System parametrization using affine derivative systems

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***Abstract*— In this paper we discuss the constructive calculation of a flat system parametrization for nonlinear implicit control systems which are quasilinear in the derivative coordinates. The proposed scheme is based on the successive reduction of derivative variables and the elimination of non-derivative variables. A key challenge is the derivation of a procedure that maintains the quasi-linearity also in the elimination steps, since this is beneficial for the subsequent reductions. Two examples demonstrate the applicability of the suggested methods.**

I. INTRODUCTION

Control systems that allow for a flat parametrization play a prominent role in the theory and the application with respect to control system analysis and synthesis. The concept of flatness has its origins in the work of M. Fliess and coworkers, see [1] and references therein. Nowadays a big amount of literature is available which is dealing with the concept of flatness using various different mathematical approaches and dealing with several system classes including systems described by partial differential equations, see beside many others [2], [3], [4], [5], [6], [7]. It is well known that for linear MIMO (multi-input multi-output) systems the concept of flatness is equivalent to controllability, see e.g. [8], and to obtain a flat parametrization it is convenient to derive so-called normal forms where the system parametrization can be read off. The derivation of the normal form is based on state and input transformations as can be found for instance in [9] and references therein. For nonlinear control systems it is in general not clear how to construct a normal form concerning the concept of flatness, unless the systems enjoys the property to be input to state linearizable, see e.g. [10], [11].

The goal of this paper is to formally apply the scheme which is used to derive a normal form in the linear scenario based on successive coordinate transformations also in the nonlinear case. We present a strategy that is based on the successive reduction and elimination of several system variables as described in [12] but with the difference that in the present work we introduce a subclass of nonlinear implicit ordinary differential equations, which are quasilinear with respect to the derivative coordinates and termed affine derivative (AD) systems, which turn out to have desirable properties with respect to the elimination and reduction process. Based on the affine derivative form we are able to discuss also a dual system representation using Pfaffian systems, which are used to proof how a structure preserving elimination and reduction can be performed. To motivate the general principle derived in [12] we want to demonstrate this machinery using MIMO control systems and then try to copy as much as possible to apply these ideas in the nonlinear scenario. The key observation will be to classify coordinate transformations that possess the property that in the transformed coordinates several quantities become non-derivative variables and can be eliminated. The elimination process in general destroys quasi-linearity with respect to the derivative variables and therefore we will exploit again the affine derivative structure to analyze this problem.

The paper is organized as follows. In section II the case of linear control systems is analyzed to motivate for the proposed elimination and reduction algorithm. In the third section we define affine derivative systems and discuss the concept of the reduction of derivative variables in this context. Section IV presents two examples, an academic one and the well known VTOL example where we show that the well known system parametrization can be derived systematically using the proposed methods.

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II. THE CASE OF LINEAR CONTROL SYSTEMS

Let us consider the class of linear control systems, described by ordinary differential equations of the form

$$\Sigma_0 \quad : \quad x_t^\alpha = A_\beta^\alpha x^\beta + B_\xi^\alpha u^\xi \tag{1}$$

with $x \in \mathcal{X}$, $A : \mathcal{X} \to \mathcal{T}(\mathcal{X})$, $B : \mathcal{U} \to \mathcal{T}(\mathcal{X})$ where we assume that $\dim(\mathrm{im}(B)) = \dim(\mathcal{U})$. To derive a normal form such that the system parametrization can be read off, we apply an algorithm based on the successive elimination of non-derivative variables (inputs) and elimination of derivative variables (states), as already discussed in [12]. The elimination of the inputs yields a system

$$M_\beta^\alpha x_t^\beta \quad = \quad N_\beta^\alpha x^\beta \tag{2}$$

with $M_\beta^\alpha B_\xi^\beta = 0_\xi^\alpha$, $N_\beta^\alpha = M_\rho^\alpha A_\beta^\rho$. A regular transformation of the type

$$x^\alpha \quad = \quad V_\tau^\alpha w^\tau + R_{\bar{\beta}}^\alpha \bar{x}^{\bar{\beta}} \tag{3}$$

such that $M_\beta^\alpha V_\tau^\beta = 0$ is met will be crucial for the following construction. From (2) we obtain

$$M_{\bar{\rho}}^\alpha \bar{x}_t^{\bar{\rho}} = N_{\bar{\rho}}^\alpha \bar{x}^{\bar{\rho}} + N_\tau^\alpha w^\tau$$

with $M_{\bar{\rho}}^\alpha = M_\beta^\alpha R_{\bar{\rho}}^\beta$, $N_{\bar{\rho}}^\alpha = N_\beta^\alpha R_{\bar{\rho}}^\beta$ and $N_\tau^\alpha = N_\beta^\alpha V_\tau^\beta$ such that the coordinates w can be eliminated. This leads to a system decomposition of the form

$$S_\alpha^i M_{\bar{\rho}}^\alpha \bar{x}_t^{\bar{\rho}} \quad = \quad S_\alpha^i N_{\bar{\rho}}^\alpha \bar{x}^{\bar{\rho}} \tag{4}$$

$$W_\alpha^j M_{\bar{\rho}}^\alpha \bar{x}_t^{\bar{\rho}} \quad = \quad W_\alpha^j N_{\bar{\rho}}^\alpha \bar{x}^{\bar{\rho}} + W_\alpha^j N_\tau^\alpha w^\tau , \tag{5}$$

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where $[S^i_\alpha]$ and $[W^j_\alpha]$ have maximal rank, which enjoys the following properties. Given a solution of the system (4) a solution of the subsystem (5) can be derived in a straightforward manner. Then the whole process can be repeated with the system (4). The case where $\dim(\ker(W^i_\alpha N^\alpha_\tau) > 0$ is met leads to additional variables that can be chosen freely. Two remarks are appropriate at this stage.

Remark 1: The key feature of this procedure is the transformation (3) which guarantees that several variables appear as non-differentiated quantities. This is achieved by the choice of $[V^\alpha_\tau] \in \ker([\mathrm{M}^\alpha_\beta])$. Obviously, the elimination process is structure preserving in the linear scenario, in the sense that we derive a sequence of implicit linear systems.

Remark 2: Based on the form of the equations as in (2) one can construct a nonlinear analogue to the presented elimination theory. This leads to the class of so called affine derivative systems, which will be analyzed in the forthcoming part of this paper. However, in the nonlinear scenario the question if elimination is structure preserving will be the key task to be tackled.
This short motivation concerning the basic idea of the algorithm will be illustrated by a short example before we turn to the nonlinear scenario.

Example 1: Let us consider the system

$$x_t = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} u$$

which is controllable. Eliminating the inputs leads to an implicit system of the form

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} x_t = \begin{bmatrix} -2 & 2 & -3 & 2 \\ -1 & 1 & -1 & 1 \end{bmatrix} x.$$

Let us apply a transformation of the type, see (3)

$$x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \qquad (6)$$

which leads to a system

$$M \begin{bmatrix} \bar{x}^1_t \\ \bar{x}^2_t \end{bmatrix} = N \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}$$

with

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} -2 & -3 \\ -1 & -1 \end{bmatrix}.$$

Eliminating the quantities w^1 and w^2 we obtain

$$\bar{x}^2_t = \bar{x}^1 + \bar{x}^2$$

and the remaining system follows to

$$\bar{x}^1_t = -2\bar{x}^1 - 3\bar{x}^2 - w^2$$

from which we conclude that $\bar{x}^2$ and w^1 are the flat outputs since form $\bar{x}^2$ we are able to compute $\bar{x}^1$ and successively w^2. It is obvious that in this example $\dim(\ker(W^i_\alpha N^\alpha_\tau) > 0$ is met which leads to the fact that w^1 can be chosen freely. From the regular transformation (6) we can compute $x(t)$ easily and $u(t)$ follows from the system equations in a straightforward manner.

III. Affine Derivative System and Elimination

The scope of this section is to analyze nonlinear systems in the spirit as we did it for linear ones in the last part. We will try to copy as much as possible of the presented scheme and we will see that in this context quasi-linearity of the derivative coordinates will play a prominent role. Two questions arise quite naturally which make the nonlinear scenario much more complicated than the linear one. Firstly, which transformation plays the role of (3) and secondly how can the elimination process be performed in the nonlinear case. To start with, we consider implicit nonlinear systems of the form

$$S^1 : f^i(t, z^\alpha, z^\alpha_t) = 0\,, \quad i = 1, \ldots, n_e \qquad (7)$$

which are modeled on a bundle $\pi : \mathcal{E} \to \mathcal{T}$, $(t, z^\alpha) \to (t)$, with $\dim(\mathcal{E}) = n_z + 1$. We assume that the equations (7) form a regular submanifold of $\mathcal{J}^1(\mathcal{E})$, where the first jet bundle $\pi^1_0 : \mathcal{J}^1(\mathcal{E}) \to \mathcal{E}$, $(t, z^\alpha, z^\alpha_t) \to (t, z^\alpha)$ can be seen as a container for first prolongations of sections of the bundle π. In the forthcoming a special case of (7) is of importance, namely systems in AD-structure.

Definition 1: A nonlinear implicit system is called in AD (affine derivative) form, if it can be written as

$$M^i_\alpha(z, t) z^\alpha_t = N^i(z, t)$$

with $n_z > n_e$ and $i = 1, \ldots, n_e$.

Remark 3: Also time-invariant systems, will be modeled on the bundle $\pi : \mathcal{E} \to \mathcal{T}$ as described before. Then AD-systems take the form

$$M^i_\alpha(z) z^\alpha_t = N^i(z).$$

For the presented theory it makes no difference if the systems are time-variant or time-invariant. The only restriction is that we do not consider time re-parametrization. For simplicity we will analyze only time-invariant systems in the sequel but we model them on the bundle $\pi : \mathcal{E} \to \mathcal{T}$, $(t, z^\alpha) \to (t)$, with $\dim(\mathcal{E}) = n_z + 1$.

A. The construction of the transformation

The central point, how to transform coordinates such that certain coordinates appear as non-differentiated has been treated in [12] and we will repeat the main construction here, such that we are able to see the advantage of the AD form. Let us consider a transformation, of the form

$$z^\alpha = \phi^\alpha_\tau(\bar{z}^{\bar{\alpha}}) \circ \psi_{tr} = \varphi(\bar{z}, \tau), \qquad (8)$$

with a flow ϕ_τ to be specified and where ψ_{tr} is chosen in a way that the map $z = \varphi(\bar{z}, \tau)$ is a diffeomorphism with inverse $(\bar{z}, \tau) = \varphi^{-1}(z)$.

Lemma 1: Given the system (7) and the transformation (8). The transformed system (7) is independent of the jet variable τ_t iff the solutions of

$$\partial^t_\alpha f^i \hat{v}^\alpha_\tau \stackrel{S^1}{=} 0 \qquad (9)$$

are such that $v_\tau = \rho(\hat{v}_\tau)$ generates the flow ϕ_τ on $\mathcal{E}$, i.e.

$$\partial_\tau \phi_\tau = v_\tau \circ \phi_\tau$$

is met, where the map $\rho : \mathcal{V}(\mathcal{J}^1(\mathcal{E})) \to \mathcal{V}(\mathcal{E})$ is based on the isomorphism $\mathcal{V}(\mathcal{J}^1(\mathcal{E})) \approx \mathcal{V}(\mathcal{E})$, i.e. $\partial_\alpha^t \to \partial_\alpha$. Here the vertical tangent bundle of $\mathcal{J}^1(\mathcal{E})$ is regarded with respect to the fibration $\mathcal{J}^1(\mathcal{E}) \to \mathcal{E}$.

Proof: The proof of this lemma rests upon the direct coordinate calculation

$$\partial_\tau^t (f^i \circ j^1\varphi) = (\partial_\tau^t (f^i \circ j^1\phi_\tau)) \circ \psi_{tr} \quad \overset{\mathcal{S}^1}{=} \quad 0$$

and

$$\left((\partial_\alpha^t f^i) \circ j^1\phi_\tau\right) \partial_\tau \phi_\tau^\alpha = (\partial_\alpha^t f^i \underbrace{\partial_\tau \phi_\tau^\alpha \circ \phi_\tau^{-1}}_{\hat{v}_\tau^\alpha}) \circ j^1 \phi_\tau \overset{\mathcal{S}^1}{=} 0,$$

which deliverers the desired result. ■

The crucial point is that $v = \rho(\hat{v})$ has to generate a flow on $\mathcal{E}$, which is only possible if no derivative variables are involved in the components of v. It is readily observed that this cannot be guaranteed in general by solving the expression (9) since $\partial_\alpha^t f^i$ has no prescribed structure. Here the class of AD systems comes into the play quite naturally since AD systems automatically generate these vector fields that are of the desired shape by construction. This can be seen easily because for AD-systems the equation (9) takes the form

$$M_\alpha^i(z)\hat{v}_\tau^\alpha = 0. \tag{10}$$

Remark 4: For linear implicit systems of the form

$$M_\alpha^i z_t^\alpha = N_\alpha^i z^\alpha$$

the criterion gives $M_\alpha^i \hat{v}_\tau^\alpha$ which coincides with the results of the previous section and it is readily observed that of course in the linear scenario $v = \rho(\hat{v})$ always generates a flow with the desired properties leading to a coordinate transformation of the form $z^\alpha = v_\tau^\alpha w^\tau + R_{\bar{\beta}}^\alpha \bar{z}^{\bar{\beta}}$ as discussed.

One problem however remains in the nonlinear scenario, which is concerned with the elimination of non-derivative coordinates.

B. The elimination process

It is by far not assured that elimination of non-derivative variables is preserving the AD-structure. However, this property is extremely beneficial for successive reduction steps. Therefore it is of interest to preserve the AD structure also while performing the eliminations. To achieve this desired behavior we suggest to consider a kind of dual form to AD systems. One key property of AD-systems is that they allow for a description as a Pfaffian system of the form

$$\omega^i = M_\alpha^i(z)\mathrm{d}z^\alpha - N^i(z)\mathrm{d}t$$

with $\Delta_0 = \mathrm{span}\{\omega^i\}$ and $\omega^i \in \mathcal{T}^*(\mathcal{E})$. The elimination process based on the system representation Δ_0 rests on the construction of the annihilator $\Delta_0^\perp$ (modulo $\mathrm{d}t$) as well as on the criterion

$$v(\Delta_1) \subset \Delta_1 \tag{11}$$

with $\Delta_1 \subseteq \Delta_0$ and $v \in \Delta_0^\perp$, where $v(\xi)$ denotes the Lie-derivative of the form ξ in direction v. If a solution of (11) can be constructed, it is assured that the elimination of the variable corresponding to the flow parameter of v leads again to a system in affine derivative structure.

The procedure is the following: We have to compute $\Delta_0^\perp$ whose elements v_κ meet $M_\alpha^i(z)v_\kappa^\alpha = 0$ and then if possible choose elements of $\Delta_0^\perp$ such that (11) can be fulfilled. This means we combine the desire to reduce derivative coordinates with the goal to preserve the affine derivative structure after we have eliminated the variables that appear non-differentiated.

Remark 5: It should be observed that $v(\Delta_1) \subset \Delta_0$ is a necessary condition which has to hold for (11) and is much simpler to check as (11) and provides candidates to be considered subsequently.

C. A constructive algorithm

In this section we will combine the reduction and the elimination process described above. The desire to obtain a sequence of affine derivative systems is mainly motivated on the fact that AD systems allow for the construction of the transformation (8) in a simple manner, the only point left to discuss is the case when non-derivative coordinates appear that can not be eliminated without destroying the affine derivative structure. In the following we make use of two well known relations.

Fact 1: Given a manifold $\mathcal{M}$, a covectorfield $\omega \in \mathcal{T}^*(\mathcal{M})$ as well as vectorfields $v, w \in \mathcal{T}(\mathcal{M})$ it holds that

$$v(w\rfloor\omega) = w\rfloor v(\omega) + [v,w]\rfloor\omega \tag{12}$$

where $[\cdot,\cdot]$ denotes the Lie-bracket and $\rfloor$ the natural contraction.

Fact 2: Let $\phi_t^f(\cdot)$ and $\phi_s^g(\cdot)$ denote the flows generated by f and g on a manifold $\mathcal{M}$, respectively, then it holds that

$$\partial_s(\phi_t^f \circ \phi_s^g \circ \phi_{-t}^f)\Big|_{s=0} = \sum_{k=0}^{\infty} ad_f^k(g)\frac{t^k}{k!} \tag{13}$$

with

$$ad_f^k(g) = [ad_f^{k-1}(g), f], \quad ad_f^0(g) = g.$$

We study AD systems of the type

$$M_\alpha^i(z,\tilde{z})z_t^\alpha = N^i(z,\tilde{z}) \tag{14}$$

with a minimal number of equations. It should be noted that the coordinates $\tilde{z}$ (possibly the empty set) are non-derivative variables but they could not have been eliminated without destroying the affine derivative structure. Based on the system description we can apply the reduction and elimination scheme, but care has to be taken since it has to be guaranteed that the derivative variables are reduced in each step, where the role of $\tilde{z}$ needs further attention. The critical point is the fact, that the coordinate transformation used to reduce the derivative coordinates concerning z may generate derivatives of $\tilde{z}$, i.e. no reduction could have been achieved.

Obviously, we can write the system (14) as

$$\Delta_0 : M_\alpha^i(z,\tilde{z})\mathrm{d}z^\alpha - N^i(z,\tilde{z})\mathrm{d}t$$

where its annihilator (modulo $\mathrm{d}t$) meets

$$\Delta_0^\perp = \mathrm{span}\{\partial_{\tilde{z}}\} \oplus \mathrm{span}\{v_\kappa | M^i_\alpha v^\alpha_\kappa = 0\}. \tag{15}$$

Based on (15) of the system Δ_0 we are interested in the annihilator of the next system in the sequence Δ_1.

Lemma 2: Suppose we find a solution of $v(\Delta_1) \subset \Delta_1$ with $\Delta_1 \subseteq \Delta_0$ and $v \in \Delta_0^\perp$ then

$$v,\, ad^k_v(\partial_{\tilde{z}}) \in \Delta_1^\perp\,, \quad k = 0, \ldots \tag{16}$$

is met.

Proof: The proof of the lemma is based on a successive application of (12) since it is easily seen that from

$$v(\partial_{\tilde{z}} \rfloor \omega) = \partial_{\tilde{z}} \rfloor v(\omega) + [v, \partial_{\tilde{z}}] \rfloor \omega\,, \quad \omega \in \Delta_1$$

we have $[v, \partial_{\tilde{z}}] \in \Delta_1^\perp$ together with $\partial_{\tilde{z}} \in \Delta_1^\perp$. Then we repeat the procedure with

$$v([v, \partial_{\tilde{z}}] \rfloor \omega) = [v, \partial_{\tilde{z}}] \rfloor v(\omega) + [v, [v, \partial_{\tilde{z}}]] \rfloor \omega\,, \quad \omega \in \Delta_1$$

which leads to $[v, [v, \partial_{\tilde{z}}]] \in \Delta_1^\perp$ and so on. ■

We have proved that based on the annihilator of the system Δ_0 given in (15), the structure of $\Delta_1^\perp$ is given by (16) provided a solution of $v(\Delta_1) \subset \Delta_1$ can be found.

Remark 6: It should be noted that

$$\partial_{\tilde{z}}(\Delta_1) \nsubseteq \Delta_1$$

since otherwise $\tilde{z}$ could be eliminated preserving the affine derivative structure.

The reduction is now performed by choosing a transformation of the type $(z, \tilde{z}) = \varphi(\tilde{z}, \bar{z}, \tau)$

$$\begin{aligned} z^\alpha &= \phi^\alpha(\tilde{z}^{\tilde{\alpha}}, \bar{z}^{\bar{\alpha}}, \tau) \circ \psi_{tr} \\ \tilde{z}^{\tilde{\alpha}} &= \phi^{\tilde{\alpha}}(\tilde{z}^{\tilde{\alpha}}) = \tilde{z}^{\tilde{\alpha}} \end{aligned} \tag{17}$$

such that ϕ^α is generated by v from (15) where in the new coordinates $(\tilde{z}, \bar{z}, \tau)$ it is assured that τ can be eliminated. The question is now if $\tilde{z}$ which is not affected by the transformation is again a non-derivative variable also in the new coordinates. This is the case if

$$\partial_{\tilde{z}} \rfloor \phi^*(\Delta_1) = 0 \tag{18}$$

holds.

Remark 7: It should be noted that in (18) the fields $\partial_{\tilde{z}}$ are to be considered in the coordinate system $(\tilde{z}, \bar{z}, \tau)$. Furthermore it can be stated that $\partial_{\tilde{z}} \rfloor (\phi \circ \psi_{tr})^*(\Delta_1) = 0$ is equivalent to the criteria as in (18) which is easier to handle since it is based on flows.

Let us now turn to the key observation concerning the interpretation of the non-derivative variables $\tilde{z}$.

Proposition 1: Given a system (14) where a solution $v(\Delta_1) \subset \Delta_1$ with $\Delta_1 \subseteq \Delta_0$ and $v \in \Delta_0^\perp$ has been found, then $\partial_{\tilde{z}} \rfloor \phi^*(\Delta_1) = 0$ holds, with ϕ from (17).

Proof: It holds that

$$\partial_{\tilde{z}} \rfloor \phi^*(\Delta_1) = (\phi_* \partial_{\tilde{z}}) \circ \phi^{-1} \rfloor (\Delta_1) = \partial_{\tilde{z}}(\phi) \circ \phi^{-1} \rfloor (\Delta_1)$$

is met. From

$$\partial_{\tilde{z}}(\phi) \circ \phi^{-1} = \partial_s(\phi_\epsilon \circ \varphi_s^{\partial_{\tilde{z}}} \circ \phi_{-\epsilon})\big|_{s=0}$$

and

$$(\phi_* \partial_{\tilde{z}}) \circ \phi^{-1} \quad = \quad \sum_{k=0}^{\infty} ad^k_v(\partial_{\tilde{z}}) \frac{\epsilon^k}{k!}$$

where we used (13) it follows that $\partial_{\tilde{z}} \rfloor \phi^*(\Delta_1) = 0$ together with Lemma 2. ■

When we apply the transformation (17) to the system (14) we obtain a system structure of the type

$$\bar{M}^{\bar{i}}_{\bar{\alpha}}(\bar{z}, \tilde{z}) \bar{z}^{\bar{\alpha}}_t = \bar{N}^{\bar{i}}(\bar{z}, \tilde{z})$$

where we have proved that the dimension of the derivative variables is decreased, and where $\tilde{z}$ again appears non-differentiated, if the requirements of lemma 2 hold.

Remark 8: Of course it might happen that the set of the variables $\tilde{z}$ is empty. This is the case when all the non-derivative variables could have been eliminated preserving the affine derivative structure.

The scheme of successive elimination and reduction can now be repeated, if possible, until a system is derived, where the flat output can be read off. At this stage it is worth mentioning that the presented scheme is only constructive in the nonlinear scenario, but we will demonstrate in the next section that we can treat successfully examples belonging to the two cases from above (i.e. the case where all input variables can be eliminated and therefore the set $\tilde{z}$ is empty, and the general one).

IV. Examples

In this section we discuss the applicability of the proposed machinery using two examples. The first one is an academic example which is challenging since the elimination of all the control input destroys the AD-structure. The second example is the well known VTOL model, where we have to apply the algorithm successively to construct the system parametrization, which is well known from the literature, but here produced in a systematic fashion.

A. An academic one

Let us consider the following system in AD form

$$\begin{aligned} z^1_t &= z^4 \\ z^2_t &= z^5 \\ z^3_t &= \sqrt{z^4 z^5} \end{aligned}$$

where it is obvious that elimination of both inputs $u^1 = z^4$ and $u^2 = z^5$ does not maintain the AD-structure. To overcome this problem, we inspect the system written as a Pfaffian system

$$\begin{aligned} \rho^1 &: \ \mathrm{d}z^1 - z^4 \mathrm{d}t \\ \rho^2 &: \ \mathrm{d}z^2 - z^5 \mathrm{d}t \\ \rho^3 &: \ \mathrm{d}z^3 - \sqrt{z^4 z^5} \mathrm{d}t. \end{aligned}$$

We introduce $\Delta_0 = \mathrm{span}\{\rho^1, \rho^2, \rho^3\}$ as well as its annihilator $\Delta_0^\perp = \mathrm{span}\{\partial_4, \partial_5\}$ (where $\Delta_0^\perp$ is computed modulo $\mathrm{d}t$). It holds that we are able to fulfill $v_0(\Delta_1) \subset \Delta_1$ with $\Delta_1 \subseteq \Delta_0$ where the field v_0 is given as

$$v_0 = z^4 \partial_4 + z^5 \partial_5$$

and the codistribution Δ_1 reads as

$$\Delta_1 = \mathrm{span}\{\sqrt{z^4 z^5}\rho^1 - z^4\rho^3, \sqrt{z^4 z^5}\rho^2 - z^5\rho^3\}.$$

Obviously $\Delta_1 = \mathrm{span}\{\omega^1, \omega^2\}$ then follows to

$$\begin{array}{rcl} \omega^1 & : & \sqrt{z^4 z^5}\mathrm{d}z^1 - z^4\mathrm{d}z^3 \\ \omega^2 & : & \sqrt{z^4 z^5}\mathrm{d}z^2 - z^5\mathrm{d}z^3. \end{array}$$

It can be checked easily that v_0 generates the flow

$$\begin{array}{rcl} z^4 & = & e^\tau \tilde{z}^4 \\ z^5 & = & e^\tau \tilde{z}^5 \end{array}$$

and on the submanifold $\tilde{z}^5 = 1$ (this corresponds to the choice of ψ_{tr}) we obtain $z^4 = z^5\tilde{z}^4$. This leads us directly to the system

$$\begin{array}{rcl} z_t^1 & = & z^5\tilde{z}^4 \\ z_t^2 & = & z^5 \\ z_t^3 & = & z^5\sqrt{\tilde{z}^4} \end{array}$$

which can be rewritten as

$$\begin{bmatrix} z_t^1 \\ z_t^2 \\ z_t^3 \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{z}^4 \\ 1 \\ \sqrt{\tilde{z}^4} \end{bmatrix}}_{b(\tilde{z}^4)} z^5.$$

By construction the elimination of z^5 is now possible, preserving the AD-structure. Consequently, we derive a system of the form

$$\underbrace{\begin{bmatrix} 1 & -\tilde{z}^4 & 0 \\ 0 & -\sqrt{\tilde{z}^4} & 1 \end{bmatrix}}_{M(\tilde{z}^4)} \begin{bmatrix} z_t^1 \\ z_t^2 \\ z_t^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

which easily allows for solving the equation (9).

Remark 9: The system (19) follows immediately also based on calculations using the Pfaffian system Δ_1, since plugging in $z^4 = z^5\tilde{z}^4$ leads to

$$\begin{array}{rcl} \omega^1 & : & \sqrt{\tilde{z}^4}\mathrm{d}z^1 - \tilde{z}^4\mathrm{d}z^3 \\ \omega^2 & : & \sqrt{\tilde{z}^4}\mathrm{d}z^2 - \mathrm{d}z^3, \end{array}$$

where it is readily observed that $\tilde{z}^5$ is eliminated already.
We now continue our scheme based on the system (19) where we just need to inspect the solution of (10) which corresponds to

$$b(\tilde{z}^4) = v_1 = \tilde{z}^4\partial_1 + \partial_2 + \sqrt{\tilde{z}^4}\partial_3.$$

Therefore we have the flow

$$\begin{array}{rcl} z^1 & = & \tilde{z}^4\tau \\ z^2 & = & \tau + \bar{z}^2 \\ z^3 & = & \sqrt{\tilde{z}^4}\tau + \bar{z}^3 \end{array} \tag{20}$$

with $\bar{z}^1 = 0$ and the jet prolongation of the flow reads as

$$\begin{bmatrix} z_t^1 \\ z_t^2 \\ z_t^3 \end{bmatrix} = \begin{bmatrix} \tilde{z}_t^4\tau \\ \bar{z}_t^2 \\ \frac{1}{2\sqrt{\tilde{z}^4}}\tilde{z}_t^4\tau + \bar{z}_t^3 \end{bmatrix} + \underbrace{\begin{bmatrix} \tilde{z}^4 \\ 1 \\ \sqrt{\tilde{z}^4} \end{bmatrix}}_{b(\tilde{z}^4)} \tau_t.$$

Now it is easily seen that τ_t is eliminated, by construction and the following system remains

$$\begin{bmatrix} \tau & -\tilde{z}^4 & 0 \\ \frac{1}{2\sqrt{\tilde{z}^4}}\tau & -\sqrt{\tilde{z}^4} & 1 \end{bmatrix} \begin{bmatrix} \tilde{z}_t^4 \\ \bar{z}_t^2 \\ \bar{z}_t^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Elimination of τ leads to

$$\frac{1}{2}\sqrt{\tilde{z}^4}\bar{z}_t^2 = \bar{z}_t^3 \tag{21}$$

and so $h^1 = \bar{z}^2$ and $h^2 = \bar{z}^3$ are the flat outputs. From the remaining system

$$\tau\tilde{z}_t^4 = \tilde{z}^4\bar{z}_t^2$$

we are able to compute τ and the inverse of the transformation (20) allows for the calculation of the flat outputs in the original coordinates which reads as

$$h^1 = z^2 - z^1\frac{z^5}{z^4}, \quad h^2 = z^3 - z^1\sqrt{\frac{z^5}{z^4}}.$$

Remark 10: The fact that the system (21) is an affine derivative system is based on the observation, that

$$\Delta_2 : -dz^1 - \tilde{z}^4 dz^2 + 2\sqrt{\tilde{z}^4}dz^3, \quad \Delta_2 \subset \Delta_1$$

can be constructed with $v_1(\Delta_2) \subset \Delta_2$. Furthermore, it is assured by Proposition 1 that no derivative of $\tilde{z}^4$ appears in (21). Additionally, it can be checked that

$$[\partial_{\tilde{z}^4}, v_1] = \partial_1 + \frac{1}{2\sqrt{\tilde{z}^4}}\partial_3 \in \Delta_2^\perp$$

is met.

B. The VTOL example

Let us consider the well known VTOL example, see [8]

$$\begin{array}{rcl} x_t & = & v_x \\ v_{x,t} & = & -u^1\sin(\theta) + u^2\varepsilon\cos(\theta) \\ z_t & = & v_z \\ v_{z,t} & = & u^1\cos(\theta) + u^2\varepsilon\sin(\theta) - 1 \\ \theta_t & = & \omega \\ \omega_t & = & u^2, \end{array}$$

which is in affine derivative form, where ε is a constant parameter. Furthermore both inputs u^1, u^2 can be eliminated preserving the quasi-linearity such that we obtain

$$\begin{array}{rcl} x_t - v_x & = & 0 \\ v_{x,t}\cos(\theta) + v_{z,t}\sin(\theta) - \omega_t\varepsilon + \sin(\theta) & = & 0 \\ z_t - v_z & = & 0 \\ \theta_t - \omega & = & 0\,. \end{array}$$

This system can be written as

$$M_1(\theta)\begin{bmatrix} x_t \\ v_{x,t} \\ z_t \\ v_{z,t} \\ \theta_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} v_x \\ -\sin(\theta) \\ v_z \\ \omega \end{bmatrix}$$

together with

$$M_1(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & \sin(\theta) & 0 & -\varepsilon \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We choose the kernel

$$v_1 = \varepsilon\cos(\theta)\partial_{v_x} + \varepsilon\sin(\theta)\partial_{v_z} + \partial_\omega$$

and compute the flow

$$\begin{aligned} v_x &= \varepsilon\cos(\theta)\tau + \bar{v}_x \\ v_z &= \varepsilon\sin(\theta)\tau + \bar{v}_z \\ \omega &= \tau \end{aligned}$$

which enables us to derive its jet prolongation

$$\begin{aligned} v_{x,t} &= \varepsilon\cos(\theta)\tau_t - \varepsilon\sin(\theta)\tau\theta_t + \bar{v}_{x,t} \\ v_{z,t} &= \varepsilon\sin(\theta)\tau_t + \varepsilon\cos(\theta)\tau\theta_t + \bar{v}_{z,t} \\ \omega_t &= \tau_t. \end{aligned}$$

Remark 11: The crucial point is how to select an appropriate kernel. This was done in the following manner. Write the system under investigation as a Pfaffian system $\Delta_0 = \mathrm{span}\{\rho_1, \rho_2, \rho_3, \rho_4\}$

$$\begin{aligned} \rho_1 &: \mathrm{d}x - v_x\mathrm{d}t \\ \rho_2 &= \cos(\theta)\,\mathrm{d}v_x + \sin(\theta)\mathrm{d}v_z - \varepsilon\mathrm{d}\omega + \sin(\theta)\,\mathrm{d}t \\ \rho_3 &= \mathrm{d}z - v_z\mathrm{d}t \\ \rho_4 &= \mathrm{d}\theta - \omega\mathrm{d}t \end{aligned}$$

and compute the annihilator (modulo $\mathrm{d}t$) which reads as $\Delta_0^\perp = \mathrm{span}\{\varepsilon\partial_{v_x} + \cos(\theta)\partial_w, \varepsilon\partial_{v_z} + \sin(\theta)\partial_w\}$. Then we seek for a solution of the equation $v_1(\Delta_1) \subset \Delta_1$ with $\Delta_1 \subseteq \Delta_0$ and $v_1 \in \Delta_0^\perp$ which guarantees that after the elimination process the affine structure is preserved.

In the new coordinates the system takes the form

$$M_2(\theta)\begin{bmatrix} x_t \\ \bar{v}_{x,t} \\ z_t \\ \bar{v}_{z,t} \\ \theta_t \end{bmatrix} = \begin{bmatrix} \varepsilon\cos(\theta)\tau + \bar{v}_x \\ -\sin(\theta) \\ \varepsilon\sin(\theta)\tau + \bar{v}_z \\ \tau \end{bmatrix}$$

with

$$M_2(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The elimination of τ preserves again the affine derivative form as desired

$$M_3(\theta)\begin{bmatrix} x_t \\ \bar{v}_{x,t} \\ z_t \\ \bar{v}_{z,t} \\ \theta_t \end{bmatrix} = \begin{bmatrix} \bar{v}_x \\ -\sin(\theta) \\ \bar{v}_z \end{bmatrix}$$

with

$$M_3(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & -\epsilon\cos(\theta) \\ 0 & \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 0 & 1 & 0 & -\epsilon\sin(\theta) \end{bmatrix}.$$

We compute again the kernel

$$v_2 = \varepsilon\cos(\theta)\partial_x + \varepsilon\sin(\theta)\partial_z + \partial_\theta$$

based on the same considerations as in the previous remark and derive the flow

$$\begin{aligned} x &= \varepsilon\sin(\tau) + \bar{x} \\ z &= -\varepsilon\cos(\tau) + \bar{z} \\ \theta &= \tau \end{aligned}$$

as well as the jet prolongation

$$\begin{aligned} x_t &= \varepsilon\cos(\tau)\tau_t + \bar{x}_t \\ z_t &= \varepsilon\sin(\tau)\tau_t + \bar{z}_t \\ \theta_t &= \tau_t \end{aligned}$$

which leads to a system of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\tau) & 0 & \sin(\tau) \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_t \\ \bar{v}_{x,t} \\ \bar{z}_t \\ \bar{v}_{z,t} \end{bmatrix} = \begin{bmatrix} \bar{v}_x \\ -\sin(\tau) \\ \bar{v}_z \end{bmatrix}.$$

Elimination of τ is again structure preserving and we are left with the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{x}_t \\ \bar{z}_t \end{bmatrix} = \begin{bmatrix} \bar{v}_x \\ \bar{v}_z \end{bmatrix}.$$

The flat outputs are $h^1 = \bar{x} = x - \epsilon\sin(\theta)$ and $h^2 = \bar{z} = z + \epsilon\cos(\theta)$ since from $\bar{x}_t = \bar{v}_x$ and from $\bar{z}_t = \bar{v}_z$ we can compute $\bar{v}_x$ and $\bar{v}_z$. From the relation

$$\cos(\tau)\bar{v}_{x,t} + \sin(\tau)\bar{v}_{z,t} + \sin(\tau) = 0$$

which can be rewritten as

$$\cot(\tau)h^1_{tt} + h^2_{tt} + 1 = 0$$

as well as $\theta = \tau$ we derive the relation

$$\tan(\theta) = -\frac{h^1_{tt}}{h^2_{tt} + 1},$$

see again [8]. This example shows that the constructive algorithm leads to the system parametrization in a systematic manner, if we are able to perform the reduction and the elimination process, preserving the AD structure in every step. Furthermore it should be noted that no non-derivative variables appeared after the first elimination of the two inputs.

V. Conclusion

In this paper we have discussed a constructive algorithm that generates a system parametrization for a special subclass of nonlinear implicit control systems, if possible. The main idea is to copy the principles reduction and elimination known from the linear scenario. The crucial point is to derive a sequence of systems that are in affine derivative form, since then the reduction process is simpler. However the structure preserving elimination is a challenging task in general.

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