r \in U$ . For notational simplicity, we denote

$$\begin{aligned} (-B, B)_h &= (-B, B) \cap S_h, \\ [-B, B]_h &= (-B, B)_h \cup \{B, -B\}. \end{aligned} \quad (5)$$

Then  $\bar{V}_i^{B,h}(x)$ , the discretization of  $V_i^B(x)$  with step size  $h > 0$ , is the solution of

$$\begin{aligned} \bar{V}_i^{B,h}(x) &= \inf_{r \in U} \left\{ \bar{p}_i^{h,+}(x, r) \bar{V}_i^{B,h}(x+h) \right. \\ &\quad \left. + \bar{p}_i^{h,-}(x, r) \bar{V}_i^{B,h}(x-h) + \sum_{j \neq i} \bar{p}_{ij}^h(x) \bar{V}_j^{B,h}(x) \right. \\ &\quad \left. + f_i(x, r) \Delta \bar{t}_i^h(x) \right\}, \forall x \notin (-B, B)_h, i \in \mathcal{M}, \\ \bar{V}_i^{B,h}(x) &= 0, \forall x \notin (-B, B)_h, i \in \mathcal{M}, \end{aligned} \quad (6)$$

where transition probabilities is given by, for some  $\gamma \in (2, 3]$

$$\begin{aligned}
 \bar{p}_i^{h,\pm}(x, r) &= p_{ii}^h(x)p_i^{h,\pm}(x, r) + O(h^\gamma) \\
 &= \frac{1}{2} \pm \frac{b_i(x, r)h}{2\sigma_i^2(x)} + \frac{q_{ii}h^2}{2\sigma_i^2(x)} + O(h^\gamma), \\
 \bar{p}_{ij}^h(x) &= p_{ij}^h(x) + O(h^{\gamma+1}) \\
 &= \frac{q_{ij}h^2}{\sigma_i^2(x)} + O(h^{\gamma+1}), \quad \forall j \neq i, \\
 \Delta \bar{t}_i^h(x) &= p_{ii}^h(x)\Delta t_i^h(x) + O(h^{\gamma+1}) \\
 &= \frac{h^2}{\sigma_i^2(x)} + O(h^{\gamma+1}),
 \end{aligned} \tag{7}$$

and

$$P(\alpha_{t+\Delta} = j | \alpha_t = i) = q_{ij}\Delta + O(\Delta^2), \quad \forall j \neq i. \tag{8}$$

To proceed, our problem is state as follows.

*Problem 2.1:* find the upper bound of  $\|\bar{V}^{B,h} - V^B\|_\infty$ , where  $V^B$  is the value function (3) with the underlying two-component controlled process (1), and  $\bar{V}^{B,h}$  is the piecewise constant interpolation of the value function corresponding to the Markov chain approximation with the two-component Markov chain generated by transition probability (7) satisfying the dynamic programming equation (6).

Note that the numerical approximation of the Markov chain approximation methods for regime-switching diffusions has been developed in [12]. However, the rates of convergence of the approximation has not been studied until our recent work [11], to the best of our knowledge.

*Remark 2.2:* Owing to the presence of the regime switching, instead of one cost function and one value function as in the setup of controlled diffusion processes, a collection of cost functions and value functions must be taken into consideration. The inclusion of the random switching process enables us to incorporate various applications involving random environment and other stochastic behaviors. On the other hand, the coupling due to the switching process causes much difficulty in the analysis as well as numerical approximation.

*Remark 2.3:* Note that in the traditional setup for controlled diffusions, the cost function and value function are written as  $J(x, u)$  and  $V(x)$ , respectively. Here, we modify the notion by introducing the dependence of the boundary of the state, namely,  $B$ . Such a notation will facilitate the analysis in use of boundary perturbations.

Throughout the paper, we use  $K$  to denote a generic positive constant, and  $K_t$  a generic positive constant depending on  $t$ . We use the following assumptions.

- (H1) Functions  $b(\cdot)$ ,  $\sigma(\cdot)$ , and  $f(\cdot)$  are bounded and Lipschitz continuous on  $G \triangleq (-B - \varepsilon, B + \varepsilon)$  for some  $\varepsilon > 0$ .
- (H2)  $\sigma_i(x) > 0$ ,  $\forall (i, x) \in \mathcal{M} \times G$ .
- (H3)  $Q$  is irreducible in the sense that the system of equations

$$\nu Q = 0, \quad \sum_{i \in \mathcal{M}} \nu_i = 1 \tag{9}$$

has a unique solution  $\nu = (\nu_1, \dots, \nu_{m_0}) \in \mathbb{R}^{1 \times m_0}$  satisfying  $\nu_i > 0$  for each  $i \in \mathcal{M}$ .

### III. STRONG MARKOV CHAIN APPROXIMATION UNDER RELAXED CONTROLS

To facilitate the analysis, we introduce the relaxed control representation; see [8]. We consider the same optimal control problem by extending real-valued control space  $\mathcal{U}$  to the measure-valued control space  $\Gamma$ . It allows us to work with convergence analysis in relaxed control space. Moreover, we construct strong approximation under relaxed controls in the same probability space of solutions of the controlled switching diffusions, which have approximately the same value functions as in the weak approximation given in the previous section, and which make comparisons of different functions possible within the same probability space in the subsequent sections.

Let  $\mathcal{B}(U \times [0, \infty))$  be the  $\sigma$ -algebra of Borel subsets of  $U \times [0, \infty)$ . An *admissible relaxed control* (or deterministic relaxed control)  $m(\cdot)$  is a measure on  $\mathcal{B}(U \times [0, \infty))$  such that  $m(U \times [0, t]) = t$  for each  $t \geq 0$ . Given a relaxed control  $m(\cdot)$ , there is an  $m_t(\cdot)$  such that  $m(dr dt) = m_t(dr)dt$ . In fact, we can define  $m_t(B) = \lim_{\delta \rightarrow 0} \frac{m(B \times [t-\delta, t])}{\delta}$  for  $B \in \mathcal{B}(U)$ . With the given probability space, we say that  $m(\cdot)$  is an admissible relaxed (stochastic) control for  $(W(\cdot), \alpha(\cdot))$  or  $(m(\cdot), W(\cdot), \alpha(\cdot))$  is admissible, if  $m(\cdot, \omega)$  is a deterministic relaxed control with probability one and if  $m(A \times [0, t])$  is  $\mathcal{F}_t$ -adapted for all  $A \in \mathcal{B}(U)$ . There is a derivative  $m_t(\cdot)$  such that  $m_t(\cdot)$  is  $\mathcal{F}_t$ -adapted for all  $A \in \mathcal{B}(U)$ .

Let  $\mathcal{P}(U)$  be the collection of probability measures on  $\mathcal{B}(U)$ . Then a relaxed control  $\{m_t : t > 0\}$  can be considered as an  $\mathcal{F}_t$ -adapted control taking values in  $\mathcal{P}(U)$ . Let  $\Gamma$  be the collection of all admissible relaxed controls. Let  $\mathcal{U}$  be the collection of admissible  $U$ -valued controls, which are sometimes referred to as ordinary controls in contrast to relaxed controls. Then  $\Gamma$  is a convex hull of  $\mathcal{U}$ . Define

$$\phi(\cdot, \mu) = \int_U \phi(\cdot, r)\mu(dr) \text{ for } \mu \in \mathcal{P}(U). \tag{10}$$

Let  $m(\cdot)$  be a relaxed control, which is an  $\mathcal{F}_t$ -adapted control taking values in  $\mathcal{P}(U)$ . The coupled random process  $(X_t^{x,i,m}, \alpha_t)$  with control  $m(\cdot)$  satisfies

$$X_t = x + \int_0^t b_{\alpha_s}(X_s, m_s)ds + \int_0^t \sigma_{\alpha_s}(X_s)dW_s, \tag{11}$$

where  $\alpha_t$  is a continuous-time Markov chain generated by  $Q$  with  $\alpha_0 = i$ .

For each  $i \in \mathcal{M}$ , the associated stopping time is given by  $\tau_B^{x,i,m} = \inf\{t : X^{x,i,m}(t) \notin (-B, B)\}$ , and for  $i \in \mathcal{M}$ , the objective function is given by

$$\begin{cases} J_i^B(x, m) = E[\int_0^{\tau_B^{x,i,m}} f_\alpha(X_s, m_s)ds], & \forall x \in (-B, B), \\ J_i^B(x, m) = 0, & \forall x \notin (-B, B). \end{cases} \tag{12}$$

and the value function is  $V_i^B(x) = \inf_{m \in \Gamma} J_i^B(x, m)$ , where  $\Gamma$  is the relaxed control space. Since  $\mathcal{P}(U)$  is a convex hull of  $\mathcal{U}$ , it is well known that the value function under relaxed control is also equal to the value function under ordinary control, i.e.,  $V_i^B = \bar{V}_i^B(x)$ .

To study the rate of the convergence of the Markov chain approximation, it is equivalent to consider the convergence rate of solutions of dynamic programming equation under relaxed control

$$\begin{aligned} \bar{V}_i^{B,h}(x) = & \inf_{\mu \in \mathcal{P}(U)} \left\{ \bar{p}_i^{h,+}(x, \mu) \bar{V}_i^{B,h}(x+h) \right. \\ & + \bar{p}_i^{h,-}(x, \mu) \bar{V}_i^{B,h}(x-h) + \sum_{j \neq i} \bar{p}_{ij}^h(x) \bar{V}_j^{B,h}(x) \\ & \left. + f_i(x, \mu) \Delta \bar{t}_i^h(x) \right\}, \quad \forall x \in (-B, B)_h, \quad i \in \mathcal{M}, \end{aligned} \quad (13)$$

where for given  $\gamma \in (2, 3]$  and  $(x, i) \in (-B, B)_h \times \mathcal{M}$ ,

$$\begin{aligned} \bar{p}_i^{h,\pm}(x, \mu) &= p_{ii}^h(x) p_i^{h,\pm}(x, \mu) + O(h^\gamma) \\ &= \frac{1}{2} \pm \frac{b_i(x, \mu)h}{2\sigma_i^2(x)} + \frac{q_{ii}h^2}{2\sigma_i^2(x)} + O(h^\gamma), \\ \bar{p}_{ij}^h(x) &= p_{ij}^h(x) + O(h^{\gamma+1}) \\ &= \frac{q_{ij}h^2}{\sigma_i^2(x)} + O(h^{\gamma+1}), \quad \forall j \neq i, \\ \Delta \bar{t}_i^h(x) &= p_{ii}^h(x) \Delta t_i^h(x) + O(h^{\gamma+1}) \\ &= \frac{h^2}{\sigma_i^2(x)} + O(h^{\gamma+1}). \end{aligned} \quad (14)$$

Using relaxed control setup, Problem 2.1 is equivalent to the following problem.

*Problem 3.1:* Find the upper bound of  $\|\bar{V}^{B,h} - V^B\|_\infty$ , where  $V^B$  is the above value function under relaxed control, and  $\bar{V}^{B,h}$  is the piecewise constant interpolation of the value function under relaxed control.

*Construction 3.2:* (Strong Markov Chain Approximation) Define a sequence of discrete random variables  $\{(x_n^h, \alpha_n^h, m_n^h)\}$  with  $\Delta \tau_n^h = \tau_{n+1}^h - \tau_n^h$  in the same probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P, W, \alpha)$ . For convenience, let us use  $(x^h(t), \alpha^h(t), m_t^h)$  to denote the piecewise constant interpolation of  $\{(x_n^h, \alpha_n^h, m_n^h)\}$  with  $\{\tau_n^h\}$  used, and let the space of such adapted controls be  $\Gamma^h$ . For a given  $\mathcal{F}_{\tau_n^h}$ -adapted control  $m^h$ ,  $\{(x_n^h, \alpha_n^h)\}$  is a Markov chain defined by

- 1) Set  $x_0^h = x, \alpha_0^h = i, m_0^h = m_0^h(i, x)$  in  $\mathcal{F}_0$ .
- 2) Given  $(x_n^h, \alpha_n^h, m_n^h) \in \mathcal{F}_{\tau_n^h}$ , let  $\alpha_{n+1}^h = \alpha(\tau_{n+1}^h)$ .
  - (a) if  $\alpha_{n+1}^h = \alpha_n^h$ , then  $\Delta \tau_n^h = \inf\{t : |b_{\alpha_n^h}(x_n^h, m_n^h)t + \sigma_{\alpha_n^h}(x_n^h)(W(\tau_n^h+t) - W(\tau_n^h))| \geq h\}$ ,  $x_{n+1}^h = x_n^h + b_{\alpha_n^h}(x_n^h, m_n^h)\Delta \tau_n^h + \sigma_{\alpha_n^h}(x_n^h)(W(\tau_{n+1}^h) - W(\tau_n^h))$ ,  $m_{n+1}^h = m_{n+1}^h(\alpha_{n+1}^h, x_{n+1}^h)$ ,
  - (b) else if  $\alpha_{n+1}^h \neq \alpha_n^h$ , then  $\tau_n^h = 0, x_{n+1}^h = x_n^h, m_{n+1}^h = m_{n+1}^h(\alpha_{n+1}^h, x_{n+1}^h)$ .

The above technique is in the spirit of Skorohod representation; see [3, Theorem A.1], also [13, Theorem 4.3]. We also note that similar processes are used for the same purpose in [9] and [15].

For each  $i \in \mathcal{M}$ , let the value function be

$$\begin{aligned} V_i^{B,h}(x) &= \inf_{m^h \in \Gamma^h} \sum_{k=0}^{N_B-1} f_{\alpha_n^h}(x_n^h, m_n^h) \Delta \tau_n^h \\ &\triangleq \inf_{m^h \in \Gamma^h} J_i^{B,h}(x, m^h), \end{aligned} \quad (15)$$

where  $N_B = \inf\{n : x_n^h \notin B^h\}$ . The corresponding dynamic

programming equation is: For each  $i \in \mathcal{M}$ ,

$$\begin{aligned} V_i^{B,h}(x) = & \inf_{\mu \in \mathcal{P}(U)} \left\{ \bar{p}_i^{h,+}(x, \mu) V_i^{B,h}(x+h) \right. \\ & + \bar{p}_i^{h,-}(x, \mu) V_i^{B,h}(x-h) \\ & \left. + \sum_{j \neq i} \bar{p}_{ij}^h(x) V_j^{B,h}(x) + f_i(x, \mu) \Delta \bar{t}_i^h(x) \right\}. \end{aligned} \quad (16)$$

*Theorem 3.3:* Let  $V_i^{B,h}(\cdot)$  and  $\bar{V}_i^{B,h}(\cdot)$  be the solutions of (13) and (16), respectively. Then

$$|V_i^{B,h}(x) - \bar{V}_i^{B,h}(x)| \leq Kh^{\gamma-2}, \quad \forall (i, x) \in \mathcal{M} \times (-B, B)_h,$$

for some constant  $K$ .

Thanks to Theorem 3.3 and triangle inequality, we have

$$\|\bar{V}^{B,h} - V^B\|_\infty \leq Kh^{\gamma-2} + \|V^{B,h} - V^B\|_\infty.$$

Thus, Problem 3.1 can be reduced to following problem:

*Problem 3.4:* Find upper bound of  $\|V^{B,h} - V^B\|_\infty$ , where  $V^B$  is the value function under relaxed control, and  $V^{B,h}$  is the value function corresponding to strong Markov chain approximation with the two-component Markov chain generated by Construction 3.2 satisfying the dynamic programming equation (16).

#### IV. CONVERGENCE RATE

Define

$$z^h(t) \triangleq \max\{j : \tau_j^h \leq t\}, \quad \tau_t^h \triangleq \tau_{z^h(t)}^h. \quad (17)$$

*Lemma 4.1:* Let the discrete random sequence  $\{(x_n^h, \alpha_n^h, m_n^h)\}$  be defined in the same probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P, W, \alpha)$  as Construction 3.2. Suppose that there exists an  $m \in \Gamma$  such that

$$x^h(t) = x + \int_0^{\tau_t^h} b_{\alpha_s^h}(x^h(s), m_s) ds + \int_0^{\tau_t^h} \sigma_{\alpha_s^h}(x^h(s)) dW_s,$$

and

$$X_t = x + \int_0^t b_{\alpha_s}(X_s, m_s) ds + \int_0^t \sigma_{\alpha_s}(X_s) dW_s.$$

Then, for  $\forall \theta \in (0, 1]$ ,  $E|x^h(t) - X_t|^\theta \leq K_t h^\theta$ .

By applying Lemma 4.1 with  $m_t \equiv m_t^h$ , it immediately follows that  $E|x^h(t) - X^{i,x,m^h}(t)|^\theta \leq K_t h^\theta, \forall \theta \in (0, 1]$ . To proceed, we assume another condition holds.

(H4) For any  $\mathcal{F}_t$ -adapted process  $m(\cdot)$ , there exists constant  $K$  such that  $|J_i^B(x, m) - J_i^B(y, m)| \leq K|x - y|$ .

*Theorem 4.2:* Assume (H4). For any  $m \in \Gamma$ , there exists  $m^h \in \Gamma^h$  such that  $|J_i^{B,h}(x, m^h) - J_i^B(x, m)| \leq Kh^{\frac{1}{2}}$ .

*Sketch of proof.* Without loss of generality, for each  $i \in \mathcal{M}$ , we assume  $f_i$  is a nonnegative function. Otherwise, set  $f_i = f_i^+ - f_i^-$ , consequently we can prove Theorem 4.2 for nonnegative functions  $f_i^+$  and  $f_i^-$  separately.

Given  $(\alpha_n^h, x_n^h) \in \mathcal{F}_{\tau_n^h}$ , and  $\alpha_n^h \neq \alpha_{n-1}^h$ , define  $\{m_n^h\} \in \Gamma^h$ , such that

$$m_n^h(A) = \frac{E_{\tau_n^h}[\int_{\tau_n^h}^{\tau_{n+1}^h} m_t(A) dt]}{E_{\tau_n^h}[\Delta \tau_n^h]}, \quad \forall A \in \mathcal{B}(U). \quad (18)$$

Note that  $m_n^h(\cdot)$  can be considered as an averaged occupation measure of  $m(\cdot)$  within the interval  $[\tau_n^h, \tau_{n+1}^h)$ . Therefore, we can use  $J_i^{B,h}(x, m)$  to denote  $J_i^{B,h}(x, m^h(\cdot))$ , where  $m^h(\cdot)$  is piecewise constant interpolation of  $\{m_n^h\}$  given by (18). Also, it can be verified for a function  $\varphi = b_i, f_i$

$$E_{\tau_n^h}[\varphi(x_n^h, m_n^h)\Delta\tau_n^h] = E_{\tau_n^h}\left[\int_{\tau_n^h}^{\tau_{n+1}^h}\varphi(x_n^h, m_t)dt\right]. \quad (19)$$

Using the constructed  $\{m_n^h\}$  with its property (19), we can write the strong approximation  $x^h(\cdot)$  as

$$\begin{aligned} x^h(t) &= x + \sum_{n=0}^{z^h(t)-1} [b_{\alpha_n^h}(x_n^h, m_n^h)\Delta\tau_n^h + \sigma_{\alpha_n^h}(x_n^h)\Delta W(\tau_n^h)] \\ &= x + \int_0^{\tau_t^h} b_{\alpha_s^h}(x_s^h, m_s)ds + \sigma_{\alpha_s^h}(x_s^h)dW_s, \end{aligned} \quad (20)$$

and by Lemma 4.1, we have

$$E|x^h(t) - X_t|^\theta \leq Kth^\theta, \quad \text{for } \forall\theta \in (0, 1]. \quad (21)$$

In what follows, we divide the work into two steps. Step 1 derives a lower bound on  $J_i^{B,h}(x, m^h) - J_i^B(x, m)$ , and Step 2 further obtains an upper bound.

*Step 1:* In this part, we will obtain a lower bound of  $J_i^{B,h}(x, m^h) - J_i^B(x, m)$ . Let  $B_h = B - h^\theta$  for some  $\theta \in (0, 1]$ , and  $e_i(x, m) = J_i^{B,h}(x, m^h) - J_i^{B_h}(x, m)$ . We can show  $\inf\{e_i(x, m)\} \geq -Kh^{1-\theta}$ . This leads to  $J_i^{B,h}(x, m^h) - J_i^B(x, m) \geq -Kh^{(1-\theta)\wedge\theta}$ . Taking  $\theta = \frac{1}{2}$  gives  $J_i^{B,h}(x, m^h) - J_i^B(x, m) \geq -Kh^{\frac{1}{2}}$ .

*Step 2:* In this part, we obtain an upper bound of  $J_i^{B,h}(x, m^h) - J_i^B(x, m)$ . Let  $B^h = B + h^\theta$  for some  $\theta \in (0, 1]$ , and  $e_i(x, m) = J_i^{B,h}(x, m^h) - J_i^{B^h}(x, m)$ . Define  $\tau = \tau_{B^h}^{i,x,m}$  and  $\tau^h = \inf\{t : x^h(t) \notin (-B, B)\}$ . Parallel to Step 1, we can obtain  $J_i^{B,h}(x, m^h) - J_i^B(x, m) \leq Kh^{\frac{1}{2}}$ .

The rate of convergence result is a consequence of the above theorem. The result is presented next.

**Theorem 4.3:** Assume (H4). The convergence rate is  $(\gamma - 2) \wedge \frac{1}{2}$ . That is,  $|\bar{V}_i^{B,h}(x) - V_i^B(x)| \leq Kh^{\frac{1}{2}\wedge(\gamma-2)}, \quad \forall(i, x) \in \mathcal{M} \times G$ .

**Remark 4.4:** The rate  $(\gamma - 2) \wedge \frac{1}{2}$  is specific for using Markov chain approximation approach. This is slightly different from the finite difference approach. As is known that the approach of Markov chain approximation method is useful in the actual computation since little prior information of the HJB equations need to be known. In fact, in the actual computation, in lieu of discretizing the PDEs directly, policy improvement methods are used; see [12] for the numerical examples.

## V. DISCUSSION

This section is divided into three parts. In the first part, we examine a couple of specific cases for rates of convergence. In the second part, we propose another assumption (H5), which is a PDE problem leading to the verification of (H4). In the last part, we consider the tangency problem in numerical approximation of stopping time problems.

### A. Remarks on Convergence Rates for Special Cases

In this section, we consider a couple of special cases and discuss related convergence rates. In the first case, the drift  $b$  is independent of control. Such systems may arise in certain financial engineering problems. For example, in [16], stock liquidation problems for regime-switching diffusion models are considered, where neither the drift nor the diffusion coefficients depend on control. The objective is to choose the stopping time so as to take profit or cut loss. When the underlying Markov chain has more than two states, no closed-form solution has been found. Thus numerical approximation is a natural choice.

**Theorem 5.1:** Suppose  $b_i(x, r) = b_i(x)$ . Then, the convergence rate is  $(\gamma - 2) \wedge \frac{1}{2}$ . That is,

$$|\bar{V}_i^{B,h}(x) - V_i^B(x)| \leq Kh^{\frac{1}{2}\wedge(\gamma-2)}, \quad \forall(i, x) \in \mathcal{M} \times G.$$

In the second case,  $b$  and  $\sigma$  are independent  $x$ . We present two motivations. In the first one, consider a regime-switching diffusion model that is linear in the continuous state variable and that the diffusion coefficient is independent of control. Taking a logarithm transformation, we obtain an equivalent model in which the drift and diffusion coefficients are free of  $x$  dependence. The second motivation stems from a controlled Markov chain model that is perturbed by an additional white noise. In both cases, the following result holds.

**Theorem 5.2:** Suppose  $b_i(x, r) = b_i(r)$ ,  $\sigma_i(x) = \sigma_i$ . Then, the convergence rate is  $(\gamma - 2) \wedge \frac{1}{2}$ . That is,

$$|\bar{V}_i^{B,h}(x) - V_i^B(x)| \leq Kh^{\frac{1}{2}\wedge(\gamma-2)}, \quad \forall(i, x) \in \mathcal{M} \times G.$$

### B. Remark on Condition (H4)

For simplicity, the discussion is confined to the case of controlled diffusions. The extension to regime-switching diffusion is straightforward. In this paper, we assumed (H4), which could be verified if the following condition (H5) holds. Let  $b(\cdot, r)$  and  $\sigma(\cdot)$  be Lipschitz continuous and  $\sigma(\cdot) > c > 0$  on  $G = (-1, 1)$ , and  $U$  be a compact set.

(H5) Consider

$$\begin{aligned} &\frac{1}{2}\frac{\partial^2\Phi}{\partial x^2}\sigma^2(x) + \frac{1}{2}\frac{\partial^2\Phi}{\partial y^2}\sigma^2(y) + \frac{\partial^2\Phi}{\partial x\partial y}\sigma(x)\sigma(y) \\ &+ \inf_{r \in U} \left\{ \frac{\partial\Phi}{\partial x}b(x, r) + \frac{\partial\Phi}{\partial y}b(y, r) + f(x, y, r) \right\} = 0, \end{aligned} \quad (22)$$

for  $\forall(x, y) \in G^2 = G \times G$  with boundary condition

$$\Phi(x, y) = Z_1(x)\mathbb{1}_{\{x \in G\}} + Z_2(y)\mathbb{1}_{\{y \in G\}}, \quad \forall(x, y) \in \partial G^2, \quad (23)$$

where  $\partial G^2$  denotes the boundary of  $G^2$ ,  $Z_1$  and  $Z_2$  are smooth functions on  $G$  with  $Z_1(\pm 1) = Z_2(\pm 1) = 0$ . Then,  $\exists!$  viscosity solution of  $C^{0,1}(G^2)$ .

The following theorem shows that under (H5), condition (H4) holds.

**Theorem 5.3:** Assume (H5). Let  $G = (-1, 1)$ , and  $m_t \in \mathcal{F}_t$  taking values in  $U$ . Consider the stochastic process

$$X_t^x = x + \int_0^t b(X_s^x, m_s)ds + \sigma(X_t^x)dW_t \quad (24)$$

and related objective function, for  $\tau^x = \inf\{t : X_t^x \notin G\}$   $J(x, m) = E[\int_0^{\tau^x} f(X_t^x, m_t) dt]$ . Then  $|J(x, m) - J(y, m)| \leq K|x - y|$ .

### C. Tangency Problem

For controlled diffusions without switching with a stopping time in the cost function, a well-known approach for proving the convergence of numerical scheme is based on weak convergence a properly designed Markov chain. One of the difficulties is so-called tangency problem as explained in [8, p. 278]. As a result, an assumption is added to complete the proof of convergence in [8], which can be described as follows. Using Markov chain approximation techniques, one constructs finite difference schemes for stochastic control problems in which the cost functions involve a first exit time from a bounded region. The true exit time of the diffusion process is replaced by an approximating sequence as well. The sequence of functions converges to a limit function, but the limit may not be the first exit time of the desired diffusion process and it could be tangent to the boundary of the first contact.

Consider  $X_t$  of (1). Let  $x^h(t)$  be a piecewise constant process in Construction 3.2 which approximate  $X_t$  in the same probability space. Let  $\tau$  and  $\tau^h$  be the first hitting time of  $X_t$  and  $x^h(t)$  to the boundary. The problem is: The sequence of  $\tau^h$  does not converge to  $\tau$ , even if  $x^h$  converges to  $X$ . Therefore, it is difficult to estimate the convergence rate  $|E\tau^h - E\tau|$ , which is the special case of value function (2) when the running cost  $f(\cdot)$  is a constant. The problem described is a more fundamental in the stochastic control literature as a continuity problem.

By our convergence rate result of the previous section, we conclude that the tangency problem will not occur in probability and in  $L^1$ . To see this, consider the control-independent dynamic system. Assumptions (H4) is satisfied in this situation (see previous section). Therefore, we can conclude  $|E\tau^h - E\tau| < Kh^{\frac{1}{2}}$  by Theorem 5.1. In fact, we can obtain stronger result if we carefully examine the proof of Theorem 4.3. In this case,  $f \equiv 1$ . To get the upper bound of  $(\tau^h - \tau)^+$ , we follow the procedure of Step 3 in Theorem 4.3, we can obtain the estimates for  $I_1$ ,  $I_3$ ,  $I_4$ , and  $I_5$  as previously. Note that  $I_2 = 0$ . We thus conclude  $E(\tau^h - \tau)^+ \leq Kh$ . Similarly, we obtain  $E(\tau^h - \tau)^- \leq Kh^{1/2}$  by Step 2. So, we have  $E|\tau^h - \tau| \leq Kh^{1/2}$ , which implies  $\tau^h \rightarrow \tau$  in  $L^1$ . Then the Tchebyshev's inequality implies that  $\tau^h \rightarrow \tau$  in probability as desired.

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