Accessibility and observability for a class of first-order PDE systems with boundary control and observation

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***Abstract*— A group-theoretical approach is used to tackle both the problem of local accessibility and observability along a trajectory for a class of first-order PDE systems with boundary control and observation. Based on an intrinsic formulation including boundary terms (local) criteria are derived in form of equivalence problems, where existence and/or non-existence of (pointwise) transformation groups and their invariants is related to (non-)observability and/or (non-)accessibility of PDE systems, respectively. Examples demonstrate the theory and results.**

***Index Terms*— differential geometry, infinite-dimensional systems, nonlinear systems, accessibility, observability.**

I. Introduction

Control systems described by partial differential equations, or PDE systems for short, can arise in several different disciplines if certain phenomena are modeled, for instance transportation delays, spatially-distributed parameters and the like. Examples can be found in various fields such as continuum mechanics, thermodynamics, economics and finance. A system-theoretical analysis of PDE systems is much more involved in comparison to systems described by ordinary differential equations, or ODE systems for short, because new effects appear which are unknown in the finite-dimensional case, see, e.g., [1], [2], [3], [4]. In this article we propose an approach by means of transformation groups and their invariants to tackle both the problem of observability and accessibility along a trajectory. The basic questions for the problems are different in principle, because invariants of groups are either known or unknown, but both cases can be treated with the same mathematical tool. In previous publications, see, e.g., [5], [6], [7] and references therein, a similar approach was successfully applied to dynamic systems, whose evolution along continuous or discrete time is governed by nonlinear (explicit and implicit) ordinary differential equations or difference equations. Especially, we highlight that the concepts and ideas can also be transferred to PDE systems, and observability/accessibility criteria can be derived in a successive manner. Of course, the check of these criteria becomes much more comprehensive in contrast to ODE systems.

In the literature observability and accessibility are well-studied for particular (nonlinear) PDE systems, see, e.g., [1], [2], [3], [8], [4] and references therein, just to name a few. To the knowledge of the authors there are not many results for broader classes of (nonlinear) PDE systems. Differential geometric methods have emerged as a useful tool to study PDE systems, see, e.g., [9], [10], [11], [12] and references therein, where all results rely on an intrinsic formulation of dynamic systems. However, mostly important aspects are missing, for instance, boundary conditions, boundary system inputs and outputs. In this contribution we first present a coordinate-independent formulation for a class of first-order PDE systems including boundary terms, namely we associate the system (equations) with a so-called generalized vector field with boundary conditions, or alternatively, with a pair of submanifolds, containing all possible solutions of the system. Then, in the main part of the work we discuss the observability and the accessibility along a trajectory in general. It is shown that based upon the intrinsic picture of PDE systems observability and accessibility criteria can be derived in the same manner by using a so-called infinitesimal criterion for invariance. It is worth mentioning that neither a systematic method for checking the criteria nor the topic of optimal sensor and actuator placement are addressed within this contribution. With respect to previous publications, see, e.g., [13], [14], [15] we provide additional results for PDE systems, especially, with boundary control and observation. It is worth mentioning that the formulation and the analysis approach can be adopted to consider (coupled, higher-order) PDE systems with various boundary conditions as well.

The article is organized as follows. In Section II the intrinsic formulation for dynamic systems under consideration is introduced. In Section III observability and accessibility are discussed and an approach is motivated by using transformation groups. Based on the geometric picture of systems in Section IV it is outlined how (local) criteria for (non-)observability and (non-)accessibility along a trajectory can be derived by studying group invariants and utilizing an infinitesimal criterion for invariance. To illustrate the theory nonlinear examples are arranged in Section V. Finally, the contribution finishes with some conclusions. The utilized notation and relevant mathematical preliminaries are introduced when necessary and/or can be found in the appendix. In the contribution we particularly apply well-known concepts from the field of differential geometry, where the interested reader is referred to, e.g., [9], [11], [16] for an introduction and much more details.

II. PDE Systems and Intrinsic Representation

In the article we confine ourselves to control systems Σ described (in coordinates) by a set of n first-order PDEs

$$\dot{x}^{\beta} = f^{\beta}(X^i, x^{\alpha}, x^{\alpha}_j)\,, \qquad \beta = 1, \ldots, n\,. \tag{1a}$$

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Unlike for ODE systems a success of the group approach for PDE systems depends here on a considerable extent to the accompanying boundary conditions

$$g^{\nu}(X^i, x^{\alpha}, u^{\kappa}) = 0\,, \qquad \nu = 1, \ldots, b \tag{1b}$$

and boundary output functions

$$y^{\zeta} = h^{\zeta}(X^i, x^{\alpha}, u^{\kappa})\,, \qquad \zeta = 1, \ldots, l\,. \tag{1c}$$

Here, $X = \left(X^i\right)$, $i = 1, ..., q$ are the (spatial) coordinates on the domain $\mathcal{D}$, which is supposed to be a compact manifold with global volume form and coherently orientable boundary $\partial\mathcal{D}$. Moreover, we assume that $X_{\partial} = \left(X^i\right)$, $i = 1, \ldots, q-1$ are the (spatial) coordinates on $\partial\mathcal{D}$. Basically, the bundles $\mathcal{X} \to \mathcal{D}$, $\mathcal{U} \to \partial\mathcal{D}$, $\mathcal{Y} \to \partial\mathcal{D}$ are applied with independent coordinates $X = \left(X^i\right)$ and dependent coordinates $x = (x^{\alpha})$, $\alpha = 1, ..., n$, $u = (u^{\kappa})$, $\kappa = 1, ..., m$ and $y = \left(y^{\zeta}\right)$, respectively. In this setting the time t is considered as an evolution parameter and not as a coordinate. In addition, the jet bundle $J^1(\mathcal{X}) \to \mathcal{X}$ is introduced to include derivative coordinates x^{α}_j, $j = 1, ..., q$ in the formalism. To consider boundary terms[1] the restricted bundle $\iota^*(\mathcal{X}) \to \partial\mathcal{D}$ is utilized with an embedding $\iota : \partial\mathcal{D} \to \mathcal{D}$, see, e.g., [11, Definition 1.4.8], where a section $\gamma \in \Gamma(\mathcal{D}, \mathcal{X})$ implies a section $\gamma_{\partial} = \gamma \circ \iota \in \Gamma\left(\partial\mathcal{D}, \iota^*\left(\mathcal{X}\right)\right)$, which is also denoted γ for short.

In order to avoid any mathematical irregularities from now on all manifolds are smooth manifolds and the functions f^{β}, g^{ν}, h^{ζ} depend smoothly on their arguments[2]. If the latter assumptions do not hold, the presented methods do not loose their validity; hence often many distinction of cases can be avoided. For brevity and readability, the Einstein summation convention is utilized throughout the article and the ranges of the indices as well as the arguments of functions are not always stated explicitly if they are clear from the context.

The intrinsic formulation of the system Σ is obtained by the assumption that the equations (1a) define (locally) a generalized vector field[3] $J^1\left(\mathcal{X}\right) \to \pi_0^{1,*}\left(\mathcal{V}\left(\mathcal{X}\right)\right)$ or, alternatively, a submanifold $\mathcal{S} \subset \pi_0^{1,*}\left(\mathcal{V}\left(\mathcal{X}\right)\right)$ with $f^{\beta} \in C^{\infty}\left(J^1\left(\mathcal{X}\right)\right)$, and the equations (1a), (1b) (locally) a fibred (regular) submanifold $\mathcal{S}_{\partial} \subset \iota^*(\mathcal{X}) \times_{\partial\mathcal{D}} \mathcal{U}$ with $g^{\nu} \in C^{\infty}\left(\iota^*(\mathcal{X}) \times_{\partial\mathcal{D}} \mathcal{U}\right)$. In addition, we suppose that locally the output equations (1c) define a fibred (regular) submanifold $\mathcal{N} \subset \iota^*(\mathcal{X}) \times_{\partial\mathcal{D}} \mathcal{U} \times_{\partial\mathcal{D}} \mathcal{Y}$ with $h^{\zeta} \in C^{\infty}\left(\iota^*(\mathcal{X}) \times_{\partial\mathcal{D}} \mathcal{U}\right)$.

Remark 1 The formalism includes boundary conditions and, thus, leads to the definition of a pair of submanifolds $(\mathcal{S}, \mathcal{S}_{\partial})$ in comparison to, e.g., [9], [10], where no boundary conditions are considered.

A solution of the system Σ on the time interval $[0, T] \subset \mathbb{R}$ is given by maps

$$(\gamma, \eta) : [0, T] \to \Gamma(\mathcal{D}, \mathcal{X}) \times \Gamma(\partial\mathcal{D}, \mathcal{U}) \tag{2}$$

such that $\gamma(t) : \mathcal{D} \to \mathcal{X}$, $\eta(t) : \partial\mathcal{D} \to \mathcal{U}$ are sections of the bundles $\mathcal{X} \to \mathcal{D}$, $\mathcal{U} \to \partial\mathcal{D}$ at time instance $t \in [0, T]$, respectively, and the equations

$$\begin{aligned} \frac{\partial}{\partial t}\gamma^{\beta}(t)(X) &= f^{\beta} \circ j^1(\gamma)(t)(X) \\ 0 &= g^{\nu} \circ (\gamma, \eta)(t)(X_{\partial}), \end{aligned}$$

are satisfied with the initial condition $\gamma(0)$ and the input $\eta(t)$. The output follows to

$$y^{\zeta} = \vartheta\left(t\right)\left(X_{\partial}\right) = h^{\zeta} \circ (\gamma, \eta)(t)(X_{\partial})$$

with $\vartheta = h \circ (\gamma, \eta) : [0, T] \to \Gamma\left(\partial\mathcal{D}, \mathcal{Y}\right)$ such that $\vartheta\left(t\right) \in \Gamma\left(\partial\mathcal{D}, \mathcal{Y}\right)$ is a section of the bundle $\mathcal{Y} \to \partial\mathcal{D}$.

For the analysis we suppose that solutions of the system Σ are well-posed in the sense of Hadamard, see, e.g., [17], i.e., there are appropriate function spaces for trajectories $(\gamma^{\alpha}) \in \mathcal{F}_t\left(\left[0, T\right], \mathcal{F}_{\mathcal{D}}\right)$, for inputs $(\mu^{\kappa}) \in \mathcal{F}_{\mathcal{U}}$ and for outputs $\left(h^{\zeta} \circ (\gamma, \eta)\right) \in \mathcal{F}_{\mathcal{Y}}$, where $\mathcal{F}_{\mathcal{D}}$ denotes a function space for functions defined on the domain $\mathcal{D}$ and $\mathcal{F}_t\left(\left[0, T\right], \mathcal{F}_{\mathcal{D}}\right)$ for the evolution in time. In addition, we suppose a locally solvable system, i.e., there are no hidden equations leading to the definition of a smaller submanifold $\bar{\mathcal{S}}_{\partial} \subseteq \mathcal{S}_{\partial}$.

III. Definitions, Motivation and Transformation Groups

For the sake of completeness let us recall well-known definitions of observability and accessibility along a fixed trajectory for PDE systems. For ODE systems it is well known that in general both properties depend on the system trajectory, see, e.g., [18], [7], and, thus, for PDE systems we associate the observability property and the accessibility property with a fixed trajectory. In particular, it is assumed that there exists a solution[4] (γ, η) of the system Σ on the time interval $[0, T]$ with initial condition $\gamma\left(0\right) = \gamma_0$ and end condition $\gamma\left(T\right) = \gamma_T$.

Let us first introduce the following definitions.

Definition 1 *An initial condition* $\bar{\gamma}_0$ *is said to be* indistinguishable along the trajectory (γ, η), *written* $\gamma_0 \sim \bar{\gamma}_0$, *if for all* $t \in [0, T]$ *the same output* $h \circ (\gamma, \eta)\left(t\right) = h \circ (\bar{\gamma}, \eta)\left(t\right)$ *(for the same input) follows.*

Definition 2 *An end condition* $\bar{\gamma}_T$ *is said to be* reachable along the trajectory (γ, η), *written* $\gamma_T \rightsquigarrow \bar{\gamma}_T$, *if there is an input* $\bar{\eta}$ *to steer the system* Σ *from* γ_0 *to* $\bar{\gamma}_T$, *i.e., there is a solution* $(\bar{\gamma}, \bar{\eta})$ *with* $\bar{\gamma}\left(0\right) = \gamma_0$ *and* $\bar{\gamma}\left(T\right) = \bar{\gamma}_T$.

By means of the previous definitions we able to define the properties observability and accessibility along a trajectory.

Definition 3 (Observability along a trajectory) *A system* Σ *is said to be (locally)* observable along the trajectory (γ, η), *if there is an open neighborhood* $\mathcal{V}_{\gamma_0}$ *of* γ_0 *such that for every open neighborhood* $\mathcal{U}_{\gamma_0}$ *of* γ_0 *contained in* $\mathcal{V}_{\gamma_0}$

[1] Note the abuse of notation since we also write $\partial\mathcal{D}$ when the boundary conditions are only defined on a part of $\partial\mathcal{D}$.

[2] Solutions do not have to be smooth.

[3] The concept of a pull-back bundle is applied here, see, e.g., [11, Definition 1.4.5], whereof a restricted bundle is a special case.

[4] Otherwise we have to study a two-point boundary problem for (nonlinear) systems Σ, which might not have a solution at all, to determine whether all subsequent investigations are reasonable or not.

and $\bar{\gamma}_0 \in \mathcal{U}_{\gamma_0}$ with $\bar{\gamma}_0 \sim \gamma_0$ implies $\bar{\gamma}_0 = \gamma_0$, otherwise the system is said to be (locally) non-observable.

Definition 4 (Accessibility along a trajectory) *A system Σ is said to be (locally)* accessible along the trajectory *(γ, η), if there is an open neighborhood $\mathcal{V}_{\gamma_T}$ of γ_T such that for every open neighborhood $\mathcal{U}_{\gamma_T}$ of γ_T contained in $\mathcal{V}_{\gamma_T}$ and $\bar{\gamma}_T \in \mathcal{U}_{\gamma_T}$ it follows $\gamma_T \rightsquigarrow \bar{\gamma}_T$, otherwise the system is said to be (locally)* non-accessible.

According to the previous definitions both properties are defined locally with respect to a fixed trajectory and serve as the basis for our subsequent investigations. Hence, it is worth noting that also different (stronger and weaker) notions of observability and accessibility along a trajectory can be used, see, e.g., [13], [19].

In [20] the concept of so-called similarity transformations is used to study properties of (nonlinear) ODE systems. Here, we follow a similar approach for PDE systems.

Remark 2 Let (γ, η) be a solution of the system Σ on the time interval $[0, T]$ and consider all possible (pointwise) variations of the form[5]

$$\begin{aligned} (X^i, \bar{\gamma}^\alpha(t)(X^i)) &= \varphi_\mathcal{X}\left(t, X^i, \gamma^\alpha(t)(X^i)\right) \\ (X^i, \bar{\eta}^\kappa(t)(X^i)) &= \varphi_\mathcal{U}\left(t, X^i, \eta^\kappa(t)(X^i)\right). \end{aligned} \tag{3}$$

such that the distorted solution $(\bar{\gamma}, \bar{\eta})$ is another solution of the system Σ. If there is one variation (3) such that the input and the output remain invariant,

$$\begin{aligned} \eta(t)(X_\partial) &= \bar{\eta}(t)(X_\partial) \\ h \circ (\gamma, \eta)(t)(X_\partial) &= h \circ (\bar{\gamma}, \bar{\eta})(t)(X_\partial)\,, \end{aligned} \tag{4}$$

the system Σ is non-observable because the initial conditions[6] $\gamma(0) = \gamma_0$, $\bar{\gamma}(0) = \bar{\gamma}_0$ are indistinguishable along the trajectory (γ, η). If there is an invariant[7] of the form

$$\mathcal{I}(\gamma(t)) = I_t \circ \gamma(t)(X) = I_t \circ \bar{\gamma}(t)(X) \tag{5}$$

or

$$\mathcal{I}(\gamma(t)) = \textstyle\int_\mathcal{D} I_t \circ \gamma(t)(X)\,\mathrm{dX} = \int_\mathcal{D} I_t \circ \bar{\gamma}(t)(X)\,\mathrm{dX} \tag{6}$$

with $I : [0, T] \to C^\infty(\mathcal{X})$, $I(t) = I_t \in C^\infty(\mathcal{X})$ and $\mathrm{dX} = \mathrm{d}X^1 \wedge \cdots \wedge \mathrm{d}X^q$, remaining invariant despite all possible variations, the system Σ is non-accessible because end conditions $\bar{\gamma}_T$ with $I_T \circ \gamma_T(X) \neq I_T \circ \gamma_T(X)$ or $\int_\mathcal{D} I_T \circ \gamma_T(X)\,\mathrm{dX} \neq \int_\mathcal{D} I_T \circ \gamma_T(X)\,\mathrm{dX}$ are non-reachable along the trajectory (γ, η). Since (5) implies an invariant (6), we consider only invariants of the form (6).

Let us consider the output map $\mathcal{O}$ and the accessibility map $\mathcal{A}_T$ for the observability problem and the accessibility problem, respectively, which are defined by[8]

$$\begin{aligned} \mathcal{O}: \quad \Gamma(\mathcal{D}, \mathcal{X}) \times \Gamma_{[0,T]}(\partial\mathcal{D}, \mathcal{U}) &\to \Gamma_{[0,T]}(\partial\mathcal{D}, \mathcal{Y})\,; \\ (\gamma_0, \eta) &\mapsto h \circ (\gamma, \eta) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \mathcal{A}_T: \quad \Gamma(\mathcal{D}, \mathcal{X}) \times \Gamma_{[0,T]}(\partial\mathcal{D}, \mathcal{U}) &\to \Gamma(\mathcal{D}, \mathcal{X})\,; \\ (\gamma_0, \eta) &\mapsto \gamma_T \end{aligned} \tag{8}$$

where (γ, η) solves the system Σ. The map $\mathcal{O}$ relates the initial condition and the input to the associated output, whereas the map $\mathcal{A}_T$ relates the initial condition and the input to the associated end condition at time instance T. The local observability problem is equivalent to the local injectivity of $\mathcal{O}_\eta = \mathcal{O}(\cdot, \eta) : \mathcal{V}_{\gamma_0} \subset \Gamma(\mathcal{D}, \mathcal{X}) \to \Gamma(\mathcal{D}, \mathcal{X})$ and in case of non-observability the set of indistinguishable initial conditions reads $\mathcal{IS}_{\gamma_0} = \{\bar{\gamma}_0 \in \mathcal{V}_{\gamma_0} : \mathcal{O}_\eta(\gamma_0) = \mathcal{O}_\eta(\bar{\gamma}_0)\}$. The local accessibility problem is equivalent to the local surjectivity of $\mathcal{A}_{T,\gamma_0} = \mathcal{A}_T(\gamma_0, \cdot) : \mathcal{V}_\eta \subset \Gamma_{[0,T]}(\partial\mathcal{D}, \mathcal{U}) \to \Gamma(\mathcal{D}, \mathcal{X})$ and in case of non-accessibility the set of reachable end conditions reads $\mathcal{RS}_{\gamma_0} = \{\bar{\gamma}_T \in \mathcal{V}_{\gamma_T} : \exists \bar{\eta} \in \mathcal{V}_\eta\, \mathcal{A}_{T,\gamma_0}(\bar{\eta}) = \bar{\gamma}_T\}$ and we may look for invariants $\mathcal{IV} \subset C^\infty(\mathcal{X})$ describing $\mathcal{RS}_{\gamma_0}$. The key idea is now to parametrize the sets $\mathcal{IS}_{\gamma_0}$ and $\mathcal{RS}_{\gamma_0}$ via 1-parameter (pointwise) Lie groups of transformation fulfilling the properties of admissible variations (3) for each group parameter and to consider group invariants of interest.

Remark 3 The approach for tackling both problems is similar, hence, the basic questions are different in principle. Besides the system equations all invariants of the groups are known for the observability problem, namely the system inputs and outputs, whereas the invariants of interest are unknown for the accessibility problem, namely $\mathcal{IV}$.

For the subsequent analysis, locally both maps $\mathcal{O}|_{\mathcal{V}_{\gamma_0} \times \mathcal{V}_\eta}$ and $\mathcal{A}_T|_{\mathcal{V}_{\gamma_0} \times \mathcal{V}_\eta}$ (with $\mathcal{V}_{\gamma_0} \times \mathcal{V}_\eta \subset \Gamma(\mathcal{D}, \mathcal{X}) \times \Gamma_{[0,T]}(\partial\mathcal{D}, \mathcal{U})$ and $(\gamma_0, \eta) \in \mathcal{V}_{\gamma_0} \times \mathcal{V}_\eta$) are supposed to be of class C^1 and that the linearized maps $D\mathcal{O}$ and $D\mathcal{A}_T$ of the system Σ are equivalent to the maps of the linearized system.

IV. Accessibility and Observability

Next, we pursue the ideas of Section III and apply the theory of symmetry groups (Lie groups of transformation), because variations of solutions (γ, η) are identified by 1-parameter symmetry groups, which generate a flow on $\mathcal{X}, \mathcal{U}$, respectively, and allow to transform solutions into solutions. To formulate the above considerations in the context of symmetry groups and to derive criteria for both properties, again we suppose to have a solution (γ, η) of the system Σ, and consider the set of local (pointwise) transformation groups[9]

$$\Phi_\varepsilon = (\Phi_\varepsilon^\mathcal{X}, \Phi_\varepsilon^\mathcal{U}) : [0, T] \to \mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{U}) \tag{9}$$

[5] Note that input functions are considered as functions of time, which are produced, for instance, by feedback control laws, and the properties observability and accessibility are invariant under feedback.

[6] The initial conditions are different because of the uniqueness of solutions.

[7] Also different types of invariants can be studied.

[8] The notation $\Gamma_{[0,T]}(\mathcal{N}, \mathcal{M})$ denotes a set of functions $[0, T] \to \Gamma(\mathcal{N}, \mathcal{M})$.

[9] $\mathcal{G}(\mathcal{M})$ denotes 1-parameter transformation groups on a manifold $\mathcal{M}$.

with

$$\begin{aligned} \Phi_{\varepsilon}^{\mathcal{X}}(t) &= \Phi_{t,\varepsilon}^{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \\ \Phi_{\varepsilon}^{\mathcal{U}}(t) &= \Phi_{t,\varepsilon}^{\mathcal{U}} : \mathcal{U} \to \mathcal{U} \end{aligned} \tag{10}$$

and the group parameter $\varepsilon \in \mathbb{R}$. The groups are supposed not to operate on X^i such that $\Phi_{t,\varepsilon}$ fulfills

$$\begin{aligned} \left(X^i, \bar{\gamma}^{\alpha}(t)\left(X^i\right)\right) &= \Phi_{t,\varepsilon}^{\mathcal{X}}\left(X^i, \gamma^{\alpha}(t)(X^i)\right) \\ \left(X^i, \bar{\eta}^{\kappa}(t)(X^i)\right) &= \Phi_{t,\varepsilon}^{\mathcal{U}}\left(X^i, \eta^{\kappa}(t)(X^i)\right), \end{aligned} \tag{11}$$

wherewith we have variations of the form (3) for fixed group parameter $\varepsilon \neq 0$. To ensure that Φ_{ε} transform solutions into solutions of the system Σ, the conditions

$$\begin{aligned} \frac{\partial}{\partial t}\gamma_{\varepsilon}^{\beta}(t)(X) &= f^{\beta} \circ j^1(\gamma_{\varepsilon})(t)(X) \\ 0 &= g^{\nu} \circ (\gamma_{\varepsilon}, \eta_{\varepsilon})(t)(X_{\partial}) \end{aligned} \tag{12}$$

must be satisfied with

$$\begin{aligned} \gamma_{\varepsilon}(t) &= \Phi_{t,\varepsilon}^{\mathcal{X}} \circ \gamma(t) \\ \eta_{\varepsilon}(t) &= \Phi_{t,\varepsilon}^{\mathcal{U}} \circ \eta(t). \end{aligned} \tag{13}$$

Remark 4 Following the ideas of Remark 2, if there is at least one group Φ_{ε}, where the input and the output remain invariant,

$$\begin{aligned} \eta(t)(X_{\partial}) &= \eta_{\varepsilon}(t)(X_{\partial}), \\ h \circ (\gamma, \eta)(t)(X_{\partial}) &= h \circ (\gamma_{\varepsilon}, \eta_{\varepsilon})(t)(X_{\partial}) \end{aligned} \tag{14}$$

the system Σ is non-observable because the initial conditions $\gamma(0) \neq \gamma_{\varepsilon}(0)$, $\varepsilon \neq 0$ cannot be distinguished by means of the system output. On the contrary, if there is at least one invariant

$$\int_{\mathcal{D}} I_t \circ \gamma(t)(X)\mathrm{dX} = \int_{\mathcal{D}} I_t \circ \gamma_{\varepsilon}(t)(X)\mathrm{dX} \tag{15}$$

for any group Φ_{ε}, the system Σ is non-accessible because there is no neighborhood $\mathcal{V}_{\gamma_T}$ of γ_T where any end condition $\bar{\gamma}_T \in \mathcal{V}_{\gamma_T}$ is reachable.

The derivation of (local) criteria mainly relies on the fact that we can use the infinitesimal generators of the transformation groups $\Phi_{t,\varepsilon}$,

$$\begin{aligned} v_{\mathcal{X}}(t) = v_{\mathcal{X},t} &= \left(\frac{\partial}{\partial\varepsilon}\Phi_{t,\varepsilon}^{\mathcal{X},\alpha}\right)\bigg|_{\varepsilon=0} = \partial_{\alpha} v_{\mathcal{X},t}^{\alpha}\partial_{\alpha} \\ v_{\mathcal{U}}(t) = v_{\mathcal{U},t} &= \left(\frac{\partial}{\partial\varepsilon}\Phi_{t,\varepsilon}^{\mathcal{U},\kappa}\right)\bigg|_{\varepsilon=0}\partial_{\kappa} = v_{\mathcal{U},t}^{\kappa}\partial_{\kappa}, \end{aligned} \tag{16}$$

which are (vertical) vector fields $v_{\mathcal{X},t} = v_{\mathcal{X},t}^{\alpha}\partial_{\alpha} : \mathcal{X} \to \mathcal{V}(\mathcal{X})$, $v_{\mathcal{U},t} = v_{\mathcal{U},t}^{\kappa}\partial_{\kappa} : \mathcal{U} \to \mathcal{V}(\mathcal{U})$ on the manifolds $\mathcal{X}$, $\mathcal{U}$, respectively. For a system Σ we also require the group actions on the jet manifold $J^1(\mathcal{X})$. Hence, instead of prolonging the group actions,

$$j^1\left(\Phi_{t,\varepsilon}^{\mathcal{X}}\right) = \left(\Phi_{t,\varepsilon}^{\mathcal{X},i}, \Phi_{t,\varepsilon}^{\mathcal{X},\alpha}, d_j\left(\Phi_{t,\varepsilon}^{\mathcal{X},\alpha}\right)\right) : J^1(\mathcal{X}) \to J^1(\mathcal{X}), \tag{17}$$

we prolong their infinitesimal generators,[10]

$$j^1(v_{\mathcal{X},t}) = v_{\mathcal{X},t}^{\alpha}\partial_{\alpha} + d_j\left(v_{\mathcal{X},t}^{\alpha}\right)\partial_{\alpha}^{j} : J^1(\mathcal{X}) \to \mathcal{T}\left(J^1(\mathcal{X})\right). \tag{18}$$

In addition, the group actions and the corresponding infinitesimal generators with respect to $\iota^*(\mathcal{X})$ can easily be obtained by restricting the group actions and the infinitesimal generators, respectively.

Then, we have that $\Phi_{t,\varepsilon}$ are symmetry groups iff the infinitesimal criterion of invariance

$$\frac{\partial}{\partial\varepsilon}\left(\frac{\partial}{\partial t}\gamma_{\varepsilon}^{\beta}(t)(X)\right)\bigg|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\left(f^{\beta} \circ j^1(\gamma_{\varepsilon})(t)(X)\right)\bigg|_{\varepsilon=0} \tag{19a}$$

and

$$0 = \frac{\partial}{\partial\varepsilon}\left(g^{\nu} \circ (\gamma_{\varepsilon}, \eta_{\varepsilon})(t)(X_{\partial})\right)\bigg|_{\varepsilon=0} \tag{19b}$$

resp.[11]

$$\begin{aligned} \dot{v}_{\mathcal{X},t}^{\beta} &= j^1(v_{\mathcal{X}})\left(f^{\beta}\right) \\ &= v_{\mathcal{X},t}^{\alpha}\partial_{\alpha}f^{\beta} + d_j\left(v_{\mathcal{X},t}^{\alpha}\right)\partial_{\alpha}^{j}f^{\beta} \end{aligned} \tag{20a}$$

and

$$\begin{aligned} 0 &= v_{\mathcal{X},t}(g^{\nu}) + v_{\mathcal{U},t}(g^{\nu}) \\ &= v_{\mathcal{X},t}^{\alpha}\partial_{\alpha}g^{\nu} + v_{\mathcal{U},t}^{\kappa}\partial_{\kappa}g^{\nu}. \end{aligned} \tag{20b}$$

is satisfied. Thus, we have conditions on the coefficients $v_{\mathcal{X}.t}^{\alpha}$, $v_{\mathcal{U},t}^{\kappa}$ of the vertical vector fields.

According to (4) for the observability there are the additional conditions

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}\left(\eta(t)(X_{\partial})\right)\bigg|_{\varepsilon=0} &= \frac{\partial}{\partial\varepsilon}\left(\eta_{\varepsilon}(t)(X_{\partial})\right)\bigg|_{\varepsilon=0} \\ \frac{\partial}{\partial\varepsilon}\left(h \circ (\gamma, \eta)(t)(X_{\partial})\right)\bigg|_{\varepsilon=0} &= \\ &\frac{\partial}{\partial\varepsilon}\left(h \circ (\gamma_{\varepsilon}, \eta_{\varepsilon})(t)(X_{\partial})\right)\bigg|_{\varepsilon=0} \end{aligned} \tag{21}$$

resp.

$$\begin{aligned} 0 &= v_{\mathcal{U},t}^{\kappa}, \\ 0 &= v_{\mathcal{X},t}\left(h^{\zeta}\right) + v_{\mathcal{U},t}\left(h^{\zeta}\right), \\ &= v_{\mathcal{X},t}^{\alpha}\partial_{\alpha}h^{\zeta} + v_{\mathcal{U},t}^{\kappa}\partial_{\kappa}h^{\zeta} \end{aligned} \tag{22}$$

Apparently, for the observability problem we have a vector field $v_{\mathcal{U},t} = v_{\mathcal{U},t}^{\kappa}\partial_{\kappa} = 0$.

Remark 5 By means of the output equations we have induced group actions on $\mathcal{Y}$ defined by

$$\Phi_{\varepsilon}^{\mathcal{Y}} = h \circ (\gamma_{\varepsilon}, \eta_{\varepsilon}) : [0, T] \to \mathcal{G}(\mathcal{Y}) \tag{23}$$

[10] The prolongation simplifies because of vertical vector fields, see, e.g., [11, p. 124ff].

[11] Actually, we should write $\iota^*\left(v_{\mathcal{X},t}(g^{\nu})\right) + \iota^*\left(v_{\mathcal{U},t}(g^{\nu})\right)$, hence, the pull-back is neglected since it should be clear from the context.

with $\Phi^{\mathcal{Y}}_{t,\varepsilon} = \Phi^{\mathcal{Y}}_{\varepsilon}(t) : \mathcal{Y} \to \mathcal{Y}$. Using the infinitesimal generators

$$v_{\mathcal{Y}}(t) = v_{\mathcal{Y},t} = \left(\frac{\partial}{\partial\varepsilon}\Phi^{\mathcal{Y},\zeta}_{t,\varepsilon}\right)\bigg|_{\varepsilon=0}\partial_\zeta = v^{\zeta}_{\mathcal{Y},t}\partial_\zeta \tag{24}$$

and $v_{\mathcal{U},t} = 0$ for the observability problem the case

$$v^{\zeta}_{\mathcal{Y},t} = v_{\mathcal{X},t}\left(h^{\zeta}\right) = v^{\alpha}_{\mathcal{X},t}\partial_\alpha h^{\zeta} = 0 \tag{25}$$

is considered.

With these insights at our disposal, we are able state a non-observability criterion.

Theorem 1 *If locally the conditions (20) and (22) allow a non-trivial vector field* $0 \neq v_{\mathcal{X},t} : \mathcal{X} \to \mathcal{V}(\mathcal{X})$ *along the trajectory* (γ, η)*, the system* Σ *is locally non-observable.*

Proof: A non-trivial vector field $v_{\mathcal{X},t}$ together with $v_{\mathcal{U},t} = 0$ induces a 1-parameter symmetry group $\Phi_{t,\varepsilon}$, which allows to derive another solution $(\bar{\gamma}, \eta)$ of the system Σ, having the same input and the same output. Hence, the different initial conditions $\gamma(0) \neq \bar{\gamma}(0)$ are indistinguishable. ■

Remark 6 It is worth noting that the approach can also be extended straightforward to investigate different types of system outputs. Moreover, if the observability conditions are evaluated along the trajectory, they coincide with the linearized system Σ_Δ (along the trajectory)

$$\begin{aligned}\Delta\dot{x}^{\beta} &= \left(\partial_\alpha f^{\beta}\circ j^1(\gamma)(t)(X)\right)\Delta x^{\alpha} \\ &\quad + \left(\partial^j_\alpha f^{\beta}\circ j^1(\gamma)(t)(X)\right)\Delta x^{\alpha}_j\end{aligned} \tag{26a}$$

with boundary conditions

$$\left(\partial_\alpha g^{\nu}\circ(\gamma,\eta)\right)\Delta x^{\alpha} + \left(\partial_\kappa g^{\nu}\circ(\gamma,\eta)\right)\Delta u^{\kappa} = 0 \tag{26b}$$

and outputs

$$\Delta y^{\zeta} = \left(\partial_\alpha h^{\zeta}\circ(\gamma,\eta)\right)\Delta x^{\alpha} + \left(\partial_\kappa h^{\zeta}\circ(\gamma,\eta)\right)\Delta u^{\kappa}\,, \tag{26c}$$

by identifying $v^{\alpha}_{\mathcal{X},t}$, $v^{\kappa}_{\mathcal{U},t}$, $v^{\zeta}_{\mathcal{Y},t}$ with Δx^{α}, Δu^{κ}, Δy^{ζ}, respectively, where the inputs and the outputs have to vanish for the considered time interval $[0, T]$.

For the accessibility problem we are interested in (local) invariants of the symmetry groups Φ_ε of the form

$$\mathcal{I}(\gamma_\varepsilon(t)) = \int_{\mathcal{D}}\left(I_t\circ\gamma_\varepsilon(t)(X)\right)\mathrm{dX} \tag{27}$$

satisfying

$$\mathcal{I}(\gamma_\varepsilon(t)) = \mathrm{const.}$$

or

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\mathcal{I}(\gamma_\varepsilon(t))\bigg|_{\varepsilon=0} &= \frac{\partial}{\partial\varepsilon}\mathcal{I}\left(\Phi^{\mathcal{X}}_{t,\varepsilon}(\gamma(t))\right)\bigg|_{\varepsilon=0} \\ &= \int_{\mathcal{D}}\gamma(t)^*\left(v_{\mathcal{X},t}\left(I_t\mathrm{dX}\right)\right) \\ &= 0\end{aligned} \tag{28}$$

if $v_{\mathcal{X},t}$ is an admissible vector field. By using the relation $v_{\mathcal{X},t}\left(I_t\mathrm{dX}\right) = \mathrm{d}\left(v_{\mathcal{X},t}\rfloor\left(I_t\mathrm{dX}\right)\right) + v_{\mathcal{X},t}\rfloor\mathrm{d}\left(I_t\mathrm{dX}\right)$ and Stokes' Theorem, see, e.g., [16], one can write the functional (28) in the form

$$\begin{aligned}&\underbrace{\int_{\mathcal{D}}\gamma(t)^*\left(v_{\mathcal{X},t}\rfloor\mathrm{d}I_t\wedge\mathrm{dX}\right)}_{=\mathcal{I}_1(\gamma(t))=0} \\ &\quad + \underbrace{\int_{\partial\mathcal{D}}\iota^*\left(\gamma(t)^*\left(v_{\mathcal{X},t}\rfloor\left(I_t\mathrm{dX}\right)\right)\right)}_{=\mathcal{I}_2(\gamma(t))=0} = 0\end{aligned} \tag{29}$$

with the embedding $\iota : \partial\mathcal{D} \to \mathcal{D}$ and where $\mathrm{d}I_t : \mathcal{X} \to \mathcal{T}^*(\mathcal{X})$ is the differential of I_t. The functional $\mathcal{I}_2(\gamma(t))$ vanishes since $v_{\mathcal{X},t}\rfloor\left(I_t\mathrm{dX}\right) = 0$. Because of $\mathcal{I}_1(\gamma(t)) = 0$ we have

$$\mathcal{I}_1(\gamma(t)) = \int_{\mathcal{D}}\gamma(t)^*\left(v^{\alpha}_{\mathcal{X},t}\omega^{\mathcal{X}}_{t,\alpha}\mathrm{dX}\right) = 0 \tag{30}$$

with

$$\begin{aligned}\omega_t = \mathrm{d}I_t &= \left(\partial_i I_t\right)\mathrm{d}X^i + \left(\partial_\alpha I_t\right)\mathrm{d}x^{\alpha} \\ &= \omega^{\mathcal{X}}_{t,i}\mathrm{d}X^i + \omega^{\mathcal{X}}_{t,\alpha}\mathrm{d}x^{\alpha}\end{aligned}$$

and $\omega^{\mathcal{X}}_{t,i} = \omega^{\mathcal{X}}_{i}(t)$, $\omega^{\mathcal{X}}_{t,\alpha} = \omega^{\mathcal{X}}_{\alpha}(t)$.

To incorporate the information that $v_{\mathcal{X},t}$ is the generator of a symmetry group we differentiate (30) with respect to the evolution parameter t,

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\mathcal{D}}\gamma(t)^*\left(v^{\alpha}_{\mathcal{X},t}\omega^{\mathcal{X}}_{t,\alpha}\mathrm{dX}\right) \\ &\quad = \int_{\mathcal{D}}\gamma(t)^*\left(\left(\dot{v}^{\alpha}_{\mathcal{X},t}\omega^{\mathcal{X}}_{t,\alpha} + v^{\alpha}_{\mathcal{X},t}\dot{\omega}^{\mathcal{X}}_{t,\alpha}\right)\mathrm{dX}\right) = 0\,,\end{aligned} \tag{31}$$

and use the infinitesimal criterion for invariance to substitute for $\dot{v}^{\alpha}_{\mathcal{X},t}$,

$$\begin{aligned}&\int_{\mathcal{D}}\gamma(t)^*\left(\dot{v}^{\beta}_{\mathcal{X},t}\omega^{\mathcal{X}}_{t,\beta}\mathrm{dX}_{\mathcal{D}}\right) = \\ &= \int_{\mathcal{D}}j^1(\gamma(t))^*\left(\left(v^{\alpha}_{\mathcal{X},t}\partial_\alpha f^{\beta} + d_j(v^{\alpha}_{\mathcal{X},t})\partial^j_\alpha f^{\beta}\right)\omega^{\mathcal{X}}_{t,\beta}\mathrm{dX}\right)\,.\end{aligned} \tag{32}$$

For PDE systems the latter functional is not appropriate in its actual form because it depends on the derivatives $d_j\left(v^{\alpha}_{\mathcal{X},t}\right)$ of components of the vector field $v_{\mathcal{X},t}$, and, thus, can contribute to the so far vanishing boundary integral $\mathcal{I}_2(\gamma(t))$. This is a well known problem in the calculus of variations, which alternatively could be solved by using Lepagian forms, see, e.g., [9]. In order to obtain a form depending only on the components of the vector field $v_{\mathcal{X},t}$, but not on their derivatives, we apply the identity

$$\begin{aligned}&\int_{\mathcal{D}}j^1(\gamma(t))^*\left(d_j(v^{\alpha}_{\mathcal{X},t})\partial^j_\alpha f^{\beta}\omega^{\mathcal{X}}_{t,\beta}\mathrm{dX}\right) = \\ &\quad = -\int_{\mathcal{D}}j^1(\gamma(t))^*\left(v^{\alpha}_{\mathcal{X},t}d_j\left(\partial^j_\alpha f^{\beta}\omega^{\mathcal{X}}_{t,\beta}\right)\mathrm{dX}\right) \\ &\qquad + \int_{\partial\mathcal{D}}(\gamma(t)\circ\iota)^*\left(v_{\mathcal{X},t}\rfloor\bar{\omega}^j_t\wedge\partial_j\rfloor\mathrm{dX}\right)\end{aligned} \tag{33}$$

with $\bar{\omega}_t^j = \partial_\alpha^j f^\beta \omega_{t,\beta}^{\mathcal{X}} \mathrm{d}x^\alpha$, where Stokes' Theorem is used. We get the following terms on the domain

$$\begin{aligned} &\int_{\mathcal{D}} j^1(\gamma(t))^* \left(\left(\dot{\omega}_{t,\beta}^{\mathcal{X}} + \omega_{t,\alpha}^{\mathcal{X}} \partial_\beta f^\alpha \right) v_{\mathcal{X},t}^\beta \mathrm{dX} \right) \\ &\quad - \int_{\mathcal{D}} j^1(\gamma(t))^* \left(d_j \left(\partial_\beta^j f^\alpha \omega_{t,\alpha}^{\mathcal{X}} \right) v_{\mathcal{X},t}^\beta \mathrm{dX} \right) \end{aligned} \tag{34}$$

and on the boundary

$$\int_{\partial\mathcal{D}} (\gamma(t) \circ \iota)^* \left(\partial_\beta^j f^\alpha \omega_{t,\alpha}^{\mathcal{X}} v_{\mathcal{X},t}^\beta \partial_j \rfloor \mathrm{dX} \right) = 0\,. \tag{35}$$

The equations (34) must now hold along a given trajectory (γ, η) and independently from the choice of $v_{\mathcal{X},t}$ since there are no further restrictions on the domain. Thus, for equations (34) to hold we require that the corresponding braced term vanishes along a trajectory (γ, η) and so the following conditions for an invariant can be extracted,

$$v_{\mathcal{X},t}^\beta : \; \dot{\omega}_{t,\beta}^{\mathcal{X}} = -\omega_{t,\alpha}^{\mathcal{X}} \partial_\beta f^\alpha + d_j \left(\partial_\beta^j f^\alpha \omega_{t,\alpha}^{\mathcal{X}} \right)\,. \tag{36}$$

In order to obtain the boundary conditions for an invariant, we pay attention to the vector field restrictions (20) on the boundary. Considering all non-trivial terms

$$\partial_\alpha g^\nu v_{\mathcal{X},t}^\alpha = -\partial_\kappa g^\nu v_{\mathcal{U},t}^\kappa\,, \tag{37}$$

we have to successively substitute corresponding terms in (35) to incorporate all present boundary conditions of the vector field. Following this procedure, this may require some distinctions of cases on (parts of) the boundary, hence, by considering the special coordinate system we finally get boundary conditions of the form

$$v_{\mathcal{U},t}^\kappa : \; 0 \;=\; p_\kappa^\alpha \omega_{t,\alpha}^{\mathcal{X}} \tag{38a}$$

$$v_{\mathcal{X},t}^\beta : \; 0 \;=\; \bar{p}_\beta^\alpha \omega_{t,\alpha}^{\mathcal{X}} \tag{38b}$$

with suitable functions $p_\kappa^\alpha, \bar{p}_\beta^\alpha \in C^\infty \left(\iota^* \left(J^1 (\mathcal{X}) \right) \times_{\partial\mathcal{D}} \mathcal{U} \right)$.

Theorem 2 *If locally the conditions (36) and (38) together with the condition* $\mathrm{d}\tilde{\omega}_t \wedge \mathrm{dX} = 0$, $\tilde{\omega}_t = \omega_{t,\alpha}^{\mathcal{X}} \mathrm{d}x^\alpha$ *allow a non-trivial solution leading to an invariant* $\mathcal{I}(\gamma(t)) = \int_{\mathcal{D}} (I_t \circ \gamma(t)(X))\, \mathrm{dX}$ *with* $\partial_\alpha I_t(X, x) = \omega_{t,\alpha}^{\mathcal{X}}(X, x) \neq 0$, *the system is (locally) non-accessible.*

Proof: If we assume a non-trivial solution $\omega_{t,\alpha}^{\mathcal{X}}$, then due to the Frobenius Integrability theorem, see, e.g., [16], and $\mathrm{d}\tilde{\omega}_t \wedge \mathrm{dX} = 0$ we have a (local) solution I_t for the PDEs $\omega_{t,\alpha}^{\mathcal{X}}(X, x) = \partial_\alpha I_t(X, x)$, depending on x^α, and a (local) non-trivial functional $\mathcal{I}(\gamma(t)) = \int_{\mathcal{D}} (I_t \circ \gamma(t)(X))\, \mathrm{dX}$. Then, we can select a point $\tilde{\gamma}_T$ in the arbitrarily small neighborhood $\mathcal{V}_{\gamma_T}$ of γ_T, which fulfills $\mathcal{I}(\tilde{\gamma}_T) \neq \mathcal{I}(\gamma_T)$ and, obviously, the mapping $\mathcal{A}_{T,\gamma_0}$ is locally not onto since there is no input η steering the system to the point $\tilde{\gamma}_T$. I.e., any neighborhood $\mathcal{V}_{\gamma_T}$ of γ_T will not be open and so the system is non-accessible. ■

Remark 7 If the accessibility conditions are evaluated along the trajectory, they coincide with the adjoint system of the linearized system Σ_Δ, see, e.g., [19], by identifying $\omega_{t,\alpha}^{\mathcal{X}}$, $\omega_{t,\zeta}^{\mathcal{U}}$, $\omega_{t,\kappa}^{\mathcal{Y}}$ with Δx_α, Δu_ζ, Δy_κ, respectively, where the (adjoint) inputs Δu_ζ and the (adjoint) outputs Δy_κ (with $\omega_\kappa^{\mathcal{Y}} = p_\kappa^\alpha \omega_\alpha^{\mathcal{X}}$) have to vanish for the considered time interval $[0, T]$.

In summary, necessary criteria in form of (homogeneous) linear PDE systems are obtained, where a non-trivial solution implies a non-trivial transformation group / invariant and non-observability / non-accessibility, respectively. In addition, sufficient conditions are provided in form of a priori inequalities, which can be seen as the integral version of the previous conditions.

Theorem 3 *The system* Σ *is locally observable along a trajectory* (γ, η), *if the following inequality is satisfied,*

$$\left\| \left(v_{\mathcal{Y}}^\zeta \right) (\gamma(\cdot)) \right\|_{\mathcal{F}_{\mathcal{Y}}} \geq c \left\| \left(v_{\mathcal{X},0}^\alpha \right) (\gamma_0(\cdot)) \right\|_{\mathcal{F}_{\mathcal{D}}} \tag{39}$$

with $v_{\mathcal{Y}}^\zeta = v_{\mathcal{X}} \left(h^\zeta \right)$ *and* $c \in \mathbb{R}^+$, *where* $v_{\mathcal{X},t}^\alpha$ *solves (20).*

Proof: The proof relies on the Local Injectivity Theorem, see, e.g., [21, Theorem 2.5.10, p. 123] and the fact that the system Σ_Δ is the generator of $D\mathcal{O}_\eta(\gamma_0)$. Because of the Closed Range Theorem and the inequality (39) the range $\mathcal{R}(D\mathcal{O}_\eta(\gamma_0))$ of the derivative $D\mathcal{O}_\eta(\gamma_0)$ at γ_0 is closed, see, e.g., [22]. Thus, there is a neighborhood $\mathcal{U}_{\gamma_0}$ of γ_0, $\mathcal{U}_{\gamma_0} \subset \mathcal{V}_{\gamma_0}$, on which $\mathcal{O}_\eta$ is injective, and the system Σ is observable along this trajectory. ■

Theorem 4 *The system* Σ *is locally accessible along a trajectory* (γ, η), *if the following inequality is satisfied,*

$$\left\| \left(\omega_\kappa^{\mathcal{Y}} \right) \left(j^1(\gamma)(\cdot) \right) \right\|_{\mathcal{F}_{\mathcal{U}}^*} \geq c \left\| \left(\omega_{\alpha,0}^{\mathcal{X}} \right) \left(j^1(\gamma_0)(\cdot) \right) \right\|_{\mathcal{F}_{\mathcal{D}}^*} \tag{40}$$

with $\omega_\kappa^{\mathcal{Y}} = p_\kappa^\alpha \omega_\alpha^{\mathcal{X}}$ *and* $c \in \mathbb{R}^+$, *where* $\omega_{t,\alpha}^{\mathcal{X}}$ *solves (36) and (38b).*

Proof: The proof relies on the Local Surjectivity Theorem, see, e.g., [21, Theorem 2.5.9, p. 122] and the fact that the system Σ_Δ is a generator of $D\mathcal{A}_{T,\gamma_0}(\eta)$. Because of the inequality (40) the adjoint operator $D\mathcal{A}_{T,\gamma_0}^*(\eta)$ of $D\mathcal{A}_{T,\gamma_0}(\eta)$ at η is one-to-one and has closed range, see, e.g., [22]. Due to the Closed Range Theorem the map $D\mathcal{A}_{T,\gamma_0}(\eta)$ is onto at η, and, thus, there are open neighborhoods $\mathcal{U}_\eta$ and $\mathcal{U}_{\gamma_T}$ such that $\mathcal{A}_{T,\gamma_0}|_{\mathcal{U}_\eta} \to \mathcal{U}_{\gamma_T}$ is onto, and the system Σ is accessible along this trajectory.

It is worth mentioning that, for instance, [3], [8] contain equivalent criteria for linear PDE systems.. ■

V. Nonlinear Examples

Next, nonlinear examples are studied to illustrate the theory and results. Let us consider the nonlinear example

$$\begin{aligned} \dot{x}^1 &= x^2 \\ \dot{x}^2 &= x_1^4 x^3 - x_1^3 \\ \dot{x}^3 &= \left(1 + x^4 \right) x_1^2 \\ \dot{x}^4 &= 0 \end{aligned} \tag{41a}$$

on $\mathcal{D} = [0,1]$ for $x^4 > 0$ with the boundary terms

$$0 = x^3 + u^1 \,,\; y^1 = x^1 \,,\; y^2 = x^2 \tag{41b}$$

on $\partial\mathcal{D}$.

Proposition 1 *The system (41) is locally non-observable along the trajectory (γ, η) with $(\gamma^\alpha)(t)(X) = (0,0,0,1)$, $\eta^\kappa(t)(X) = 0$ and any $T > 0$.*

Proof: According to Theorem 1 the observability conditions follow to

$$\begin{aligned}\dot{v}^1_{\mathcal{X},t} &= v^2_{\mathcal{X},t}\\ \dot{v}^2_{\mathcal{X},t} &= x^3 d_1\left(v^4_{\mathcal{X},t}\right) + x^4_1 v^3_{\mathcal{X},t} - d_1\left(v^3_{\mathcal{X},t}\right)\\ \dot{v}^3_{\mathcal{X},t} &= x^2_1 v^4_{\mathcal{X},t} + \left(1 + x^4\right) d_1\left(v^2_{\mathcal{X},t}\right)\\ \dot{v}^4_{\mathcal{X},t} &= 0\end{aligned} \tag{42}$$

on $\mathcal{D}$ and

$$0 = v^1_{\mathcal{X},t} \,,\; 0 = v^2_{\mathcal{X},t} \,,\; 0 = v^3_{\mathcal{X},t} \,,\; 0 = v^1_{\mathcal{U},t} \tag{43}$$

on $\partial\mathcal{D}$. There is a vector field $v_{\mathcal{X},t} = \partial_4$ with $v^1_{\mathcal{X},t} = v^2_{\mathcal{X},t} = v^3_{\mathcal{X},t} = 0$ solving the equations along the trajectory such that the system (41) is non-observable. $v_{\mathcal{X},t}$ is the generator of a symmetry group with $\gamma_\varepsilon = \Phi^{\mathcal{X}}_{t,\varepsilon}(\gamma) = \left(\gamma^1, \gamma^2, \gamma^3, \gamma^4 + \varepsilon\right)$. Hence, all solutions $(\gamma_\varepsilon, \eta)$ with different initial conditions lead to the same outputs $y^1 = \gamma^1_\varepsilon(t)(X_\partial) = 0$, $y^2 = \gamma^2_\varepsilon(t)(X_\partial) = 0$. ■

Next, let us consider the nonlinear example, see, e.g., [19],

$$\begin{aligned}\dot{x}^1 &= \frac{(x^1x^2 - x^1)x^2_1 + x^1_1 x^2}{x^2}\,,\\ \dot{x}^2 &= -x^2 x^2_1\end{aligned} \tag{44a}$$

with $\mathcal{D} = [0,1]$ for $x^2 > 0$ and the boundary conditions

$$0 = x^1 + u^1 \,, \quad 0 = x^2 + u^1 \tag{44b}$$

on $\partial\mathcal{D}_1 = \{0\} \subset \partial\mathcal{D}$ for $u^1 > 0$.

Proposition 2 *The system (44) is locally non-accessible along any trajectory for any $T > 0$.*

Proof: The infinitesimal principle of invariance delivers the conditions

$$\begin{aligned}\dot{v}^1_{\mathcal{X},t} &= -\frac{(x^2-1)x^2_1}{x^2} v^1_{\mathcal{X},t} - \frac{x^1 x^2_1}{(x^2)^2} v^2_{\mathcal{X},t}\\ &\quad -d_1(v^1_{\mathcal{X},t}) - \frac{x^1x^2 - x^1}{x^2} d_1(v^2_{\mathcal{X},t})\,,\\ \dot{v}^2_{\mathcal{X},t} &= -x^2_1 v^2_{\mathcal{X},t} - x^2 d_1(v^2_{\mathcal{X},t})\end{aligned} \tag{45}$$

on the domain $\mathcal{D}$ and

$$0 = v^1_{\mathcal{X},t} + v^1_{\mathcal{U},t} \,,\; 0 = v^2_{\mathcal{X},t} + v^1_{\mathcal{U},t} \tag{46}$$

on the boundary $\partial\mathcal{D}_1$. According to Theorem 2 the (local) accessibility conditions follow as

$$\begin{aligned} v^1_{\mathcal{X},t}:\;\; \dot{\omega}^{\mathcal{X}}_{t,1} &= \frac{(x^2-1)x^2_1}{x^2}\omega^{\mathcal{X}}_{t,1} - d_1(\omega^{\mathcal{X}}_{t,1})\,,\\ v^2_{\mathcal{X},t}:\;\; \dot{\omega}^{\mathcal{X}}_{t,2} &= \frac{x^1 x^2_1}{(x^2)^2}\omega^{\mathcal{X}}_{t,1} + x^2_1 \omega^{\mathcal{X}}_{t,2}\\ &\quad -d_1\left(\frac{x^1x^2 - x^1}{x^2}\omega^{\mathcal{X}}_{t,1}\right) - d_1\left(x^2 \omega^{\mathcal{X}}_{t,2}\right)\end{aligned} \tag{47}$$

on $\mathcal{D}$ and

$$v^1_{\mathcal{U},t} : 0 = \left(-1 - \frac{x^1x^2 - x^1}{x^2}\right)\omega^{\mathcal{X}}_{t,1} - x^2 \omega^{\mathcal{X}}_{t,2} \tag{48}$$

on $\partial\mathcal{D}_1$ and

$$v^1_{\mathcal{X},t} :\; 0 = \omega^{\mathcal{X}}_{t,1}\,, \quad v^2_{\mathcal{X},t} :\; 0 = \omega^{\mathcal{X}}_{t,2} \tag{49}$$

on $\partial\mathcal{D}_2 = \{1\}$. A non-trivial solution is given by

$$\begin{aligned}\omega^{\mathcal{X}}_{1,t} &= \frac{\phi_b(t - X - (T - 1 + b))}{x^2}\,,\\ \omega^{\mathcal{X}}_{2,t} &= -\frac{\phi_b(t - X - (T - 1 + b))x^1}{(x^2)^2}\,,\end{aligned} \tag{50}$$

which fulfills $\mathrm{d}\left(\omega^{\mathcal{X}}_{1,t}\mathrm{d}x^1 + \omega^{\mathcal{X}}_{2,t}\mathrm{d}x^2\right) \wedge \mathrm{dX} = 0$ and leads to the invariant

$$I_t(X, x) = -\phi_b(t - X - (T - 1 + b))\frac{x^2 - x^1}{x^2} \tag{51}$$

with

$$\phi_b(a) = \begin{cases} e^{\frac{b^2}{a^2 - b^2}} & |a| < b\\ 0 & |a| \geqslant b\end{cases}$$

and $b \in \mathbb{R}^+$ for any $T > 0$ such that the system (44) is non-accessible along any trajectory for any $T > 0$. ■

Finally, let us consider the nonlinear example, see, e.g., [23],

$$\begin{aligned}\dot{x}^1 &= x^3\\ \dot{x}^2 &= x^3_1\\ \dot{x}^3 &= \alpha^2_1(X, x)\, x^2_1 + \alpha_2(X, x)\end{aligned} \tag{52a}$$

with $\alpha_1, \alpha_2 \in C^\infty(\mathcal{X})$ and $\alpha_1(X, x) > 0$, $\alpha_2(X, 0) = 0$ on $\mathcal{D} = [0, L]$, $L \in \mathbb{R}^+$ with the boundary terms

$$0 = x^2 + u^1 \,,\; y^1 = x^1 \tag{52b}$$

on $\partial\mathcal{D}$. Let $\mathcal{F}_{\mathcal{D}} = C^2[0, L]$, $\mathcal{F}_t = C^2[0, L] \times [0, T]$, $\mathcal{F}_{\mathcal{Y}} = C^1[0, T]$, see, e.g., [23] for a discussion of the well-posedness of solutions, then we have the following observability result.

Proposition 3 *The system (52) is locally observable along a trajectory (γ, η) with $T > L/\inf_{0 \le X^1 \le L} \alpha_1$.*

Proof: According to Theorem 1 the observability conditions follow to

$$\begin{aligned}\dot{v}^1_{\mathcal{X},t} &= v^3_{\mathcal{X},t}\\ \dot{v}^2_{\mathcal{X},t} &= d_1\left(v^3_{\mathcal{X},t}\right)\\ \dot{v}^3_{\mathcal{X},t} &= 2x^2_1\left(\partial_1\alpha_1 v^1_{\mathcal{X},t} + \partial_2\alpha_1 v^2_{\mathcal{X},t} + \partial_3\alpha_1 v^3_{\mathcal{X},t}\right)\\ &\quad +\partial_1\alpha_2 v^1_{\mathcal{X},t} + \partial_2\alpha_2 v^2_{\mathcal{X},t} + \partial_3\alpha_2 v^3_{\mathcal{X},t}\\ &\quad +\alpha^2_1 d_1\left(v^2_{\mathcal{X},t}\right)\end{aligned} \tag{53}$$

on $\mathcal{D}$ and

$$0 = v^1_{\mathcal{X},t} \,,\; 0 = v^2_{\mathcal{X},t} \,,\; 0 = v^1_{\mathcal{U},t} \tag{54}$$

on $\partial\mathcal{D}$. Using the result of [23] the inequality

$$\left\|\left(v^1_{\mathcal{X}}\right)(\gamma(\cdot))\right\|_{\mathcal{F}_{\mathcal{Y}}} \geq c\left\|\left(v^\alpha_{\mathcal{X},0}\right)(\gamma_0(\cdot))\right\|_{\mathcal{F}_{\mathcal{D}}}\,,$$

is satisfied, where c depends on T, and because of Theorem 3 the system (52) is observable along such a trajectory. ■

VI. Conclusions

In this article first an intrinsic formulation for a class of first-order PDE systems with boundary conditions and boundary interaction was provided by using differential geometrical methods. Then, a group-theoretical approach was proposed to tackle both the problem of accessibility and observability along a fixed trajectory and we illustrated that (local) observability and accessibility criteria could be provided by using an infinitesimal principle of invariance. In particular, group invariants were studied, which are known for the observability problem and unknown for the accessibility problem. It is worth mentioning that due to the underlying geometric structures the methods can be extended straightforward to higher-order (and coupled) PDE systems with various boundary conditions.

Acknowledgments

This work has been partially funded by the Austrian Center of Competence in Mechatronics (ACCM).

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Appendix

Let $\mathcal{E}$ and $\mathcal{M}$ be smooth manifolds, then a bundle is a triple $(\mathcal{E}, \pi, \mathcal{M})$ with the total manifold $\mathcal{E}$, the base manifold $\mathcal{M}$ and the projection (or fibration) $\pi : \mathcal{E} \to \mathcal{M}$, where $\pi^{-1}(p)$ for any $p \in \mathcal{M}$ denotes the fiber over p. If there is no danger for confusion, a bundle is denoted $\pi : \mathcal{E} \to \mathcal{M}$ or simply $\mathcal{E} \to \mathcal{M}$ for short. The manifold $\mathcal{M}$ has the coordinates $\left(Z^i\right)$ and $\mathcal{E}$ the adapted coordinates (Z^i, z^α). Z^i, $i = 1, \ldots, m$ are the independent coordinates and z^α, $\alpha = 1, \ldots, n$ the dependent coordinates. A section σ of the bundle $\mathcal{E} \to \mathcal{M}$ is a map $\sigma : \mathcal{M} \to \mathcal{E}; (Z^i) \mapsto (Z^i, \sigma^\alpha(Z^i))$ such that $\pi \circ \sigma = \mathrm{id}_\mathcal{M}$ with the identity map $\mathrm{id}_\mathcal{M}$ on $\mathcal{M}$. The set of sections $\sigma : \mathcal{M} \to \mathcal{E}$ is denoted $\Gamma(\mathcal{M}, \mathcal{E})$.

The tangent and cotangent bundle of a smooth n-dimensional manifold $\mathcal{N}$ (with coordinates X^i) are denoted by $\mathcal{T}(\mathcal{N}) \to \mathcal{N}$ and $\mathcal{T}^*(\mathcal{N}) \to \mathcal{N}$, which are equipped with the induced coordinates $(X^i, \dot{X}^i)$ and $(X^i, \dot{X}_i)$ with respect to the holonomic bases $\{\partial_i\}$ and $\{\mathrm{d}X^i\}$. The exterior algebra over an n-dimensional manifold $\mathcal{N}$ is denoted by $\wedge(\mathcal{T}^*(\mathcal{N}))$ with the exterior derivative $\mathrm{d} : \wedge_k(\mathcal{T}^*(\mathcal{N})) \to \wedge_{k+1}(\mathcal{T}^*(\mathcal{N}))$, the interior product $\rfloor : \wedge_{k+1}(\mathcal{T}^*(\mathcal{N})) \to \wedge_k(\mathcal{T}^*(\mathcal{N}))$ written as $v \rfloor \omega$ with $v : \mathcal{N} \to \mathcal{T}(\mathcal{N})$ and $\omega : \mathcal{N} \to \wedge_{k+1}(\mathcal{T}^*(\mathcal{N}))$, and the exterior product $\wedge$. Further, $\wedge_k(\mathcal{T}^*(\mathcal{N})) \to \mathcal{N}$ is the exterior k-form bundle on $\mathcal{N}$. The Lie derivative of $\omega : \mathcal{N} \to \wedge(\mathcal{T}^*(\mathcal{N}))$ along a vector field $f : \mathcal{N} \to \mathcal{T}(\mathcal{N})$ is identified by $f(\omega)$.

Let $s^1_p(\sigma)$ denote the equivalence of all sections $\bar{\sigma} \in \Gamma(\mathcal{M}, \mathcal{E})$ at $p \in \mathcal{M}$ such that $\sigma^\alpha(p) = \bar{\sigma}^\alpha(p)$ as well as $\partial_j \sigma^\alpha(p) = \partial_j \bar{\sigma}^\alpha(p)$ with $\partial_j = \frac{\partial}{\partial X^j}$. Then, the set of equivalence class $s^1_p(\sigma)$ can be endowed with the structure of a manifold $J^1(\mathcal{E}) = \{s^1_p(\sigma) : p \in \mathcal{M}, \sigma \in \Gamma(\mathcal{M}, \mathcal{E})\}$, which is called the first jet manifold and has the adapted coordinates $(Z^i, z^\alpha, z^\alpha_j)$. According to the previous construction we have the bundles $\pi^1_0 : J^1(\mathcal{E}) \to \mathcal{E}$ and $\pi^1 : J^1(\mathcal{E}) \to \mathcal{M})$. Then a section $\sigma \in \Gamma(\mathcal{M}, \mathcal{E})$ can be extended to a section $j^1(\sigma) \in \Gamma(\mathcal{M}, J^1(\mathcal{E}))$ with $j^1(\sigma)(Z) = (Z^i, \sigma^\alpha(Z), \partial_j \sigma^\alpha(Z^i))$, which is called the first jet of σ. The operator $d_j : J^1(\mathcal{E}) \to \pi^{1,*}_0(\mathcal{T}(\mathcal{E}))$ is called the total derivative with respect to the independent coordinate Z^j, which fulfills $(d_j f) \circ j^1(\sigma)(Z) = \partial_j f(\sigma)(Z)$ for any $f \in C^\infty(\mathcal{E})$ and $\sigma : \mathcal{M} \to \mathcal{E}$. In adapted coordinates $(Z^i, z^\alpha, z^\alpha_j)$ it is defined by $d_j = \partial_j + z^\alpha_j \partial_\alpha$, $\partial^j_\alpha = \frac{\partial}{\partial z^\alpha_j}$.