Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

A Unified Solution of a Class of Continuous/Discrete-time H^2 Control

Kotaro Hashikura and Yoshito Ohta

***Abstract*— Explicit solutions for the H^2 control problem for systems with non-minimum phase property at the input such as input time-delay systems attract the interest of many researchers. The existing solution method cannot be seen as a natural generalization of closed-loop reduction of the finite-dimensional standard H^2 problem. We give the method of solving two-block problems twice based on analysis of a state-space representation. The implication of the method is that the optimal controller of Smith form is obtained directly in both continuous- and discrete-time systems. The truncation operator introduced in the literature is utilized in solving the problem. We propose an alternative definition that is consistent for both continuous- and discrete-time cases. Furthermore, we show that the new definition is more appropriate to characterize the optimal control cost.**

I. Introduction

In this paper, we study a solution method of the H^2 control problem under the setup of Fig. 1, where P_+ is a plant which consists of a rational P and a general scalar inner function m at the control input, and K_+ is a controller to design. We consider both continuous- and discrete-time cases. This class of systems has applications in control performance analysis of time-delay and flexible structure systems.

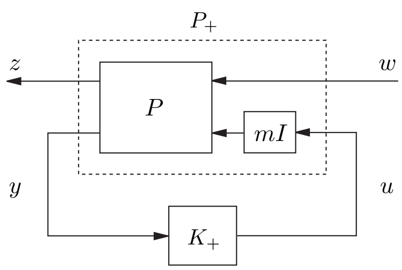

Fig. 1. control system

The problem was solved for $m(s) = e^{-sl}$ by Mirkin [5] by finding constraints on the Youla parameter of the delay-free case, and for general inner functions by Kashima [4] (the continuous-time case), Nishio-Kashima [6] (the discrete-time case with the proper controller) and Kashima-Nishio [3] (the discrete-time case with the strictly proper controller). Unlike them, we solve the problem by closed-loop reductions. Our approach has two benefits: First, the difficult task of finding constraints on the Youla parameter is not necessary. Second, when finding the increase of the optimal cost due to non-minimum phase property, the same orthogonality in H^2 space appears in both continuous- and discrete-time cases.

K. Hashikura is with the Department of Human Mechatronics Systems, Graduate School of System Design, Tokyo Metropolitan University, Japan hashikura-kotaro@sd.tmu.ac.jp

Y. Ohta is with the Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Japan yoshito_ohta@i.kyoto-u.ac.jp

II. Consideration on the Smith predictor

There are two well-known types of controllers for time-delay systems. One is the Smith predictor and another is the finite-spectrum assignment controller. In [8], the equivalence of them is shown using transfer function representation. In contrast to it, we show that they are equivalent using representation in [7].

We try to stabilize the input time-delay system

$$\begin{aligned} x(t) &= Ax(t) + Bu(t-l) \\ y(t) &= Cx(t). \end{aligned}$$

The solution of the PDE

$$\frac{\partial}{\partial t}\phi(\theta,t) = \frac{\partial}{\partial \theta}\phi(\theta,t) \ (0<\theta<l,\, t>0)$$

with an initial value $\phi(\theta,0)=\phi_0(\theta)$ and a boundary value $\phi(l,t)=u(t)$ is given by

$$\phi(\theta,t) = \begin{cases} \phi_0(t+\theta) & t+\theta<l \\ u(t+\theta-l) & t+\theta>l \end{cases}.$$

Therefore, the input time-delay system is representable as a serial connection of a distributed element and a lumped element:

$$\frac{d}{dt}\begin{bmatrix} E_\Delta & \\ & I \end{bmatrix}\begin{bmatrix} \phi(\theta,t) \\ x(t) \end{bmatrix} = \begin{bmatrix} A_\Delta & \\ B\Gamma_0 & A \end{bmatrix}\begin{bmatrix} \phi(\theta,t) \\ x(t) \end{bmatrix} + \begin{bmatrix} B_\Delta \\ O \end{bmatrix} u(t), \tag{1}$$

where

$$E_\Delta := \begin{bmatrix} I \\ O \end{bmatrix}, A_\Delta := \begin{bmatrix} \frac{\partial}{\partial\theta} \\ -\Gamma_l \end{bmatrix}, B_\Delta := \begin{bmatrix} O \\ I \end{bmatrix},$$

and Γ_0, Γ_l are operators which evaluate the values of $\phi(\theta,t)$ at $\theta=0,l$, respectively. The domains of operators are defined as follows:

$$\mathcal{D}(E_\Delta), \mathcal{D}(A_\Delta), \mathcal{D}(B\Gamma_0) := \left\{ \phi \in L^2([0,l]) \,\middle|\, \frac{\partial}{\partial\theta}\phi \in L^2([0,l]) \right\}.$$

Paying attention to $B\Gamma_0$ in Eq. (1), we consider the following transformation

$$\begin{aligned} &\begin{bmatrix} I & O \\ V & I \end{bmatrix}\begin{bmatrix} A_\Delta - sE_\Delta & O \\ B\Gamma_0 & A-sI \end{bmatrix}\begin{bmatrix} I & O \\ -VE_\Delta & I \end{bmatrix} \\ &= \begin{bmatrix} A_\Delta - sE_\Delta & O \\ B\Gamma_0 - (AVE_\Delta - VA_\Delta) & A-sI \end{bmatrix}, \end{aligned} \tag{2}$$

where $V : L^2([0,l]) \times \mathbb{R} \to \mathbb{R}$ is a parameter to determine. If we let the $(2,1)$ block of Eq. (2) be zero, then we obtain the Sylvester equation

$$AVE_\Delta - VA_\Delta = B\Gamma_0.$$

ISBN 978-963-311-370-7

Using Krein's formula [1], we can find the solution:

$$V\begin{bmatrix}u_\Delta\\u_l\end{bmatrix}=\frac{1}{2\pi j}\oint_{\partial\sigma(A)}(sI-A)^{-1}B\Gamma_0(sE_\Delta-A_\Delta)^{-1}\begin{bmatrix}u_\Delta\\u_l\end{bmatrix}ds$$
$$=e^{-Al}Bu_l+\int_0^l e^{-A\alpha}Bu_\Delta(\alpha)d\alpha,$$

where we have used

$$(sE_\Delta-A_\Delta)^{-1}:\begin{bmatrix}u_\Delta\\u_l\end{bmatrix}\in L^2([0,l])\times\mathbb{R}\to\phi\in\mathcal{D}(A_\Delta),$$
$$\phi(\theta)=e^{s(\theta-l)}u_l+\int_\theta^l e^{s(\theta-\alpha)}u_\Delta(\alpha)d\alpha.$$

Introducing a new variable $x^R(t)$ by the equation

$$\begin{bmatrix}\phi(\cdot,t)\\x(t)\end{bmatrix}=\begin{bmatrix}I&O\\-VE_\Delta&I\end{bmatrix}\begin{bmatrix}\phi(\cdot,t)\\x^R(t)\end{bmatrix},\tag{3}$$

we see that the input time-delay system is represented as

$$\frac{d}{dt}\begin{bmatrix}E_\Delta&\\&I\end{bmatrix}\begin{bmatrix}\phi(t)\\x^R(t)\end{bmatrix}=\begin{bmatrix}A_\Delta&\\&A\end{bmatrix}\begin{bmatrix}\phi(t)\\x^R(t)\end{bmatrix}+\begin{bmatrix}B_\Delta\\e^{-Al}B\end{bmatrix}u(t)$$
$$y(t)=C\begin{bmatrix}-VE_\Delta&I\end{bmatrix}\begin{bmatrix}\phi(t)\\x^R(t)\end{bmatrix}.\tag{4}$$

Referring to Eq. (4), we construct the following controller:

$$\frac{d}{dt}\begin{bmatrix}E_\Delta&\\&I\end{bmatrix}\begin{bmatrix}\phi_c(t)\\x_c^R(t)\end{bmatrix}=\begin{bmatrix}A_\Delta&\\&A\end{bmatrix}\begin{bmatrix}\phi_c(t)\\x_c^R(t)\end{bmatrix}+\begin{bmatrix}B_\Delta\\e^{-Al}B\end{bmatrix}u(t)$$
$$+\begin{bmatrix}L_\Delta\\L\end{bmatrix}\left(C\begin{bmatrix}-VE_\Delta&I\end{bmatrix}\begin{bmatrix}\phi_c(t)\\x_c^R(t)\end{bmatrix}-y(t)\right)$$
$$u(t)=\begin{bmatrix}F_\Delta^R&F^R\end{bmatrix}\begin{bmatrix}\phi_c(t)\\x_c^R(t)\end{bmatrix}.$$

We try to stabilize possibly unstable dynamics of $x^R(t)$. We let $F_\Delta^R:=0$, $L_\Delta:=0$ and $F^R:=Fe^{Al}$, and choose F and L such that $A+e^{-Al}BF^R=e^{-Al}(A+BF)e^{Al}$ and $A+LC$ become stable. Then we have the Smith predictor:

$$\frac{d}{dt}\begin{bmatrix}E_\Delta&\\&I\end{bmatrix}\begin{bmatrix}\phi_c(t)\\x_c^R(t)\end{bmatrix}=\begin{bmatrix}A_\Delta&\\-LCVE_\Delta&A+LC\end{bmatrix}\begin{bmatrix}\phi_c(t)\\x_c^R(t)\end{bmatrix}$$
$$+\begin{bmatrix}B_\Delta\\e^{-Al}B\end{bmatrix}u(t)-\begin{bmatrix}O\\L\end{bmatrix}y(t)$$
$$u(t)=F^Rx_c^R(t).$$

If we change the state-variables of the above controller by the equation

$$\begin{bmatrix}\phi_c(\cdot,t)\\x_c(t)\end{bmatrix}=\begin{bmatrix}I&O\\-VE_\Delta&I\end{bmatrix}\begin{bmatrix}\phi_c(\cdot,t)\\x_c^R(t)\end{bmatrix},\tag{5}$$

then we have the finite-spectrum assignment controller.

III. Truncation Operator

A. Alternative definition in the discrete-time case

For a given scalar inner function m, $\mathcal{H}(m)$ denotes the orthogonal complement of mH^2 in H^2. This space is characterized by

$$\mathcal{H}(m)=\left\{f\in H^2\;\middle|\;m^\sim f\in H^{2\perp}\right\}.$$

In the continuous-time case, for a given $G(s)=\left[\begin{array}{c|c}A&B\\\hline C&O\end{array}\right]$, Kashima [4] defined the truncation operator $\tau_m[G](s)$ as follows:

$$G_m(s):=\left[\begin{array}{c|c}A&B\\\hline Cm^\sim(A)&O\end{array}\right],$$
$$\tau_m[G](s):=G(s)-m(s)G_m(s)$$

and it belongs to $\mathcal{H}(m)$.

In the discrete-time case, for a given $G(z)=\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]$, we define the truncation operator $\tau_m[G](z)$ as follows:

$$G_m(z):=\left[\begin{array}{c|c}A&B\\\hline C_m&D_m\end{array}\right],$$
$$\tau_m[G](z):=G(z)-m(z)G_m(z),$$

where

$$C_m:=Cm^\sim(A),$$
$$D_m:=m^\sim(0)D+C(m^\sim(A)-m^\sim(0)I)A^{-1}B.$$

We modified the definition of it given by Nishio-Kashima [6] so that it belongs to $\mathcal{H}(m)$. To see this, we note that $\tau_m[G](z)$ and $m^\sim(z)\tau_m[G](z)$ are analytic outside and inside the unit circle, respectively and

$$m^\sim(z)\tau_m[G](z)\Big|_{z=0}=O.$$

B. Computation of H^2 norm

In this subsection, we give a computation method of the H^2 norm of the truncation operator based on a Lyapunov equation. The difference with [4] and [6] is that even if the "A" matrix of a rational function G is unstable, it is not necessary to make additional assumptions on the solvability of the Lyapunov equation. We note that in [4] and [6] another expression is given to the norm using the block components of a Hamiltonian matrix.

Theorem 1 (discrete-time case): Let (A,B) be a stabilizable pair, $\Lambda>O$ be a weight matrix, $X\geq O$ be the stabilizing solution of the Riccati equation (6), and Y be the unique solution of the Lyapunov equation (7).

$$A^*X(I+B\Lambda B^*X)^{-1}A-X=O,\tag{6}$$
$$F:=-\Lambda B^*X(I+B\Lambda B^*X)^{-1}A,\ A_r:=A+BF,$$
$$\Lambda_r:=\Lambda(I+B^*XB\Lambda)^{-1}>O,$$
$$A_rYA_r^*-Y+B\Lambda_rB^*=O.\tag{7}$$

Then we have

$$\frac{1}{2\pi j}\oint\tau_m[G](z)\Lambda\tau_m[G]^\sim(z)\frac{dz}{z}$$
$$=(C_rYC_r^*+D\Lambda_rD^*)-(C_{rm}YC_{rm}^*+D_{rm}\Lambda_rD_{rm}^*)$$
$$+(C_{mr}-C_{rm})Y(C_{mr}-C_{rm})^*$$
$$+(D_m-D_{rm})\Lambda_r(D_m-D_{rm})^*,\tag{8}$$

where

$$C_r:=C+DF,\ C_{mr}:=C_m+D_mF,\ C_{rm}:=C_rm^\sim(A_r),$$
$$D_{rm}:=m^*(0)D+C_r(m^\sim(A_r)-m^\sim(0)I)A_r^{-1}B.$$

Proof: Using X, we define

$$G_r(z) := \left[\begin{array}{c|c} A_r & B \\ \hline C_r & D \end{array}\right],\ G_{mr}(z) := \left[\begin{array}{c|c} A_r & B \\ \hline C_{mr} & D_m \end{array}\right],$$
$$M(z) := \left[\begin{array}{c|c} A_r & B \\ \hline F & I \end{array}\right]$$

then we have

$$G(z) = G_r(z)M(z)^{-1},\ G_m(z) = G_{mr}(z)M(z)^{-1}, \quad (9)$$
$$M(z)\Lambda_r M^{\sim}(z) = \Lambda. \quad (10)$$

Using Eq. (9) and Eq. (10),

$$\begin{aligned} &\tau_m [G](z)\Lambda\tau_m [G]^{\sim}(z) \\ &= (G(z) - m(z)G_m(z))\Lambda(G(z) - m(z)G_m(z))^{\sim} \\ &= (G_r(z) - m(z)G_{mr}(z))\Lambda_r(G_r(z) - m(z)G_{mr}(z))^{\sim}. \end{aligned} \quad (11)$$

We expand Eq. (11), and integrate each term separately. We show, for example, how to compute the integral of $G_r(z)\Lambda_r G_{mr}^{\sim}(z)m^{\sim}(z)$ below. Using Y, we express $G_r(z)\Lambda_r G_{mr}^{\sim}(z)$ as the sum of stable and unstable functions and we have

$$\begin{aligned} &\frac{1}{2\pi j}\oint G_r(z)\Lambda_r G_{mr}^{\sim}(z)m^{\sim}(z)\frac{dz}{z} \\ &= \frac{1}{2\pi j}\oint \left\{ \left[\begin{array}{c|c} A_r & A_r Y C_{mr}^* + B\Lambda_r D_m^* \\ \hline C_r & O \end{array}\right] \right. \\ &\left. + \left[\begin{array}{c|c} A_r & A_r Y C_r^* + B\Lambda_r D^* \\ \hline C_{mr} & C_{mr} Y C_r^* + D_m\Lambda_r D^* \end{array}\right]^{\sim} \right\} m^{\sim}(z)\frac{dz}{z}. \end{aligned} \quad (12)$$

Since

$$\frac{1}{2\pi j}\oint (zI - A_r)^{-1}m^{\sim}(z)\frac{dz}{z} = A_r^{-1}(m^{\sim}(A_r) - m^{\sim}(0)I),$$

the first term of the integral in Eq. (12) is

$$C_r A_r^{-1}(m^{\sim}(A_r) - m^{\sim}(0)I)(A_r Y C_{mr}^* + B\Lambda_r D_m^*). \quad (13)$$

By Laurent expansion around the origin, the second term of the integral in Eq. (12) is

$$(C_r Y C_{mr}^* + D\Lambda_r D_m^*)m^{\sim}(0). \quad (14)$$

From Eq. (13) and (14),

$$\frac{1}{2\pi j}\oint G_r(z)\Lambda_r G_{mr}^{\sim}(z)m^{\sim}(z)\frac{dz}{z} = C_{rm} Y C_{mr}^* + D_{rm}\Lambda_r D_m^*.$$

The other terms can be computed similarly, and we obtain Eq. (8). ■

In the same way as in the discrete-time case, we can prove the following theorem.

Theorem 2 (continuous-time case): Let (A, B) be a stabilizable pair, $\Lambda > O$ be a weight matrix, $X \geq O$ be the stabilizing solution of the Riccati equation (15), and Y be the unique solution of the Lyapunov equation (16):

$$A^* X + XA - XB\Lambda B^* X = O, \quad (15)$$
$$F := -\Lambda B^* X,\ A_r := A + BF,$$
$$A_r Y + Y A_r^* + B\Lambda B^* = O. \quad (16)$$

Then we have

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \tau_m [G](j\omega)\Lambda\tau_m [G](j\omega)^{\sim} d\omega \\ &= C(Y - m^{\sim}(A_r)Y m^{\sim}(A_r)^*)C^* \\ &\quad + C(m^{\sim}(A) - m^{\sim}(A_r))Y(m^{\sim}(A) - m^{\sim}(A_r))^* C^*. \end{aligned}$$

IV. Solution via closed-loop reduction

T_{ab} denotes the transfer function from b to a. And for simplicity, we use the definitions

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix},$$
$$\begin{bmatrix} \acute{Q} & \acute{S}^* \\ \acute{S} & \acute{R} \end{bmatrix} := \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}^*.$$

A. Discrete-time case

We construct the H^2 optimal proper controller for P_+ in Fig. 1 of which the rational part P is assumed to be

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & O \end{array}\right] \quad (17)$$

and to satisfy the same condition as in the standard problem [2]. Furthermore, we assume $m(z)$ and its para-conjugate $m^{\sim}(z) := m(\overline{z}^{-1})^*$ are analytic on the set of eigenvalues of A so that $m(A)$ and $m^{\sim}(A)$ can be defined.

1) Full information problem: The state-variable of P, $x(n)$ follows the difference equation

$$x(n+1) = Ax(n) + B_1 w(n) + B_2 m(z)u(n).$$

Referring to the change of variables used in Sec. II, we introduce new variables $x^R(n)$ and $\epsilon(n)$, which follow Eq. (18) and Eq. (19), respectively.

$$x^R(n+1) = Ax^R(n) + B_1 w(n) + m(A)B_2 u(n) \quad (18)$$
$$\epsilon(n+1) = A\epsilon(n) + (m(z)I - m(A))B_2 u(n) \quad (19)$$

Then we can express $x(n)$ as the sum of $x^R(n)$ and $\epsilon(n)$,

$$x(n) = x^R(n) + \epsilon(n).$$

We note that $x^R(n)$ and $\epsilon(n)$ correspond to $x^R(t)$ and $-VE_\Delta v(t)$ in Sec. II, respectively.

Since the transfer function from $B_2 u$ to ϵ

$$\Pi(z) := (zI - A)^{-1}(m(z)I - m(A))$$

is causal and stable, we try to stabilize the dynamics of x^R by state feedback. We let F^R and F_{21}^R be parameters to determine and make the change of the control input

$$v(n) := -F^R x(n) - F_{21}^R w(n) + u(n). \quad (20)$$

And we represent $z(z)$ by $w(z)$ and $v(z)$ (Fig. 2):

$$z(z) = P_{+c}(z)w(z) + N_{+12}(z)v(z), \quad (21)$$

Theorem 1: Let $X \geq O$ be the stabilizing solution of the Riccati equation of the standard H^2 problem:

$$A^* XA - X - (S^* + B_2^* XA)^* R_c^{-1}(S^* + B_2^* XA) + Q = O,$$
$$R_c = R + B_2^* X B_2$$

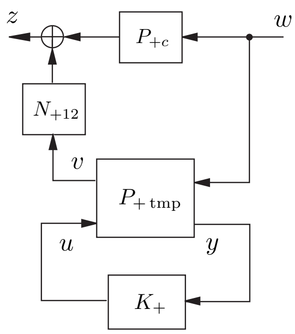

Fig. 2. decomposition of P_+ by state feedback

and F be the corresponding feedback gain

$$F := -R_c^{-1}\left(S^* + B_2^* X A\right).$$

We choose

$$F^R := F m^{\sim}(A), \tag{22}$$

$$D_{11}^R := m^{\sim}(0) D_{11} + C_1 (m^{\sim}(A) - m^{\sim}(0) I) A^{-1} B_1, \tag{23}$$

$$F_{21}^R := -R_c^{-1}(D_{12}^* D_{11}^R + B_2^* X m^{\sim}(A) B_1), \tag{24}$$

then P_{+c} and N_{+12} are given by

$$P_{+c}(z) = \tau_m \left[P_{11}\right](z) + m(z) \left[\begin{array}{c|c} A + B_2 F & m^{\sim}(A) B_1 + B_2 F_{21}^R \\ \hline C_1 + D_{12} F & D_{11}^R + D_{12} F_{21}^R \end{array}\right],$$

$$N_{+12}(z) = m(z) \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array}\right].$$

They satisfy:

- the orthogonality condition: $N_{+12}^{\sim}(z) P_{+c}(z) \in H^{2\perp}$ and
- the isometry condition: $N_{+12}^{\sim}(z) N_{+12}(z) = R_c$.

Proof: By direct calculation,

$$P_{+c}(z) = m(z) \left[\begin{array}{c|c} A + m(A) B_2 F^R & B_1 + m(A) B_2 F_{21}^R \\ \hline C_1 m^{\sim}(A) + D_{12} F^R & D_{12} F_{21}^R \end{array}\right] + D_{11} - C_1 \Pi(z) m^{\sim}(A) B_1, \tag{25}$$

$$N_{+12}(z) = m(z) \left[\begin{array}{c|c} A + m(A) B_2 F^R & m(A) B_2 \\ \hline C_1 m^{\sim}(A) + D_{12} F^R & D_{12} \end{array}\right].$$

We let F^R be as defined in Eq. (22). Since $A + m(A) B_2 F^R = m(A)(A + B_2 F) m^{\sim}(A)$, $N_{+12}(z)$ is the product of $m(z)$ and the inner function in the standard H^2 problem:

$$N_{+12}(z) = m(z) \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array}\right].$$

Therefore, it satisfies the isometry condition.

Next, we let D_{11}^R be a parameter to determine and substitute

$$D_{11} = m(z) D_{11}^R + \left(D_{11} - m(z) D_{11}^R\right)$$

into Eq. (25). Then we have

$$P_{+c}(z) = \left\{ \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right] - m(z) \left[\begin{array}{c|c} A & m^{\sim}(A) B_1 \\ \hline C_1 & D_{11}^R \end{array}\right] \right\} + m(z) \left[\begin{array}{c|c} A + B_2 F & m^{\sim}(A) B_1 + B_2 F_{21}^R \\ \hline C_1 + D_{12} F & D_{11}^R + D_{12} F_{21}^R \end{array}\right]. \tag{26}$$

We let D_{11}^R be as defined in Eq. (23), so that the first term on the right-hand side of Eq. (26) becomes $\tau_m \left[P_{11}\right](z)$:

$$P_{+c}(z) = \tau_m \left[P_{11}\right](z) + m(z) \left[\begin{array}{c|c} A + B_2 F & m^{\sim}(A) B_1 + B_2 F_{21}^R \\ \hline C_1 + D_{12} F & D_{11}^R + D_{12} F_{21}^R \end{array}\right].$$

Next, we determine F_{21}^R to satisfy the orthogonality condition.

$$N_{+12}^{\sim}(z) P_{+c}(z) = \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array}\right]^{\sim} \cdot \left(m^{\sim}(z) \tau_m \left[P_{11}\right](z) + \left[\begin{array}{c|c} A + B_2 F & m^{\sim}(A) B_1 + B_2 F_{21}^R \\ \hline C_1 + D_{12} F & D_{11}^R + D_{12} F_{21}^R \end{array}\right] \right)$$

$$= \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array}\right]^{\sim} m^{\sim}(z) \tau_m \left[P_{11}\right](z) + \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline D_{11}^{R*}(C_1 + D_{12} F) + B_1^* m^{\sim}(A)^* X (A + B_2 F) & O \end{array}\right]^{\sim} + \left\{ (D_{12}^* D_{11}^R + B_2^* X m^{\sim}(A) B_1) + R_c F_{21}^R \right\}. \tag{27}$$

Since the first and second terms on the right-hand side of Eq. (27) belong to $H^{2\perp}$, we let F_{21}^R be as defined in Eq. (24) so that the constant term in Eq. (27) becomes zero. ■

From Lemma 1, we have

$$\|T_{zw}\|_2^2 = \|P_{+c}\|_2^2 + \left\| R_c^{1/2} T_{vw} \right\|_2^2.$$

In the next section, we minimize the scaled H^2 norm of T_{vw}.

2) Output estimation problem: We consider $P_{+\mathrm{tmp}}$ of which the inputs are $(w(n), u(n))$ and the outputs are $(v(n), y(n))$:

$$\begin{aligned} \epsilon(n+1) &= A\epsilon(n) + (m(z) I - m(A)) B_2 u(n) \\ x^R(n+1) &= A x^R(n) + B_1 w(n) + m(A) B_2 u(n) \\ v(n) &= -F^R x^R(n) - F_{21}^R w(n) + u(n) \\ y(n) &= C_2 x^R(n) + C_2 \epsilon(n) + D_{12} w(n). \end{aligned}$$

Since $\epsilon(n)$ is determined by $u(n)$ causally, we consider the rational plant $P_{+\mathrm{tmp}}^R$ of which the measured output is given by

$$y^R(n) := y(n) - C_2 \epsilon(n) = C_2 x^R(n) + D_{12} w(n)$$

instead of $y(n)$ (Fig. 3). The realization of $P_{+\mathrm{tmp}}^R$ is

$$\begin{aligned} x^R(n+1) &= A x^R(n) + B_1 w(n) + m(A) B_2 u(n) \\ v(n) &= -F^R x^R(n) - F_{21}^R w(n) + u(n) \\ y^R(n) &= C_2 x^R(n) + D_{12} w(n). \end{aligned}$$

Using the stabilizing solution $Y \geq O$ of the Riccati equation of the standard H^2 problem:

$$A Y A^* - Y - (\acute{S}^* + A Y C_2^*) R_o^{-1} (\acute{S}^* + A Y C_2^*)^* + \acute{Q} = O,$$
$$\acute{R}_c := \acute{R} + C_2 Y C_2^*$$

and corresponding observer gains

$$L := -(\acute{S}^* + A Y C_2^*) \acute{R}^{-1}, L_{22}^R := (F_{21}^R D_{21}^* + F^R Y C_2^*) \acute{R}_c^{-1},$$

we can solve the finite-dimensional H^2 output estimation problem for $P_{+\mathrm{tmp}}^R$ and obtain the following theorem.

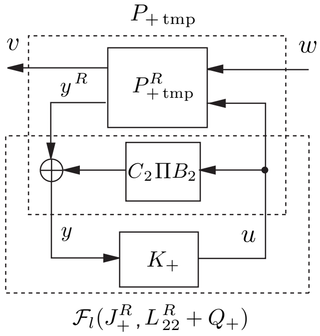

Fig. 3. output transformation

Theorem 3: The stabilizing controller K_+ is parametrized by $\forall Q_+(z) \in H^\infty$ and composed of the finite-dimensional element J^R_+:

$$\begin{aligned} x^R_c(n+1) &= (A + m(A)B_2F^R + LC_2)x^R_c(n) \\ &\quad - Ly^R(n) + m(A)B_2\mu(n) \\ u(n) &= F^R x^R_c(n) + \mu(n) \\ \nu(n) &= -C_2 x^R_c(n) + y^R(n) \\ \mu(z) &= (L^R_{22} + Q_+(z))\nu(z) \end{aligned}$$

and the infinite-dimensional element:

$$y^R(z) = y(z) - C_2\Pi(z)B_2u(z).$$

Furthermore, the optimal controller is given when $Q_+(z) = O$ and the optimal cost E_+ is given by

$$\begin{aligned} E^2_+ = \|\tau_m[P_{11}]\|^2_2 &+ \left\| \left[\begin{array}{c|c} A + B_2F & m^{\sim}(A)B_1 + B_2F^R_{21} \\ \hline C_1 + D_{12}F & D^R_{11} + D_{12}F^R_{21} \end{array} \right] \right\|^2_2 \\ &+ \left\| R_c^{1/2} \left[\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline -F^R + L^R_{22}C_2 & -F^R_{21} + L^R_{22}D_{21} \end{array} \right] \right\|^2_2 . \end{aligned}$$

The controller given in Theorem 3 is of Smith form shown in Fig. 4.

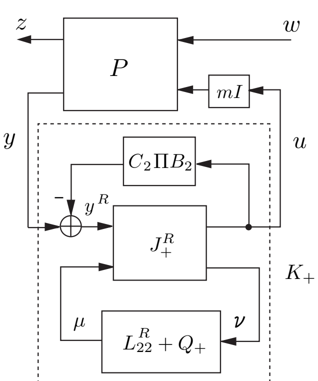

Fig. 4. Smith form

3) Increase of the optimal cost: In this section, we give a modified derivation of the increase of the optimal cost incurred by the non-minimum phase part studied in Nishio-Kashima [6]. By using the orthogonality of $\mathcal{H}(m)$ and mH^2, we derive the cost in terms of the modified truncation operator.

Theorem 4 (Nishio-Kashima [6]): The optimal cost E_+ is given by

$$E^2_+ = E^2 + \left\| R_c^{1/2} \tau_m[\Theta] \acute{R}_c^{1/2} \right\|^2_2 , \tag{28}$$

where E is the optimal cost of the standard H^2 problem (the case $m(z) = 1$) and

$$\Theta(z) := \left[\begin{array}{c|c} A & L \\ \hline F & -L_{22} \end{array} \right],$$

where

$$\begin{aligned} L_{22} &:= (F_{21}D^*_{21} + FYC^*_2)\acute{R}_c^{-1}, \\ F_{21} &:= -R_c^{-1}(D^*_{12}D_{11} + B^*_2XB_1). \end{aligned}$$

Proof: Using the manipulation employed by Mirkin [5], we can show that

$$\begin{aligned} & m(z)K_+(z) \\ =& \mathcal{F}_l(\begin{bmatrix} O & m(z)I \\ I & -C_2\Pi(z)B_2 \end{bmatrix}, \mathcal{F}_l(J^R_+, L^R_{22} + Q_+(z))) \\ =& \mathcal{F}_l(J, -F\Pi(z)m^{\sim}(A)L + m(z)(L^R_{22} + Q_+(z))), \end{aligned} \tag{29}$$

where

$$J := \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & O & I \\ -C_2 & I & O \end{array} \right].$$

We note L_{22} and J appear in parametrization of the stabilizing controller of the standard H^2 problem. Referring to Eq. (29), we let

$$\begin{aligned} & Q(z) \\ :=& -L_{22} - F\Pi(z)m^{\sim}(A)L + m(z)(L^R_{22} + Q_+(z)) \quad (30) \\ =& \left\{ \left[\begin{array}{c|c} A & L \\ \hline F & -L_{22} \end{array} \right] - m(z) \left[\begin{array}{c|c} A & L \\ \hline Fm^{\sim}(A) & -L^R_{22} \end{array} \right] \right\} \\ & + m(z)Q_+(z). \end{aligned}$$

At this point, we can confirm

$$-L^R_{22} = m^{\sim}(0)(-L_{22}) + F(m^{\sim}(A) - m^{\sim}(0)I)A^{-1}L$$

by direct calculation. Therefore, we have

$$Q(z) = \tau_m[\Theta](z) + m(z)Q_+(z). \tag{31}$$

From Eq. (29), (30) and (31),

$$m(z)K_+(z) = \mathcal{F}_l(J, L_{22} + Q(z)).$$

This means that $m(z)K_+(z)$ has the structure of the stabilizing controller of the standard H^2 problem of which the Youla parameter $Q(z)$ is constrained by Eq. (31). See Fig. 5. By the result of the standard H^2 problem,

$$\|T_{zw}\|^2_2 = E^2 + \left\| R_c^{1/2} Q \acute{R}_c^{1/2} \right\|^2_2 .$$

Since the first and second terms of the right-hand side of Eq. (31) belong to $\mathcal{H}(m)$ and mH^2, respectively, Eq. (28) holds. ■

Since the realization parameters (A, L) of $\Theta(z)$ is stabilizable, we can compute its H^2 norm by Theorem 1.

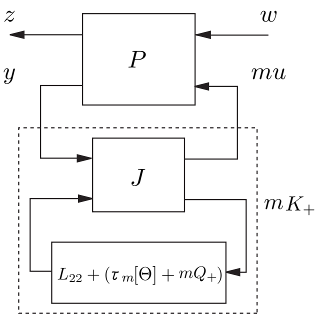

Fig. 5. constraint on the Youla parameter

B. Continuous-time case

We let the realization of the rational part P in Fig. 1 be

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right] \tag{32}$$

and impose the same condition on it as in the standard problem [9]. Furthermore, we assume $m(s)$ and its para-conjugate $m^{\sim}(s) := m(-\overline{s})^*$ are analytic on the set of eigenvalues of A so that $m(A)$ and $m^{\sim}(A)$ can be defined.

We use the stabilizing solutions $X \geq O$, $Y \geq O$ of the Riccati equations of the standard H^2 problem:

$$Q + A^*X + XA - (S^* + B_2^*X)^*R^{-1}(S^* + B_2^*X) = O,$$
$$\acute{Q} + AY + YA^* - (\acute{S}^* + YC_2^*)\acute{R}^{-1}(\acute{S}^* + YC_2^*)^* = O$$

and corresponding gains

$$F := -R^{-1}(S^* + B_2^*X), \ F^R := Fm^{\sim}(A),$$
$$L := -(\acute{S}^* + YC_2^*)\acute{R}^{-1}.$$

In the same way as in the discrete-time case, we can prove the following theorem.

Theorem 5: The stabilizing controller is parametrized by $\forall Q_+(s) \in H^2 \cap H^\infty$ and composed of the finite-dimensional element J_+^R:

$$\begin{aligned} \dot{x}_c^R(t) &= (A + m(A)B_2F^R + LC_2)x_c^R(t) \\ &\quad - Ly^R(t) + m(A)B_2\mu(t) \\ u(t) &= F^R x_c^R(t) + \mu(t) \\ \nu(t) &= -C_2 x_c^R(t) + y^R(t) \\ \mu(s) &= Q_+(s)\nu(s) \end{aligned}$$

and the infinite-dimensional element:

$$y^R(s) = y(s) - C_2\Pi(s)B_2u(s),$$

where

$$\Pi(s) := (sI - A)^{-1}(m(s)I - m(A)).$$

Furthermore, the optimal controller is given when $Q_+(s) = O$ and the optimal cost is given by

$$E_+^2 = \|\tau_m [P_{11}]\|_2^2 + \left\| \left[\begin{array}{c|c} A + B_2F & m^{\sim}(A)B_1 \\ \hline C_1 + D_{12}F & O \end{array}\right] \right\|_2^2 + \left\| R^{1/2} \left[\begin{array}{c|c} A + LC_2 & B_1 \\ \hline -F^R & O \end{array}\right] \right\|_2^2.$$

As in the proof of Theorem 4, we have

$$m(s)K_+(s) = \mathcal{F}_l(J, Q(s)),$$

where

$$J := \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & O & I \\ -C_2 & I & O \end{array}\right]$$

appears in parametrization of the stabilizing controller of the standard H^2 problem and $Q(s)$ is constrained by

$$Q(s) = \tau_m [\Theta] (s) + m(s)Q_+(s), \tag{33}$$
$$\forall Q_+(s) \in H^2 \cap H^\infty, \ \Theta(s) := \left[\begin{array}{c|c} A & L \\ \hline F & O \end{array}\right].$$

As a result, the following theorem holds.

Theorem 6 (Kashima [4]): $m(s)K_+(s)$ has the structure of the stabilizing controller of the standard H^2 problem of which the Youla parameter is constrained by Eq. (33). And by the orthogonality of $\mathcal{H}(m)$ and mH^2, the increase of the optimal cost is given by the H^2 norm of $\tau_m [\Theta]$:

$$E_+^2 = E^2 + \left\| R^{1/2}\tau_m [\Theta] \acute{R}^{1/2} \right\|_2^2.$$

In contrast to Kashima [4], we derived Theorem 6 after obtaining the controller of Smith form in Theorem 5.

V. Conclusion

We gave another solution method to the problems solved in [4] and [6]. From a viewpoint of our solution method and the representation of the increase of the optimal cost, the proposed definition of discrete-time truncation operator seems to be more natural than the one given in [6].

Establishing unified continuous/discrete-time solution methods in the corresponding H^∞ problem is a future subject of research.

References

[1] I. Gohberg, S. Goldberg and M. A. Kaashoek, *Classes of linear operators vol. I*, Birkhäuser, 1990.

[2] D. W. Gu, M. C. Tsai, S. D. O'Young and I. Postlethwaite, "State-space formulae for discrete-time H_∞ optimization," *Int. J. Control*, vol. 49, no. 5, pp. 1683–1723, 1989.

[3] K. Kashima and K. Nishio: "Performance degradation due to non-minimum phase properties, in discrete-time H^2 optimal control", Transactions of SICE, Vol. 43, No. 8, pp. 656–662, 2007 (in Japanese).

[4] K. Kashima, "A new expression for the H^2 performance limit based on state-space representation," *Automatica*, vol. 45, no. 1, pp. 283–290, 2009.

[5] L. Mirkin, "On the extraction of dead-time controllers and estimators from delay-free parametrizations," *IEEE Trans. Automat. Control*, vol. 48, no. 4, pp. 543–553, 2003.

[6] K. Nishio and K. Kashima, "A new expression for the discrete time H^2 performance limit based on state-space representation," *SICE-ICASE International Joint Conference 2006*, pp. 3959–3962, 2006.

[7] T. Reis, "Controllability and observability of infinite-dimensional descriptor systems," *IEEE. Trans. Automat. Control*, vol. 53, no. 4, pp. 929–940, 2008.

[8] Q. C. Zhong, "Bridging finite-spectrum assignment and Smith predictor," Technical Report, Imperial College London, 2003.

[9] K. Zhou, J. C. Doyle and K. Glover, *Robust and optimal control*, Prentice Hall, 1995.