Integrability and Optimal Control

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Abstract— **This paper considers left-invariant control affine systems evolving on matrix Lie groups. Any left-invariant optimal control problem (with quadratic cost) can be lifted, via the celebrated Maximum Principle, to a Hamiltonian system on the dual of the Lie algebra of the underlying state space G. The (minus) Lie-Poisson structure on the dual space $\mathfrak{g}^*$ is used to describe the (normal) extremal curves. Complete integrability of (reduced) Hamiltonian dynamical systems is discussed briefly. Some observations concerning Casimir functions and the case of semisimple (matrix) Lie groups are made. As an application, a drift-free left-invariant optimal control problem on the rotation group $\mathsf{SO}(3)$ is investigated. The reduced Hamilton equations associated with an extremal curve are derived in a simple and elegant manner. Finally, these equations are explicitly integrated by Jacobi elliptic functions.**

I. INTRODUCTION

Invariant control systems on Lie groups provide a natural geometric setting for a variety of problems of mathematical physics, classical and quantum mechanics, elasticity, differential geometry and dynamical systems. Many variational problems (with constraints) can be formulated in the geometric language of modern optimal control theory. An incomplete list of such problems includes the dynamic equations of the rigid body, the ball-plate problem, various versions of the Euler and Kirchhoff elastic rod problem, the Dubins' problem as well as the (more general) sub-Riemannian geodesic problem and the motion of a particle in a magnetic or Yang-Mills field. Some of these problems (and many other) can be found, for instance, in the monographs by Jurdjevic [13], Bloch [5] or Agrachev and Sachkov [1].

In the last two decades or so, substantial work on (applied) nonlinear control has drawn attention to (left-) invariant control systems with control affine dynamics, evolving on matrix Lie groups of low dimension. These arise in problems like the airplane landing problem [32], the motion planning for wheeled robots (subject to nonholonomic constraints) [31] or for oriented vehicles [4], the control of underactuated underwater vehicles [19], the control of quantum systems [7], and the dynamic formation of DNA [9].

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the trajectories of a given left-invariant control system, subject to appropriate boundary conditions. The Maximum Principle states that the optimal solutions are projections of the extremal curves onto the base manifold. (For invariant control systems the base manifold is a Lie group G.) The extremal curves are solutions of certain Hamiltonian systems on the cotangent bundle $T^*\mathsf{G}$. The cotangent bundle $T^*\mathsf{G}$ can be realized as the direct product $\mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. As a result, each original (left-invariant) Hamiltonian induces a reduced Hamiltonian on the dual space (which comes equipped with a natural Poisson structure).

In this paper, we consider an optimal control problem associated with a drift-free left-invariant control affine system (with two inputs) on the rotation group $\mathsf{SO}(3)$. (For the single-input case, see [12], [26].) The problem is lifted, via the Pontryagin Maximum Principle, to a Hamiltonian system on the dual of the Lie algebra $\mathfrak{so}(3)$. Now, the (minus) Lie-Poisson structure on $\mathfrak{so}(3)^*$ (identified here with $\mathbb{R}^3_\wedge$) can be used to derive, in a general and elegant manner, the equations for extrema (cf. [13], [1], [17], [28], [24], [26]). Jacobi elliptic functions are used to derive *explicit* expressions for the extremal curves (cf. [22], [23]); see also [26], [16], [14].

The paper is organized as follows. Section II contains mathematical preliminaries including invariant control systems, elements of Hamilton-Poisson formalism as well as a (coordinate-free) statement of the Maximum Principle. Section III covers briefly complete integrability as well as the Lax representation (of Hamiltonian dynamical systems). In section IV, a class of optimal control problems is identified and a particular result due to P.S. Krishnaprasad [17] is recalled. Finally, section V deals with a particular case of a left-invariant optimal control on the rotation group $\mathsf{SO}(3)$, including the derivation of explicit expressions (in terms of Jacobi elliptic functions) of the extremal curves. A final remak concludes the paper.

II. PRELIMINARIES

A. *Invariant Control Systems*

Invariant control systems on Lie groups were first considered in 1972 by Brockett [6] and by Jurdjevic and Sussmann [15]. A *left-invariant control system* is a (smooth) control system evolving on some (real) Lie group, whose dynamics is invariant under left translations. For the sake of convenience, we shall assume that the state space of the system is a matrix Lie group and that there are no constraints on the controls. Such a control system (evolving on G) is described as follows (cf. [13], [24], [26])

$$\dot{g} = g\,\Xi(\mathbf{1}, g), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{1}$$

where the parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is a (smooth) embedding. (Here $\mathbf{1} \in \mathsf{G}$ denotes the identity matrix and $\mathfrak{g}$ denotes the Lie algebra associated with G.) An admissible control is a map $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ that

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is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".) A *trajectory* for an admissible control $u(\cdot) : [0,T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0,T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(1, u(t))$ for almost every $t \in [0,T]$. The Carathéodory existence and uniqueness theorem of ordinary differential equations implies the local existence and global uniqueness of trajectories. A controlled trajectory is a pair $(g(\cdot), u(\cdot))$, where $u(\cdot)$ is an admissible control and $g(\cdot)$ is the trajectory corresponding to $u(\cdot)$.

The attainable set from $g \in \mathsf{G}$ is the set $\mathcal{A}(g)$ of all terminal points $g(T)$ of all trajectories $g(\cdot) : [0,T] \to \mathsf{G}$ starting at g. It follows that $\mathcal{A}(g) = g\,\mathcal{A}(\mathbf{1})$. Thus, $\mathcal{A}(g) = \mathsf{G}$ if and only if $\mathcal{A}(\mathbf{1}) = \mathsf{G}$. Control systems which satisfy $\mathcal{A}(\mathbf{1}) = \mathsf{G}$ are called *controllable*. Let $\Gamma \subseteq \mathfrak{g}$ be the image of the parametrisation map $\Xi(\mathbf{1}, \cdot)$, and let $\mathsf{Lie}(\Gamma)$ denote the Lie subalgebra of $\mathfrak{g}$ generated by Γ. It is well known that a necessary condition for the control system (1) to be controllable is that G be connected and that $\mathsf{Lie}(\Gamma) = \mathfrak{g}$. If the group G is compact, then the condition is also sufficient.

For many practical control applications, (left-invariant) control systems contain a drift term and are affine in controls, i.e., are of the form

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \qquad (2)$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. Usually the elements (matrices) $B_1, \ldots, B_\ell$ are assumed to be linearly independent.

B. Optimal Control Problems

Consider a left-invariant control system (1) evolving on some matrix Lie group $\mathsf{G} \leq \mathsf{GL}\,(n, \mathbb{R})$ of dimension m. In addition, it is assumed that there is a prescribed (smooth) *cost function* $L : \mathbb{R}^\ell \to \mathbb{R}_{>0}$ (which is also called a Lagrangian). Let g_0 and g_1 be arbitrary but fixed points of G. We shall be interested in finding a controlled trajectory $(g(\cdot), u(\cdot))$ which satisfies

$$g(0) = g_0, \quad g(T) = g_1 \qquad (3)$$

and which in addition *minimizes* the total cost functional $\mathcal{J} = \int_0^T L(u(t))\,dt$ among all trajectories of (1) which satisfy the same boundary conditions (3). The terminal time $T > 0$ can be either fixed or it can be free.

The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$. Explicitly, $\xi \in T^*_g\mathsf{G}$ is identified with $(g, p) \in \mathsf{G} \times \mathfrak{g}^*$ via $p = dL_g^*(\xi)$. (Here, dL_g^* denotes the dual of the tangent map $dL_g = (L_g)_{*,\mathbf{1}} : \mathfrak{g} \to T_g\mathsf{G}$.) That is, $\xi(gA) = p(A)$ for $g \in \mathsf{G}$, $A \in \mathfrak{g}$. Each element (matrix) $A \in \mathfrak{g}$ defines a (smooth) function H_A on the cotangent bundle $T^*\mathsf{G}$ defined by $H_A(\xi) = \xi\,(gA)$ for $\xi \in T^*_g\mathsf{G}$. Viewed as a function on $\mathsf{G} \times \mathfrak{g}^*$, H_A is left-invariant, which is equivalent to saying that H_A is a function on $\mathfrak{g}^*$.

The canonical symplectic form ω on $T^*\mathsf{G}$ sets up a correspondence between (smooth) functions H on $T^*\mathsf{G}$ and vector fields $\vec{H}$ on $T^*\mathsf{G}$ given by $\omega_\xi\left(\vec{H}(\xi), V\right) = dH(\xi) \cdot V$ for $V \in T_\xi(T^*\mathsf{G})$. The Poisson bracket of two functions F, G on $T^*\mathsf{G}$ is defined by $\{F, G\}(\xi) = \omega_\xi\left(\vec{F}(\xi), \vec{G}(\xi)\right)$ for $\xi \in T^*\mathsf{G}$. If (ϕ_t) is the flow of the Hamiltonian vector field $\vec{H}$, then $H \circ \phi_t = H$ (conservation of energy) and $\frac{d}{dt}(F \circ \phi_t) = \{F, H\} \circ \phi_t = \{F \circ \phi_t, H\}$. For short, for any $F \in C^\infty(T^*\mathsf{G})$,

$$\dot{F} = \{F, H\} \qquad (4)$$

(the equation of motion in Poisson bracket form).

The dual space $\mathfrak{g}^*$ has a natural *Poisson structure*, called the "minus Lie-Poisson structure" and given by

$$\{F, G\}_-(p) = -p\,([dF(p), dG(p)])$$

for $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. (Note that $dF(p)$ is a linear function on $\mathfrak{g}^*$ and hence is an element of $\mathfrak{g}$.) The (minus) Lie-Poisson bracket can be derived from the canonical Poisson structure on the cotangent bundle $T^*\mathsf{G}$ by a process called Poisson reduction (cf. [20], [17]). The Poisson manifold $(\mathfrak{g}, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. Each left-invariant Hamiltonian on the cotangent bundle $T^*\mathsf{G}$ is identified with its reduction on the dual space $\mathfrak{g}^*_-$. In the left-invariant realization of $T^*\mathsf{G}$, the equations of motion for the left-invariant Hamiltonian H are

$$\begin{aligned} \dot{g} &= g\,dH(p) &(5)\\ \dot{p} &= \mathrm{ad}^*_{dH(p)}p &(6) \end{aligned}$$

where ad^* denotes the coadjoint representation of $\mathfrak{g}$ (cf. [20], [13]). Note that for non-commutative Lie groups, the representation $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$ invariably leads to non-canonical coordinates.

If $(E_k)_{1 \leq k \leq m}$ is a basis for the Lie algebra $\mathfrak{g}$, the structure constants $\left(c_{ij}^k\right)$ are defined by $[E_i, E_j] = \sum_{k=1}^m c_{ij}^k E_k$. Any element $p \in \mathfrak{g}^*$ can be expressed uniquely as $p = \sum_{k=1}^m p_k E_k^*$, where $(E_k^*)_{1 \leq k \leq m}$ is the basis of $\mathfrak{g}^*$ dual to $(E_k)_{1 \leq k \leq m}$. Then the (minus) Lie-Poisson bracket becomes

$$\{F, G\}_-(p) = -\sum_{i,j,k=1}^m c_{ij}^k p_k \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}\cdot$$

A *Casimir function* of (the Poisson structure of) $\mathfrak{g}^*_-$ is a (smooth) function C on $\mathfrak{g}^*$ such that $\{C, F\}_- = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$. The Casimir functions have the remarkable property that they are integrals of motion for any Hamiltonian system (i.e., they are constant along the flow of any Hamiltonian vector field) on $\mathfrak{g}^*_-$.

C. The Maximum Principle

The *Pontryagin Maximum Principle* is a necessary condition for optimality expressed most naturally in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (cf. [1], [13]). To an optimal control problem (with fixed terminal time)

$$\int_0^T L(u(t))\,dt \to \min \qquad (7)$$

subject to (1) and (3), we associate, for each real number λ and each control parameter $u \in \mathbb{R}^\ell$, a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\, L(u) + \xi\, (g\, \Xi(\mathbf{1}, u)) \\ &= \lambda\, L(u) + p\, (\Xi(\mathbf{1}, u))\,, \quad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned}$$

The Maximum Principle can be stated, in terms of the above Hamiltonians, as follows :

THE MAXIMUM PRINCIPLE. *Suppose the controlled trajectory* $(\bar{g}(\cdot), \bar{u}(\cdot))$ *defined over the interval* $[0, T]$ *is a solution for the optimal control problem (1)-(3)-(7). Then, there exists a curve* $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$, $t \in [0, T]$, *and a real number* $\lambda \leq 0$, *such that the following conditions hold for almost every* $t \in [0, T]$ *:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{8}$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \tag{9}$$

$$H^\lambda_{\bar{u}(t)} = \max_u H_u^\lambda\,(\xi(t)) = \textit{constant}. \tag{10}$$

An optimal trajectory $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$ defined for all $t \in [0, T]$. A trajectory-control pair $(\xi(\cdot), u(\cdot))$ defined on $[0, T]$ is said to be an *extremal pair* if $\xi(\cdot)$ is such that the conditions (8), (9) and (10) of the Maximum Principle hold. The projection $\xi(\cdot)$ of an extremal pair is called an extremal. An extremal curve is called normal if $\lambda = -1$ and abnormal if $\lambda = 0$. In this paper, we shall be concerned only with normal extremals.

If the maximum condition (10) eliminates the parameter u from the family of Hamiltonians (H_u), and as a result of this elimination, we obtain a smooth function H (without parameters) on $T^*\mathsf{G}$ (in fact, on $\mathfrak{g}^*_-$), then the whole (left-invariant) optimal control problem reduces to the study of trajectories of a fixed Hamiltonian vector field $\vec{H}$.

III. INTEGRABILITY

A. Completely integrable systems

The integrability of a Hamiltonian dynamical system is ensured by a sufficient supply of first integrals. Let $\mathsf{G} \leq \mathsf{GL}\,(n, \mathbb{R})$ be a matrix Lie group (of dimension m). A function K on the cotangent bundle $T^*\mathsf{G}$ (or any symplectic manifold) is a first integral of a Hamiltonian system with Hamiltonian H if (and only if) the Poisson bracket $\{K, H\}$ is equal to zero. One says of functions whose Poisson bracket is equal to zero that they are in involution. A Hamiltonian system on $T^*\mathsf{G}$ is said to be *completely integrable* if there exist m first integrals $K_1, K_2, \ldots, K_{m-1}, K_m = H$ in involution which are functionally independent (almost everywhere on $T^*\mathsf{G}$). Then in each level set

$$K_1 = \text{const}, \ldots, K_m = \text{const}$$

there exists a system of coordinates in which the Hamilton equations take a particular simple form and make the solutions evident. Hence, a completely integrable system can be integrated by quadratures [21], [8]. (“Quadrature” means “integration of known functions”.) For left-invariant Hamiltonian systems, there are always extra first integrals that are in involution. The Hamiltonians of right-invariant vector fields Poisson commute with the Hamiltonian of the system. (The maximum number of such functions which Poisson commute with each other is determined by the rank of $\mathfrak{g}$, i.e., by the dimension of a maximal commutative subalgebra of $\mathfrak{g}$.) In addition to these first integrals, there may be others; for instance, the Casimir functions. (On semisimple matrix Lie groups, Casimir functions always exist.) The following result is well known (cf. [13], [8]).

Proposition 1 *All left-invariant Hamiltonian dynamical systems on three-dimensional (matrix) Lie groups are completely integrable.*

B. The Lax representation

If the matrix Lie group G is semisimple, then (and only then) the Killing form (on $\mathfrak{g}$) defined by $\mathcal{K}(A, B) = \operatorname{tr}\,(\operatorname{ad}_A \circ \operatorname{ad}_B)$ is *nondegenerate*. It is also known that the bilinear symmetric form $\mathcal{K}$ is *invariant* (under the Lie bracket) in the sense that $\mathcal{K}([A, B], C) = \mathcal{K}(A, [B, C])$. As any nondegenerate form, $\mathcal{K}$ sets up a correspondence between (the vector space) $\mathfrak{g}$ and its dual $\mathfrak{g}^*$: let P denote the element in $\mathfrak{g}$ that corresponds to $p \in \mathfrak{g}^*$ via the correspondence $p(\cdot) = \mathcal{K}(P, \cdot)$. We get (along the integral curves of the left-invariant vector field $\vec{H}$)

$$\begin{aligned} \mathcal{K}\left(\dot{P}(t), A\right) &= \dot{p}(t)(A) = \left(\operatorname{ad}^*_{dH(p)} p(t)\right)(A) = \\ &= p(t)\,([dH(p), A]) = \mathcal{K}\,(P(t), [dH(p), A]) = \\ &= \mathcal{K}\,([P(t), dH(p)], A)\,. \end{aligned}$$

It follows that the use of the Killing form puts the equation of motion (6) in the *Lax-pair form*

$$\dot{P} = [P, dH(p)]\,, \quad P \in \mathfrak{g} \tag{11}$$

It can be seen from the Lax-pair representation (11) that the spectral invariants of P (i.e., $\operatorname{tr}(P), \operatorname{tr}(P^2), \ldots, \det(P)$) are first integrals of the (reduced) Hamiltonian dynamical system with Hamiltonian H (cf. [11], [8], [13]). So, when the first integrals of the system are not known, computing the spectral invariants of P is an easy way to find them.

IV. A CLASS OF OPTIMAL CONTROL PROBLEMS

Consider now a left-invariant optimal control problem (2)-(3)-(7) with quadratic cost of the form

$$L(u_1, \ldots, u_\ell) = \frac{1}{2}\left(c_1 u_1^2 + \cdots + c_\ell u_\ell^2\right)$$

where $c_1, \ldots, c_\ell$ are (positive) constants. The terminal time $T > 0$ is fixed in advance. The maximum condition (10) of the Maximum Principle implies that (for $\lambda = -1$) the optimal controls $\bar{u}(\cdot)$ satisfy

$$-\frac{\partial L}{\partial u_i} + \frac{\partial}{\partial u_i}\left(p\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\right) = 0$$

or

$$-c_i u_i + p(B_i) = 0, \quad i = 1, \ldots, \ell.$$

The following result holds (see [17]) :

Proposition 2 *For the optimal control problem (2)-(3)-(7), every normal extremal is given by*

$$\bar{u}_i(t) = \frac{1}{c_i}\, p(t)(B_i), \quad i = 1, \ldots, \ell$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the vector field* $\vec{H}$ *on* $\mathfrak{g}^*_-$ *corresponding to the reduced Hamiltonian*

$$H(p) = p(A) + \frac{1}{2}\left(\frac{1}{c_1}p(B_1)^2 + \cdots + \frac{1}{c_\ell}p(B_\ell)^2\right).$$

Furthermore, in coordinates on $\mathfrak{g}^*_-$*, the (components of the) integral curves satisfy*

$$\dot{p}_i = -\sum_{j,k=1}^{m} c_{ij}^k p_k \frac{\partial H}{\partial p_j}, \quad i = 1, \ldots, m. \tag{12}$$

V. AN OPTIMAL CONTROL PROBLEM ON THE ROTATION GROUP $\mathsf{SO}(3)$

A. A drift-free left-invariant control problem

The rotation group

$$\mathsf{SO}(3) = \left\{a \in \mathsf{GL}(3, \mathbb{R}) \,:\, a^\top a = \mathbf{1},\, \det a = 1\right\}$$

is a three-dimensional compact and connected matrix Lie group. The associated Lie algebra is given by

$$\mathfrak{so}(3) = \left\{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \,:\, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the standard basis of $\mathfrak{so}(3)$ with the following table for the bracket operation

$[\cdot,\cdot]$	E_1	E_2	E_3
E_1	0	E_3	$-E_2$
E_2	$-E_3$	0	E_1
E_3	E_2	$-E_1$	0

The linear map $\widehat{\cdot} : \mathfrak{so}(3) \to \mathbb{R}^3$ defined by

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mapsto \widehat{A} = (a_1, a_2, a_3)$$

is a Lie algebra isomorphism. Hence, we identify $\mathfrak{so}(3)$ with (the cross-product Lie algebra) $\mathbb{R}^3_\wedge$ (cf. [10], [20]). We consider the following optimal control problem

$$\dot{g} = g\,(u_1 E_1 + u_2 E_2)\,, \quad g \in \mathsf{SO}(3),\ u \in \mathbb{R}^2 \tag{13}$$

$$g(0) = g_0, \quad g(T) = g_1 \quad (g_0, g_1 \in \mathsf{SO}(3)) \tag{14}$$

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + c_2 u_2^2(t)\right)\, dt \to \min. \tag{15}$$

This problem appears in the modelling of spacecraft dynamics [22], [3]. The case with drift can be found in [32], [27] and the "fully actuated" case in [29], [30]. Note that the underlying control system is controllable.

B. Extremal curves in $\mathfrak{so}(3)^*$

The Killing form (on $\mathfrak{so}(3)$) is given by $\langle A, B\rangle = -\frac{1}{2}\mathrm{tr}\,(AB) = \widehat{A} \bullet \widehat{B}$. We will identify $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ via the pairing

$$\left\langle \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \right\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Then each extremal curve $p(\cdot)$ is identified with a curve $P(\cdot)$ in $\mathfrak{so}(3)$ via the formula $\langle P(t), A\rangle = p(t)(A)$ for all $A \in \mathfrak{so}(3)$. Thus

$$P(t) = \begin{bmatrix} 0 & -P_3(t) & P_2(t) \\ P_3(t) & 0 & -P_1(t) \\ -P_2(t) & P_1(t) & 0 \end{bmatrix} \tag{16}$$

where $P_i(t) = \langle P(t), E_i\rangle = p(t)(E_i), \quad i = 1, 2, 3.$

The (minus) Lie-Poisson bracket on $\mathfrak{so}(3)^*$ is given by

$$\begin{aligned} \{F, G\}_-(p) &= -\sum_{i,j,k=1}^{3} c_{ij}^k p_k \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j} \\ &= -\widehat{P} \bullet (\nabla F \times \nabla g)\,. \end{aligned}$$

Here, $\mathfrak{so}(3)^*$ is identified with $\mathbb{R}^3_\wedge$. Explicitly, the covector $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$ is identified with the vector $\widehat{P} = (P_1, P_2, P_3)$. The equation of motion (4) becomes

$$\dot{F} = \{F, H\}_- = \nabla F \bullet \left(\widehat{P} \times \nabla H\right)$$

and so

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{bmatrix} = \widehat{P} \times \nabla H = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix}.$$

Hence, we get the following (scalar) equations of motion

$$\dot{P}_1 = \frac{\partial H}{\partial p_3} P_2 - \frac{\partial H}{\partial p_2} P_3 \tag{17}$$

$$\dot{P}_2 = \frac{\partial H}{\partial p_1} P_3 - \frac{\partial H}{\partial p_3} P_1 \tag{18}$$

$$\dot{P}_3 = \frac{\partial H}{\partial p_2} P_1 - \frac{\partial H}{\partial p_1} P_2. \tag{19}$$

The reduced system has a Lax-form representation $\dot{P} = [P, \Omega]$, where

$$P = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}, \ \Omega = \begin{bmatrix} 0 & 0 & \frac{1}{c_2}P_2 \\ 0 & 0 & -\frac{1}{c_1}P_1 \\ -\frac{1}{c_2}P_2 & \frac{1}{c_1}P_1 & 0 \end{bmatrix}.$$

We have $\mathrm{tr}\,(P^2) = -2\left(P_1^2 + P_2^2 + P_3^2\right)$. Hence,

$$C = P_1^2 + P_2^2 + P_3^2 \tag{20}$$

is a Casimir function.

Proposition 3 *Given the left-invariant optimal control problem (13)-(14)-(15), the extremal control is*

$$\bar{u} = \frac{1}{c_1} P_1 \quad \textit{and} \quad \bar{u}_2 = \frac{1}{c_2} P_2$$

where $P_1, P_2 : [0,T] \to \mathbb{R}$ *(together with* P_3*) is a solution of the system of differential equations*

$$\dot{P}_1 = -\frac{1}{c_2} P_2 P_3 \tag{21}$$

$$\dot{P}_2 = \frac{1}{c_1} P_1 P_3 \tag{22}$$

$$\dot{P}_3 = \left(\frac{1}{c_2} - \frac{1}{c_1}\right) P_1 P_2. \tag{23}$$

Proof : The reduced Hamiltonian (on $\mathfrak{so}(3)^* = \mathbb{R}^3_\wedge$) is

$$H = \frac{1}{2c_1} P_1^2 + \frac{1}{2c_2} P_2^2. \tag{24}$$

The result follows from **Proposition 1** and (17)-(18)-(19).

It follows that the extremal trajectories (i.e., the solution curves of the reduced Hamilton equations) are the intersections of the circular cylinders $\frac{1}{c_1} P_1^2 + \frac{1}{c_2} P_2^2 = 2H$ and the spheres $P_1^2 + P_2^2 + P_3^2 = C$.

C. Integration by Jacobi Elliptic Functions

The Jacobi elliptic functions are inverses of elliptic integrals. Given a number $k \in [0,1]$, the function $F(\varphi, k) = \int_0^\varphi \frac{dt}{\sqrt{1-k^2\sin^2 t}}$ is called an (incomplete) *elliptic integral* of the first kind. The parameter k is known as the modulus. The inverse function $\mathrm{am}(\cdot, k) = F(\cdot, k)^{-1}$ is called the amplitude, from which the basic *Jacobi elliptic functions* are derived :

$$\begin{aligned} \mathrm{sn}(x,k) &= \sin \mathrm{am}(x,k) \quad \text{(sine amplitude)} \\ \mathrm{cn}(x,k) &= \cos \mathrm{am}(x,k) \quad \text{(cosine amplitude)} \\ \mathrm{dn}(x,k) &= \sqrt{1 - k^2 \sin^2 \mathrm{am}(x,k)} \quad \text{(delta amplitude)}. \end{aligned}$$

(For the degenerate cases $k = 0$ and $k = 1$, we recover the circular functions and the hyperbolic functions, respectively.) Historically, these functions were discovered as inverses of elliptic integrals. Nine other elliptic functions are defined by taking reciprocals and quotients; in particular, we get $\mathrm{nd}(\cdot, \mathrm{k}) = \frac{1}{\mathrm{dn}(\cdot,k)}$ and $\mathrm{sd}(\cdot, k) = \frac{\mathrm{sn}(\cdot,k)}{\mathrm{dn}(\cdot,k)}$.

Simple elliptic integrals can be expressed in terms of the appropriate inverse functions. Specifically, the following four formulas hold true for $b \le x \le a$ and $0 \le x \le b$, respectively (see [2] or [18]) :

$$\int_b^x \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a} \mathrm{nd}^{-1}\left(\frac{x}{b}, \frac{\sqrt{a^2-b^2}}{a}\right) \tag{25}$$

$$\int_x^a \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a} \mathrm{dn}^{-1}\left(\frac{x}{a}, \frac{\sqrt{a^2-b^2}}{a}\right) \tag{26}$$

$$\int_0^x \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{r} \mathrm{sd}^{-1}\left(\frac{rx}{ab}, \frac{b}{r}\right) \tag{27}$$

$$\int_x^b \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{r} \mathrm{cn}^{-1}\left(\frac{x}{b}, \frac{b}{r}\right) \tag{28}$$

where $r = \sqrt{a^2+b^2}$. When $c_1 = c_2$, a straightforward computation gives explicit formulas in terms of circular functions (cf. [22], [27]).

Proposition 4 *When* $c_1 = c_2 = c$*, the reduced Hamilton equations (21)-(22)-(23) have the solutions*

$$\begin{aligned} P_1(t) &= P_1(0)\cos\left(\frac{1}{c}P_3(0)t\right) - P_2(0)\sin\left(\frac{1}{c}P_3(0)t\right) \\ P_2(t) &= P_1(0)\sin\left(\frac{1}{c}P_3(0)t\right) + P_2(0)\cos\left(\frac{1}{c}P_3(0)t\right) \\ P_3(t) &= P_3(0). \end{aligned}$$

The generic case $c_1 \neq c_2$ requires elliptic functions.

Proposition 5 *When* $c_1 \neq c_2$*, the reduced Hamilton equations (21)-(22)-(23) can be explicitly integrated by Jacobi elliptic functions. More precisely, we have*

$$\begin{aligned} P_1(t) &= \pm\sqrt{\frac{c_1}{c_1 - c_2}\left(C - 2c_2 H - P_3^2(t)\right)} \\ P_2(t) &= \pm\sqrt{\frac{c_2}{c_2 - c_1}\left(C - 2c_1 H - P_3^2(t)\right)} \end{aligned}$$

and (i) if $0 < (c_1 - c_2)P_2^2 < c_2 P_3^2$*, then*

$$P_3(t) = \sqrt{C - 2c_1 H} \cdot \mathrm{nd}\left(\sqrt{\frac{C-2c_2H}{c_1c_2}}\, t,\ \frac{\sqrt{2(c_1-c_2)H}}{\sqrt{C-2c_2H}}\right)$$

or

$$P_3(t) = \sqrt{C - 2c_2 H} \cdot \mathrm{dn}\left(\sqrt{\frac{C-2c_2H}{c_1c_2}}\, t,\ \frac{\sqrt{2(c_1-c_2)H}}{\sqrt{C-2c_2H}}\right);$$

(ii) if $c_2 P_3^2 < (c_1 - c_2)P_2^2$*, then*

$$P_3(t) = \frac{\sqrt{(2c_1H - C)(C - 2c_2H)}}{\sqrt{2(c_1-c_2)H}} \cdot \\ \cdot \mathrm{sd}\left(\sqrt{\frac{2H(c_1-c_2)}{c_1c_2}}\, t,\ \frac{\sqrt{C-2c_2H}}{\sqrt{2(c_1-c_2)H}}\right)$$

or

$$P_3(t) = \sqrt{C - 2c_2 H} \cdot \mathrm{cn}\left(\sqrt{\frac{2(c_1-c_2)H}{c_1c_2}}\, t,\ \frac{\sqrt{C-2c_2H}}{\sqrt{2(c_1-c_2)H}}\right);$$

(iii) if $0 < (c_2 - c_1)P_1^2 < c_1 P_3^2$*, then*

$$P_3(t) = \sqrt{C - 2c_2 H} \cdot \mathrm{nd}\left(\sqrt{\frac{C-2c_1H}{c_1c_2}}\, t,\ \frac{\sqrt{2(c_2-c_1)H}}{\sqrt{C-2c_1H}}\right)$$

or

$$P_3(t) = \sqrt{C - 2c_1 H} \cdot \mathrm{dn}\left(\sqrt{\frac{C-2c_1H}{c_1c_2}}\, t,\ \frac{\sqrt{2(c_2-c_1)H}}{\sqrt{C-2c_1H}}\right);$$

(iv) if $c_1 P_3^2 < (c_2 - c_1)P_1^2$*, then*

$$P_3(t) = \frac{\sqrt{(2c_2H - C)(C - 2c_1H)}}{\sqrt{2(c_2-c_1)H}} \cdot \\ \cdot \mathrm{sd}\left(\sqrt{\frac{2(c_2-c_1)H}{c_1c_2}}\, t,\ \frac{\sqrt{C-2c_1H}}{\sqrt{2(c_2-c_1)H}}\right)$$

or

$$P_3(t) = \sqrt{C - 2c_1 H} \cdot \mathrm{cn}\left(\sqrt{\frac{2(c_2-c_1)H}{c_1c_2}}\, t,\ \frac{\sqrt{C-2c_1H}}{\sqrt{2(c_2-c_1)H}}\right).$$

Proof : The reduced Hamiltonian (24) and the Casimir function (20) are constants of motion. From $P_1^2 = \frac{c_1}{c_1-c_2}\left(C - 2c_2H - P_3^2\right)$ and $P_2^2 = \frac{c_2}{c_2-c_1}\left(C - 2c_1H - P_3^2\right)$ we get $\dot{P}_3^2 = \left(\frac{c_1-c_2}{c_1c_2}\right)^2 P_1^2P_2^2$ and so

$$\dot{P}_3^2 = -\frac{1}{c_1c_2}\left(C - 2c_2H - P_3^2\right)\left(C - 2c_1H - P_3^2\right). \quad (29)$$

The right-hand side of this equation can be written in the following four ways

$$\frac{1}{c_1c_2}\left(C - 2c_2H - P_3^2\right)\left(P_3^2 - C + 2c_1H\right) \quad (30)$$

$$\frac{1}{c_1c_2}\left(C - 2c_1H - P_3^2\right)\left(P_3^2 - C + 2c_2H\right) \quad (31)$$

$$\frac{1}{c_1c_2}\left(2c_2H - C + P_3^2\right)\left(C - 2c_1H - P_3^2\right) \quad (32)$$

$$\frac{1}{c_1c_2}\left(2c_1H - C + P_3^2\right)\left(C - 2c_2H - P_3^2\right) \quad (33)$$

(so that the constant in front be positive). The first case corresponds to the elliptic integrals (25) and (26), where

$$a^2 = C - 2c_2H > 0 \quad \text{and} \quad b^2 = C - 2c_1H > 0.$$

Notice that $a^2 > 0 \iff (c_2 - c_1)P_1^2 < c_1P_3^2$ and $b^2 > 0 \iff (c_1-c_2)P_2^2 < c_2P_3^2$. (Whenever $c_2 < c_1$, the former condition is always satisfied.) Now, straightforward algebraic manipulation and integration yield explicit expressions (in terms of Jacobi elliptic functions) for the solutions of the (first-order) ordinary differential equation (29). We get

$$P_3(t) = \sqrt{C - 2c_1H}\,\cdot\,\mathrm{nd}\left(\sqrt{\frac{C-2c_2H}{c_1c_2}}\,t,\ \frac{\sqrt{2(c_1-c_2)H}}{\sqrt{C-2c_2H}}\right)$$

(corresponding to the integral (25)) or

$$P_3(t) = \sqrt{C - 2c_2H}\,\cdot\,\mathrm{dn}\left(\sqrt{\frac{C-2c_2H}{c_1c_2}}\,t,\ \frac{\sqrt{2(c_1-c_2)H}}{\sqrt{C-2c_2H}}\right)$$

(corresponding to the integral (26)). Case (ii) follows from (i). Similarly, the third and the fourth cases (corresponding to the elliptic integrals (27) and (28)) can be derived.

VI. FINAL REMARK

Investigations of other invariant optimal control problems on various matrix Lie groups of low dimnesion are in progress. In particular, it is expected that explicit integration of the reduced Hamilton equations will be possible in all these cases. This study will appear elsewhere.

REFERENCES

[1] A.A. Agrachev and Y.L. Sachkov, *Control Theory from the Geometric Viewpoint*, Springer-Verlag, Berlin; 2004.

[2] J.V. Armitage and W.F. Eberlein, *Elliptic Functions*, Cambridge University Press, Cambridge; 2006.

[3] J. Biggs and W. Holderbaum, "Integrable Hamiltonian Systems Defined on the Lie Group $SO(3)$ and $SU(2)$: an Application to the Attitude Control of a Spacecraft", *in Fifth Symposium on Automatic Control*, Wismar, Germany, 2008.

[4] Biggs, J., Holderbaum, W. and Jurdjevic, V., "Singularities of Optimal Control Problems on Some 6 Dimensional Lie Groups", *IEEE Transactions on Automatic Control*, vol. 52, 2007, pp. 1027-1038.

[5] A.M. Bloch, *Nonholonomic Mechanics and Control*, Springer-Verlag, Berlin; 2003.

[6] R.W. Brockett, System Theory on Group Manifolds and Coset Spaces, *SIAM J. Control*, vol. 10, 1972, pp. 265-284.

[7] D. D'Alessandro and M. Dahleh, Optimal Control of Two-Level Quantum Systems, *IEEE Transactions on Automatic Control*, vol. 46, 2001, pp. 866-876.

[8] A.T. Fomenko and V.V. Trofimov, *Integrable Systems on Lie Algebras and Symmetric Spaces*, Gordon and Breach, New York; 1988.

[9] S. Goyal, N.C. Perkins and C.L. Lee, Nonlinear Dynamics and Loop Formation in Kirchhoff Rods with Implications to the Mechanics of DNA and Cables, *J. Comp. Phys.*, vol. 209, 2005, pp. 371-389.

[10] D.D. Holm, *Geometric Mechanics* (Part I: Dynamics and Symmetry; Part II: Rotating, Translating and Rolling), World Scientific, Singapore; 2008.

[11] C. Ivanescu and A. Savu, The Kowalewski Top as a Reduction of a Hamiltonian System on $\mathfrak{sp}(4,\mathbb{R})^*$, *Proc. Amer. Math. Soc.*, vol. 131, 2002, pp. 607-618.

[12] V. Jurdjevic, Non-Euclidean Elastica, *Amer J. Math.*, vol. 117, 1995, pp. 93-124.

[13] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, Cambridge; 1997.

[14] V. Jurdjevic, Hamiltonian Point of View of Non-Euclidean Geometry and Elliptic Functions, *Syst. Control Lett.*, vol. 43, 2001, pp. 25-41.

[15] V. Jurdjevic and H.J. Sussmann, Control Systems on Lie Groups, *J. Diff. Equations*, vol. 12, 1972, pp. 313-329.

[16] E. Kirilova and K. Spindler, "Optimal Control of a Vehicle during Parking", *in IFAC Symposium on Nonlinear Control Systems*, Stuttgart, Germany, 2004.

[17] P.S. Krishnaprasad, "Optimal Control and Poisson Reduction", *in IEEE Conference on Decision & Control* (Workshop on Mechanics, Holonomy and Control), San Antonio, TX, 1993.

[18] D.F. Lawden, *Elliptic Functions and Applications*, Springer-Verlag, New York; 1989.

[19] N.E. Leonard and P.S. Krishnaprasad, Motion Control of Drift-Free, Left-Invariant Systems on Lie Groups, *IEEE Transactions on Automatic Control*, vol. 40, 1995, pp. 1539-1554.

[20] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Second Edition, Springer-Verlag, Berlin; 1999.

[21] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Springer-Verlag, New York; 2000.

[22] M. Puta, Stability and Control in Spacecraft Dynamics, *J. Lie Theory*, vol. 7, 1997, pp. 269-278.

[23] M. Puta and C. Lazureanu, "Integration of the Rigid Body Equations with Quadratic Controls", *in Conference on Differential Geometry & Applications*, Brno, Czech Republic, 1998, pp. 645-652.

[24] C.C. Remsing, Optimal Control and Hamilton-Poisson Formalism, *Int. J. Pure Appl. Math.*, vol. 59, 2010, pp. 11-17.

[25] C.C. Remsing, Control and Integrability on $\mathsf{SO}(3)$, *in International Conference of Applied and Engineering Mathematics*, London, U.K., 2010.

[26] C.C. Remsing, On a Class of Optimal Control Problems, (to appear).

[27] S. Sastry, "Optimal Control on Lie Groups"', *in International Conference on Industrial and Applied Mathematics*, Hamburg, Germany, 1995.

[28] S. Sastry and R. Montgomery, "The Structure of Optimal Controls for a Steering Problem", *in IFAC Symposium on Nonlinear Control Systems*, Bordeaux, France, 1992, pp. 135-140.

[29] K. Spindler, Optimal Attitude Control of a Rigid Body, *Appl. Math. Optim.*, vol. 34, 1996, pp. 79-90.

[30] K. Spindler, Optimal Control on Lie Groups with Applications to Attitude Control, *Math. Control Signals Syst.*, vol. 11, 1998, pp. 197-219.

[31] G. Walsh, A. Sarti and S. Sastry, "Algorithms for Steering on the Group of Rotations", *in American Control Conference*, San Francisco, CA, 1993, pp. 1312-1316.

[32] G. Walsh, R. Montgomery and S. Sastry, "Optimal Path Planning on Matrix Lie Groups", *in IEEE Conference on Decision & Control*, Lake Buena Vista, FL, 1994, pp. 1258-1263.