Efficient Communication Infrastructures for Distributed Control and Decision Making in Networked Stochastic Systems

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Abstract— **In networked systems, groups of agents achieve certain objectives via interaction at local levels in a decentralized manner. The performance of such systems is determined by the communication infrastructure of the network as well as the system dynamics. The interdependence of agents in a networked system is often modeled by graphs. We study the interdependence of communication and collaboration graphs in a networked system in the context of a coordination control and decision making problems. We model the decision on whether to cooperate or not in a group effort as a result of a series of two-person games between agents and their neighbors. The payoff of each agent is computed as the sum of the agent's payoffs from each of these games. Since coordination games have more than one equilibrium point, the problem is then which equilibrium point will the agents choose and whether they will settle on a Pareto-optimal equilibrium point. We consider a behavior learning algorithm and study its effect on the emergence of a collaboration graph. We also study the effect of the communication network topology on the convergence speed of the scheme.**

I. INTRODUCTION

In networked systems, groups of agents achieve certain objectives via interaction at local levels in a decentralized manner. The performance of such systems is determined by the communication infrastructure of the network as well as the system dynamics. The interdependence of agents in a networked system is often modeled by graphs. It is however crucial to mention that several graphs are involved in the modeling of a networked system. In [1], three graphs were identified to describe the network of moving vehicles: a connectivity graph, a communication graph, and an action (collaboration) graph. The first two graphs describe the information exchange in the network whereas the action graph is specific to the particular collaborative activity that the nodes perform.

In this work we study the interdependence of communication and collaboration graphs in a networked system in the context of a coordination control and decision making problem. The system consists of a group of entities referred to as agents. The agents can be machines or humans with different degrees of rationality. Each agent has to make a decision on whether to cooperate or not in a group effort. This is modeled by two-person coordination games between neighboring agents. The payoff of each agent is computed as the sum of the agent's payoffs from each of these games. Since coordination games have more than one equilibrium point, the problem is then which equilibrium point will the agents choose and whether they will settle on a Pareto-optimal equilibrium point. After the games are played, a collaboration graph is formed from all the agents who have decided to collaborate. Each agent's decision on whether to collaborate or not is based on its personal understanding of its own behavioral tendencies as well as its neighbors'. To account for this fact we provision the agents with a behavioral variable, which indicates how risk averse an agent is and models the agents' behaviors and how they learn each other's behaviors and adapt to them based on a model of learning proposed by Cucker, Smale, and Zhou [15]. Agents exchange messages before playing the game and based on these exchanges try to learn their neighbors' behavior and adapt their behavior. The effect of the agents on each other is governed by an influence matrix which is partially derived by the communication graph's topology.

If the agents are allowed to interact for a long enough period of time before the game, a consensus will be reached [under certain conditions] on which equilibrium should be played by the agents. This work focuses on three major issues: The emergence of a collaboration graph based on the behavior adaptation caused by the learning algorithm, the effect of the communication network topology on the convergence speed of the scheme, determination of the effect of the number of like-minded agents and their well-connectedness as major decisive factors on determining whether equilibrium is attained.

The study of the emergence of cooperation and related conventions has been the subject of interest for a long time since it has applications in many social, economical and political studies [2], [3], [4]. The emergence of coordination and cooperation is explained in game theory literature by considering boundedly-rational agents who play a game indefinitely against fixed, or randomly matched agents, and learn from the previous outcomes to achieve a notion of optimality or equilibrium [5]. This framework is also adapted to study cooperation in networks and network formation [6], [7], [8]. In this framework, it is usually assumed that conventions result as equilibria of coordination games [7], [6]. The trade-off between stability of the formations and costs of link establishment are also studied using cooperative game theory [9].

This work differs from the above-mentioned literature in many respects. First, we consider a given communication

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topology, i.e. we do not consider the problem of establishing communication and network formation, rather we focus on the collaboration networks possible given an underlying communication framework. In this respect, our objective is similar to that of [10] with the difference that [10] considers a $n-$person game with emphasis on deterministic formation of common knowledge in a deterministic manner, whereas we consider a set of $2-$person games and consider the agent's learning and updating their types as a result of their observations in their neighborhoods. This way common knowledge on a global mode of operation may result. On the other hand, we consider agents to learn not only the strategies as there is the case with evolutionary game theory, but the types. In other words, the adaptation provides the agents with a mechanism to understand how the other agents see the world so that they get a better chance to coordinate on a single strategy. It is important to mention that this work is relevant to but different from the concept of cheap talk in the game theory literature, which refers to communicating non-binding statements before the game is played. It is pointed out by Aumann [13] that pre-game communication should not affect the outcome of a game when the players have a strict preference in their choices.[1] Our work is motivated by the assertion that conventions "are the product of a largely unconscious process of cultural evolution" rather than formation of common knowledge [14]. To this end we consider the observations that agents make about their neighbors' behaviors and the adaptations to their behaviors as a result of the influence of their neighbors prior to the game being played.

The paper is organized in the following sections. The model of the game and the learning process is provided in Section II. Section III provides the analysis of the system and some illustrating examples. . Section IV concludes the paper.

II. System Model

The agents are modeled as the nodes of a given [communication] graph, Each node has to take a decision on whether to cooperate (C) or not cooperate (NC) in a group effort. Based on its decision when encountered by a neighbor, a given node will acquire a payoff as in a coordination game according to the pay-off table of Figure 1.

	C	NC
C	a, a	0, b
NC	b, 0	c, c

Fig. 1. Coordination game matrix, $a > b > c > 0$

The overall payoff of a node is then the sum of the payoffs it gets from playing the game with all of its neighbors. The coordination game has two pure $((C,C),(NC,NC))$ and one mixed Nash equilibria and the equilibrium (C,C) is Pareto-optimal. However, the numerical values of a, b, and c can be chosen such that the basin of attraction of the equilibrium point (NC,NC) is large and it is a risk-dominant equilibrium point. Therefore, it is not easy to predict which strategy the agents will choose without knowing any further information about the agents. This model will be formalized in section II-A.

We consider agents with a behavioral state or type, which determines the strategy they choose. We consider each agent's type to be defined as a function that maps a random input to a deterministic output. The idea is to capture the notion of how the agent interprets a random input: e.g. if the agent is told that an event occurs with a certain probability, how likely is it for them to believe that the event occurs and in general, how do they interpret this information. For example, an agent of the type 'Strictly Optimistic' believes that a certain event occurs regardless of the probability assigned to it, while an agent of the type 'Strictly Pessimistic' is risk averse and believes that the above-mentioned event never occurs. It is assumed that most agents are not strictly optimistic or pessimistic but 'somewhere in between'. This will be formalized in section II-B.

Using the Cucker-Smale-Zhou framework of language acquisition [15], we propose a model in which an agent's *strategy* will depend on what it knows about its neighbors'*behaviors* . To this end, we consider a learning model in which by observing the previous behavior of their neighbors, each agent modifies its own behavior. We emphasize that since agents do not trust each other completely, they do not weigh the information they receive from their neighbors equally. In this approach the agents value the information they receive from their neighbors through a stochastic influence matrix. They then adapt themselves by changing their behavior through minimizing a notion of distance between their behavior and the neighbors' weighted behaviors. The agents' decision on whether to cooperate or not is based on their emerging type, when the game is played. If the agents are successful to reach a consensus on their types, they will get a higher expected utility and may achieve the Pareto optimal equilibrium. The learning process model is formalized in section II-C.

A. The coordination games

We consider a set of n agents and model the interconnection between them by a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The nodes of the graph, $\mathcal{V} = \{1, 2, ..., n\}$ represent the agents and the undirected edges $\mathcal{E} = \{l_1, l_2, ..., l_e\} \subseteq \mathcal{V} \times \mathcal{V}$ represent the communication links. Each agent in $\mathcal{V}$ has to take a decision on whether to cooperate (C) or not cooperate (NC) in a group effort. This is modeled by considering each agent being engaged in a 2×2 coordination game with each of its neighbors. We denote the set of neighbors of agent i by N_i. The set of pure strategies for each agent is

$$S = \{C, NC\}.$$

[1] While the logic of Aumann's argument is valid, there are some arguments opposing the notion that communication does not help coordination in the case of cheap talk [11],[12].

The pay-off matrix for all the coordination games is given in Figure 1.

Therefore, the over all pay-off of agent i is given by:

$$u_i(s_i, s_{-i}) = \begin{cases} a\sum_{j\in N_i} 1_{\{s_j=C\}}, & \text{if } s_i = C, \\ b\sum_{j\in N_i} 1_{\{s_j=C\}} + \\ c\sum_{j\in N_i} 1_{\{s_j=NC\}}, & \text{if } s_i = NC. \end{cases} \quad (1)$$

B. Behavior modeling

Consider that each agent has a behavior (belief) system that decides on its level of optimism and that this system evolves with time. We model the evolution of this belief system in the Cucker-Smale framework of 'language evolution'. The behavior of an agent can be considered as a function $f : X = [0, 1] \to Y = [0, 1]$. Given a uniformly distributed random variable $x \in X$, $f_i(x)$ determines whether agent i expects an event that is supposed to occur with probability x, to actually happen. In this framework, we consider an agent to be strictly optimist if its behavior function f is a step function, which maps the interval $(0, 1]$ to 1, (Figure 2:(i)). Similarly, an agent is considered to be strictly pessimist, if its belief function is the step function -for which the interval $[0, 1)$ is mapped to 0, (Figure 2:(ii)). Many other types of behavior such as ambivalence ((Figure 2:(iii))) are possible. We assume that the beliefs of the agents in the network can be modeled as continuous piecewise linear functions between the two extremes of strict optimism and strict pessimism. Figure 2:(iv) shows the behavior function for agents which we call *regular*. The behavior of these agents which is modeled by continuous functions f consist of two thresholds. If $x < \beta$, the agent decides that the event does not happen and therefore assigns $y = 0$. If on the other hand $x > \gamma$, the agent decides that the event does happen and assigns $y = 1$. For the values of x between these two thresholds, the agent is not certain and therefore assigns y to a number between 0 and 1 which can be interpreted as the percentage of the agents' confidence on whether the event happens. We assume that the two thresholds are related as in Figure 2:(iv) and therefore the behavior of a regular agent can be determined given two parameters: threshold β and slope α with $\frac{\pi}{4} \le \alpha \le \alpha_{max} < \infty$.

Remark 2.1: In order for the learning algorithm to be more easily implementable, it is better to consider smooth and differentiable behavioral functions. To this end, we approximate the proposed piecewise functions by sigmoids. For example, the function

$$f(x;\theta_1,\theta_2) = \frac{1}{2}(1 + tanh(\frac{\theta_1}{2}(x - \theta_2))), \quad (2)$$

where $\theta_{1min} \le \theta_1 \le \theta_{1max}$ and $\theta_{2min} \le \theta_2 \le \theta_{2max}$ for suitable extremal values provides a good approximation for the regular agents' behavior. Here θ_1 and θ_2 determine the slope and bias of the function respectively. We denote this family of sigmoid functions by $\mathbb{F}$. An illustration with $\theta = [10 \quad 0.5]^T$ can be seen in Figure 3. In the sequel, we will use the notations $f(x)$ and $f(x;\theta_1,\theta_2)$ interchangeably.

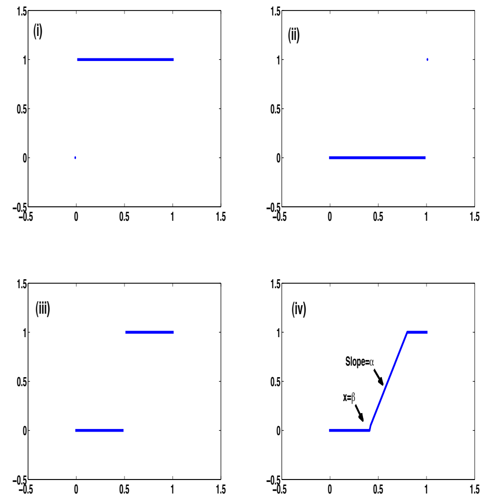

Fig. 2. Agents' sample behaviors. Graphs are numbered clockwise starting from top left: (i) Optimist, (ii) Pessimist, (iii) ambivalent, (iv) Regular

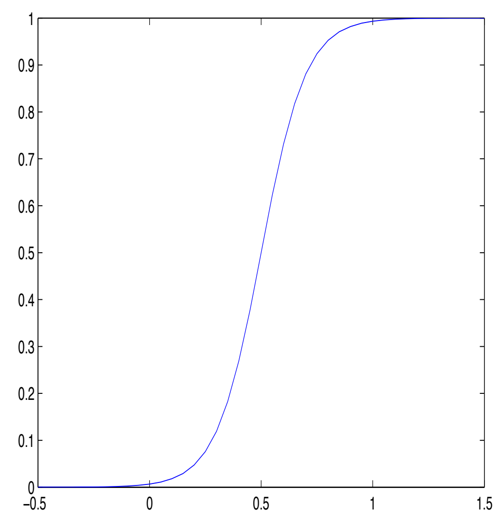

Fig. 3. Sigmoid function approximating the behavior

C. Learning algorithm

To model the agents' partial understanding of the other agents' types, we need to model the communication that happens before the game is played. In reality, people make assumptions on peers' judgement system based on their observation of how their peers react to random events. We model this aspect by assuming that that the system evolves in a synchronized manner: at each time interval t, all the agents receive data from their neighbors in the form of $\{(x_j(t), y_j(t))\}_{j\in N_i}$. We assume that $x_j(t)$ is distributed uniformly on $X = [0, 1]$ and $y_j(t) = f(x_j(t);\theta_1(t),\theta_2(t))$, where $f(x_j(t);\theta_1(t),\theta_2(t)) \in \mathbb{F}$.

We model the relative influence and credibility of the agents as perceived by other agents by a stochastic matrix $W = [w]_{ij}$, where w_{ij} denotes the relative influence of agent j on agent i (to be used as below. If an agent j does not influence agent i, $w_{ij} = 0$. This case includes situations, where agent i does not trust agent j. We also assume that $w_{ii} > \gamma > 0$ for a given $\gamma > 0$.

Having defined the influence matrix, the learning algorithm requires that at each time, all agents update their behavior function due to the following equation:

$$f(x_i(t+1)) = arg \min_{f \in \mathbb{F}} \sum_{j \in N_i} w_{ij}(f(x_j(t)) - y_j(t))^2, \quad i = 1, 2, ..., n. \tag{3}$$

The learning dynamics given by the set of equations (3) determine the evolution of the behavior of the system in the sense that it describe how the types of agents change with respect to the change of their understanding of the other agents' types. Coordination on any policy happens if the agents reach a consensus on their types. We assume that when the game is played the policy of all agents is to choose 'Coordinate' action if and only if the agent's emerging type is more towards the optimist end of the spectrum, i.e. if $\theta_2 < \frac{1}{2}$ and θ_1 is such that $f(0.5, \theta_1, \theta_2) > 1 - \epsilon$, for a predefined small $\epsilon > 0$.

III. Analysis

The learning algorithm of the previous section constitutes a stochastic dynamical system in which the dynamics are determined by the samples of random inputs provided to agents. Determining the convergence of the learning algorithm (3), requires analyzing the contraction of the distance between functions f, where the distance between the functions is defined as in [15]. Since, we consider uniform probability distribution for generating $x \in [0, 1]$, this notion of distance reduces to the deterministic distance using the induced infinity norm on the Y space. The following theorem stated without proof follows on the lines of Theorem 1 in [15].

Theorem 3.1: The learning algorithm of the system defined in Section II, converges to a consensus on the type functions, provided that the matrix W is irreducible.

In the sequel, we consider an approximate linearized model of the system described in Section II, and show how this approximation leads to a linear time varying stochastic consensus algorithm for which the dependence of convergence speed to the graph topology is well-studied. To this end, we use an affine approximation of the functions $f(x; , \theta_1, \theta_2)$. We also consider the case that each agent weighs its neighbors equally, for the ease of exposition. The results are valid for any general weighting matrix W. We consider agents with approximate type functions

$$f(x_i; \theta, \lambda) = \theta x_i + \lambda. \tag{4}$$

Writing optimality conditions, for all $t > 1$

$$\frac{d}{d\theta} f(x; \theta, \lambda) = 0, \quad \frac{d}{d\lambda} f(x; \theta, \lambda) = 0,$$

for the learning equations (3) at each time yields simple linear regression formulae:

$$\theta_i(t+1) = \frac{\sum_{j \in N_i} x_j(t) y_j(t) - \frac{1}{n_i} \sum_{j \in N_i} x_j(t) \sum_{j \in N_i} y_j(t)}{\sum_{j \in N_i} x_j(t)^2 - \frac{1}{n_i} (\sum_{j \in N_i} x_j(t))^2}$$

$$\lambda_i(t+1) = \frac{1}{n_i} \sum_{j \in N_i} y_j(t) - \frac{\theta_i(t+1)}{n_i} \sum_{j \in N_i} x_j(t), \tag{5}$$

where, N_i denotes the neighborhood of agent i and n_i denotes the number of agent i's neighbors. Using the fact that $y_j(t) = F(x_j(t)) = \theta_j(t) x_j(t) + \lambda_j(t)$, and denoting

$$S_i(t) = \sum_{j \in N_i} x_j(t)^2 - \frac{1}{n_i} (\sum_{j \in N_i} x_j(t))^2,$$

and

$$\bar{x}_i(t) = \frac{1}{n_i} \sum_{j \in N_i} x_j(t),$$

we can rewrite Equation (5) as:

$$\begin{aligned} \theta_i(t+1) &= \frac{1}{S_i} \Big[\sum_{j \in N_i} (x_j^2(t) - \bar{x}_i(t) x_j(t)) \theta_j(t)) \\ &\quad + \sum_{j \in N_i} (x_j(t) - \bar{x}_i(t)) \lambda_j(t)) \Big] \\ \lambda_i(t+1) &= \frac{1}{n_i} \sum_{j \in N_i} [x_j(t) \theta_j(t) + \lambda_j(t)] \\ &\quad - \bar{x}_i(t) \frac{1}{S_i} \Big[\sum_{j \in N_i} (x_j^2(t) - \bar{x}_i(t) x_j(t)) \theta_j(t)) \\ &\quad + \sum_{j \in N_i} (x_j(t) - \bar{x}_i(t)) \lambda_j(t)) \Big] \end{aligned} \tag{6}$$

If we denote $\forall i \in \{1, 2, ..., n\}$, $\Theta_i = [\theta_i \quad \lambda_i]^T$, and $\Theta = [\Theta_1 \quad \Theta_2 \quad ... \quad \Theta_n]^T$, then it can be readily verified that the learning dynamics of the system can be written in the form of

$$\Theta(t+1) = G(x(t)) \Theta(t), \tag{7}$$

where $G(x(t))$ is a matrix with row sums determined by edges of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and the realization of the random vector $x = [x_1 \quad x_2 \quad ... \quad x_n]^T$ at each time t. The difference with regular consensus iterations is that the entries of $G(x(t)$ can be negative as well.

The convergence of iterations of the form (7), when all the entries are nonnegative has been studied [16]. It has been shown [17] that a system with self reinforcement and stationary and ergodic sequence of matrices $\{\Theta_t, \quad t = 1, 2, ...\}$, reaches consensus almost surely if and only if $|\mu_2(E[G(x_k))]| < 1$, where μ_2 is the second largest modulus eigenvalue. Convergence of such schemes, where the weights are not necessarily positive has been addressed in [18] for a parameter estimation problem. Several results (e.g. [19], [20], [21]) address the speed of convergence of these schemes.

In general, the speed of convergence of consensus schemes and gossip algorithms is a function of the graph topology as well as the weights that are assigned to the existing links in a given topology. For probabilistic schemes bounds

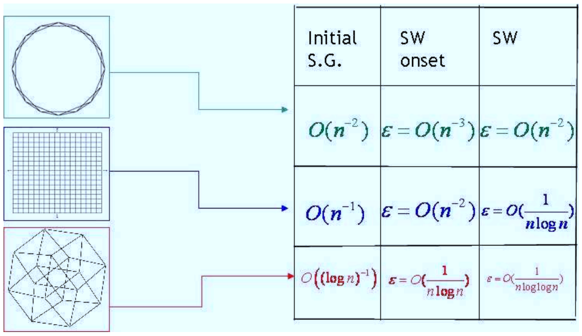

Fig. 4. Small world behavior

can be found based on expected Laplacian matrices that determine the evolution of the system. While such results mostly address the cases with i.i.d. choices of underlying graphs and weighting matrices, a necessary condition for fast convergence is to resort to graphs with the second smallest eigenvalue of the corresponding Laplacian matrix bounded away from 0. Equivalently, the random walk matrices defined on these graphs should have second largest eigenvalue modulus bounded away from 1.

Since, the speed of convergence of the learning algorithm decreases with the number of agents, the communication graph topology is a crucial factor that determines whether coordination can be achieved effectively. For large scale networks, small world topologies have proved to be efficient in the sense that while considerably sparse, distributed algorithms converge fast on them. Many large scale communication graphs can be turned into small world graph by adding a few links between distant nodes. The following procedure based on the probabilistic developments in [22], [19], [20] shows the characterization of small world phenomena in sparse networks.

Procedure 3.1:

1) To a nominal base graph, $\mathcal{G}_0$ corresponds a natural random walk matrix, F_0.
2) Capture the desired performance measure as the corresponding spectral gap of this matrix,
3) Take a probability distribution corresponding to a perturbation of the base graph, G_ϵ. In other words, perturb the graph in such a way that the random walk on graph $\mathcal{G}_0$ is modified to be able to make transitions to non-neighboring nodes, with small probability according to a perturbed matrix F_ϵ.
4) Determine if for a small perturbation parameter, there is any abrupt increase in the spectral gap.
5) If such increase can be observed for a range of perturbation parameter values, then the small world phenomenon has occured.
6) Interpret the perturbations in the weights of the random walk as structural perturbations.

Figure 4 illustrates the small world effect from ring, lattice and hypercube topologies. The first column shows the spectral gap for the initial topologies, the second column shows the amount of perturbation for which the small world effect starts, and the third column indicates the range of perturbation for which the graph can be considered as a small world.

IV. Conclusions and Future Work

In this paper we studied the interdependence of communication and collaboration graphs in a networked system in the context of a coordination control and decision making problem. We modeled the decision on whether to cooperate or not in a group effort as a result of a series of two-person games between agents and their neighbors. We considered agents with different behavioral variables (types) and proposed an adaptive scheme by which the agents' types evolve so that they can coordinate on a common strategy. We showed that the behavior evolution can be addressed in the Cucker-Smale-Zhou framework of language evolution. We mainly focused on the learning algorithm and demonstrated that since it can be considered as a gossip-type algorithm, its convergence is constrained by the spectral properties of the communication graph. We illustrated sparse large scale networks for which the small world effect can occur by adding shortcuts. In this paper we considered a learning model in which the agents use their observations of their neighbors' behaviors to determine their type. Future directions include determining how the number of like-minded agents and their connectivity play a major decisive role on the attained equilibrium. Further work also addresses how the future behavior evolution is affected once the game is played.

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