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Optimal signal reconstruction from a series of recurring delayed measurements

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***Abstract*— The modern sampled-data approach provides a general methodology for signal reconstruction. This paper discusses some implications for optimal signal reconstruction when a series of recurring measurements, some delayed, are available for the reconstruction.**

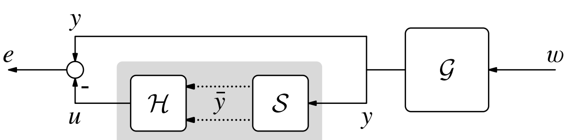

Fig. 1. Signal reconstruction setup

I. Introduction and problem formulation

The block diagram of Fig. 1 depicts a sampled data approach to signal reconstruction via multiple channels of samplers and holds,

$$\mathcal{H}\mathcal{S} = \mathcal{H}_1\mathcal{S}_1 + \mathcal{H}_2\mathcal{S}_2 + \cdots + \mathcal{H}_p\mathcal{S}_p. \tag{1}$$

Here an analog signal y is given to a $p \times 1$ sampler S (with sampling period h) which produces some discrete signal $\bar{y} \in \mathbb{R}^p$ and this, in turn, is fed to a $1 \times p$ hold device $\mathcal{H}$ that converts it back to an analog signal u. Ideally u equals y meaning that we reconstructed y error-free, and we say that a sampler and hold are optimal, with respect to some given norm, if they minimize the norm of the mapping $(I - \mathcal{H}\mathcal{S})\mathcal{G}$ from w to the reconstruction error $e := y - u$. Given $\mathcal{G}$, the design of optimal multi-channel samplers and holds among all linear h-shift invariant samplers/holds (possibly noncausal) has been solved in both L^2 and L^∞-norm [8] (it can also be derived fairly easily from the earlier paper [10].)

For the problems that we consider in this paper it is sufficient to formulate the solution only for LTI system $\mathcal{G}$ that are $p \times \omega_{\mathrm{N}}$-band dominant. These are LTI systems whose magnitude frequency response $|G(\mathrm{i}\omega)|$ satisfy

$$|G(\mathrm{i}\omega)| \geq |G(\mathrm{i}(\omega + 2kp\omega_{\mathrm{N}}))| \quad \forall \omega \in [-p\omega_{\mathrm{N}}, p\omega_{\mathrm{N}}], k \in \mathbb{Z}$$

where p is the number of sampler-hold channels, h is the sampling period and

$$\omega_{\mathrm{N}} := \frac{h}{\pi}$$

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is the corresponding Nyquist frequency. In particular any $|G(\mathrm{i}\omega)|$ that is monotonically decaying over positive frequency is $p \times \omega_{\mathrm{N}}$-band dominant. For such $\mathcal{G}$ the the linear h-shift invariant samplers and holds $\mathcal{H}_i, \mathcal{S}_i$ in (1) that minimize the L^2-norm, and L^∞-norm, of the error mapping $(I - \mathcal{H}\mathcal{S})\mathcal{G}$ are those for which their sum $\mathcal{H}\mathcal{S}$ is the LTI ideal low-pass filter with cut-off frequency $p \times \omega_{\mathrm{N}}$. So, in frequency domain,

$$(H_{\mathrm{opt}}S_{\mathrm{opt}})(\mathrm{i}\omega) = \begin{cases} 1 & \omega \in [-p\omega_{\mathrm{N}}, p\omega_{\mathrm{N}}] \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

That this ideal low-pass filter can indeed be implemented through p samplers and holds is not too hard to show. In fact this implementation can be done in many different ways and depending on the choice of implementation different reconstruction formulae result. In this paper we explore two of such implementations and corresponding reconstruction formulae. The main purpose of this paper is to show that such formulae follow without much difficulty from the general sampled-data theory. The final reconstruction formulas that we derive are not new.

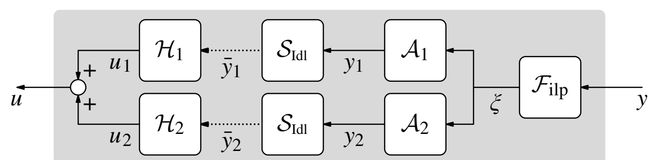

Fig. 2. An implementation of optimal $\mathcal{H}\mathcal{S}$ (for $p = 2$)

More specifically we implement the optimal samplers-holds (1) as shown in Fig. 2. Here $\mathcal{F}_{\mathrm{ilp}}$ is the ideal low-pass filter with cut-off frequency $p \times \omega_{\mathrm{N}}$, the S_{idl} are ideal samplers and the freedom is in the choice of LTI systems $\mathcal{A}_i$ and linear h-shift invariant holds $\mathcal{H}_i$. The figure depicts the case of $p = 2$ channels but the results readily extend to more channels. Within this class of implementations there is yet a wide variety of implementations and as we will see the choice of the $\mathcal{A}_i$ is almost arbitrary in the sense that for almost all $\mathcal{A}_i$ we can find hold operators $\mathcal{H}_i$ such that the overall mapping from y to u is the required optimal ideal low pass filter.

The two cases to be considered in this paper are:

- optimal signal reconstruction when besides samples $y(kh)$ also several of its derivatives $y^{(m)}(kh)$ are available for reconstruction. This follows by choosing the $\mathcal{A}_i$ to be the differentiators, $A_i(s) = s^{i-1}, i \geq 1$.
- optimal signal reconstruction when a series of recurring

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delayed measurements

$$y(kh+T_1), \quad y(kh+T_2), \ldots, y(kh+T_p), \quad k \in \mathbb{Z}$$

are available for signal reconstruction. This corresponds to the choices $A_i(s) = \mathrm{e}^{-sT_i}$, $i \geq 1$.

The latter recovers results of [12] and the first bears close resemblance with the generalized sampling theorems of [9].

This paper is an exercise in lifting and for that reason we summarize lifting first, in as far as needed to understand and prove the results to come. In Section III we summarize the results of [7] about optimal multichannel signal reconstruction and in the final two sections we turn to the two reconstruction problems.

A. Notation

Throughout, h denotes the sampling period and $\omega_{\mathrm{N}} = \pi/h$ the Nyquist frequency. The sinc with period h is denoted sinc_h, so

$$\mathrm{sinc}_h(t) = \frac{\sin(\omega_{\mathrm{N}} t)}{\omega_{\mathrm{N}} t}.$$

For any normalized frequency $\theta \in [-\pi, \pi]$ we use ω_k to denote the discrete sequence of aliased frequencies,

$$\omega_k := \frac{\theta + 2\pi k}{h} = \frac{\theta}{h} + 2\omega_{\mathrm{N}} k, \qquad k \in \mathbb{Z}. \tag{3}$$

In particular $\omega_0 = \theta/h$. The discrete unit pulse (Kronecker delta) is denoted $\bar{\delta}_k$.

II. Lifting preliminaries

In this section we review the by now familiar lifting technique [1], [2] and some of its implications. In particular we present the very useful key lifting formula [7], [6].

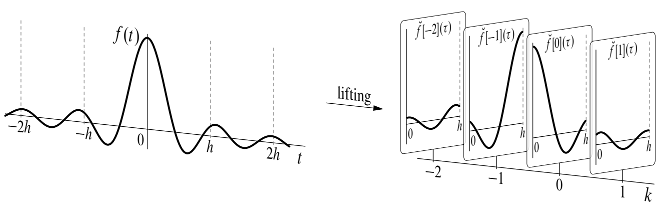


Fig. 3. Lifting analog signals (with $f(t) = \mathrm{sinc}_{0.44h}(t)$)

A. Lifting

In order to deal with the mixture of analog and discrete signals in a unified way one may represent all the analog signals as discrete signals while preserving their analog, intersample, behavior. This process is called lifting. Figure 3 explains the idea on a real-valued signal f. For arbitrary $f : \mathbb{R} \to \mathbb{C}^{n_f}$ the lifting $\breve{f} : \mathbb{Z} \to \{[0,h) \to \mathbb{C}^{n_f}\}$ is defined as

$$\breve{f}[k](\tau) = f(kh+\tau), \qquad k \in \mathbb{Z}.$$

Typically we suppress the intersample time τ and simply write $\breve{f}[k]$.

Having lifted all signals to discrete signals, *all* the systems such samplers and holds, but also the signal generator $\mathcal{G}$ can now be seen as being discrete time systems. In particular if all systems are linear and time invariant with respect multiples of the sampling period h, then in the lifted domain, as discrete systems, they are, once again, LTI. As a result one expects that Fourier analysis is once again beneficial. It is. The (lifted) z-transform of a lifted signal, say $\breve{f}$, is defined as

$$\breve{f}(z) = \sum_{k \in \mathbb{Z}} \breve{f}[k] z^{-k}.$$

When evaluated on the unit circle, $\breve{f}(\mathrm{e}^{\mathrm{i}\theta})$, we call it the (lifted) Fourier transform. The lifted z-transform equals the modified or advanced z-transform [5] and for $z = \mathrm{e}^{\mathrm{i}\theta}$ is also known as the *Zak transform* (modulo scaling) [3].

B. Key Lifting Formula

The following very useful result is a version of the Poisson Summation Formula, but then one that looses no information about the analog signal. Indeed the point of lifting is to maintain intersample behavior, also in frequency domain:

Theorem II.1 (Key Lifting Formula [6], [7])**.** *Let f be an analog signal with $f \in L^2(\mathbb{R})$. Then there is a bijection from the lifted Fourier transform $\breve{f}(\mathrm{e}^{\mathrm{i}\theta})$ and the classical Fourier transform $F(\mathrm{i}\omega)$:*

$$F(\mathrm{i}\omega_k) = \int_0^h \breve{f}(\mathrm{e}^{\mathrm{i}\theta}; \tau)\mathrm{e}^{-\mathrm{i}\omega_k \tau}\mathrm{d}\tau, \tag{4a}$$

$$\breve{f}(\mathrm{e}^{\mathrm{i}\theta}; \tau) = \frac{1}{h}\sum_{k \in \mathbb{Z}} F(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\omega_k \tau}, \tag{4b}$$

where ω_k are the aliased frequencies (3) *of θ.* ▽

C. Samplers and holds in lifted frequency domain

We allow any linear h-shift invariant hold of the form

$$u = \mathcal{H}\bar{u}: \quad u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih)\bar{u}[i], \quad t \in \mathbb{R}.$$

It maps discrete signals $\bar{u}$ to analog signals u. The function ϕ is known as the *hold function*. It defines the hold and it equals the hold's response to the discrete unit pulse. After lifting the analog output and z-transforming all signals, the hold becomes a product at each z,

$$\breve{u}(z) = \acute{H}(z)\bar{u}(z): \quad \breve{u}(z;\tau) = \breve{\phi}(z;\tau)\bar{u}(z).$$

The function $\breve{\phi}$ is the lifted z-transform of the hold function.

Similarly we allow linear h-shift invariant samplers of the form

$$\bar{y} = \mathcal{S}y: \quad \bar{y}[k] = \int_{-\infty}^{\infty} \psi(kh - s)y(s)\mathrm{d}s, \quad k \in \mathbb{Z}.$$

It maps analog signals y to discrete signals $\bar{y}$. Here ψ is its *sampling function*. After lifting the analog output and z-transforming all signals, the sampler, at each z, is an integral over intersample time,

$$\bar{y}(z) = \grave{S}(z)\breve{y}(z): \quad \bar{y}(z) = \int_0^h \breve{\psi}(z; -\sigma)\breve{y}(z;\sigma)\mathrm{d}\sigma.$$

Notice that we evaluate its z-transform over negative intersample time.

The series connection $u = \mathcal{H}\mathcal{S}y$ of a sampler and a hold in lifted frequency domain therefore has the form

$$\breve{u}(z;\tau) = \int_0^h \breve{f}(e^{i\theta};\tau,\sigma)\breve{y}(z;\sigma)\,d\sigma \tag{5}$$

where

$$\breve{f}(e^{i\theta};\tau,\sigma) = \breve{\phi}(e^{i\theta},\tau)\breve{\psi}(e^{i\theta},-\sigma).$$

By linearity then the sum of p sampler-hold channels is also of the form (5) with now the frequency response kernel a sum of p functions,

$$\breve{f}(e^{i\theta};\tau,\sigma) = \sum_{n=1}^{p}\breve{\phi}_n(e^{i\theta},\tau)\breve{\psi}_n(e^{i\theta},-\sigma).$$

The function $\breve{f}(e^{i\theta};\tau,\sigma)$ we refer to as the *frequency response kernel* of the mapping $\mathcal{H}\mathcal{S}$.

III. Multichannel signal reconstruction

In [8] it is shown that the sum $\mathcal{H}\mathcal{S}$ of p sampler-holds (1) equals the ideal low-pass filter with cut-off frequency $p \times \omega_N$ iff its frequency response kernel is

$$\breve{f}(e^{i\theta};\tau,\sigma) = \frac{1}{h}\underbrace{(e^{i\omega_0(\tau-\sigma)} + e^{i\omega_{-1}(\tau-\sigma)} + e^{i\omega_{+1}(\tau-\sigma)} + \cdots)}_{p\text{ terms}}$$

for $\theta \in [0,\pi]$. This optimal kernel naturally splits into sampler-hold channels by decomposing it as

$$\begin{aligned}\breve{f}(e^{i\theta};\tau,\sigma) &= \frac{1}{h}\left(e^{i\omega_0(\tau-\sigma)} + e^{i\omega_{-1}(\tau-\sigma)} + \cdots\right)\\ &= \begin{bmatrix} e^{i\omega_0\tau} & e^{i\omega_{-1}\tau} & \cdots \end{bmatrix}\begin{bmatrix} \frac{1}{h}e^{-i\omega_0\sigma} \\ \frac{1}{h}e^{-i\omega_{-1}\sigma} \\ \vdots \end{bmatrix}\\ &=: \begin{bmatrix} \breve{\phi}_1(e^{i\theta};\tau) & \breve{\phi}_2(e^{i\theta};\tau) & \cdots \end{bmatrix}\begin{bmatrix} \breve{\psi}_1(e^{i\theta};-\sigma) \\ \breve{\psi}_2(e^{i\theta};-\sigma) \\ \vdots \end{bmatrix}\end{aligned} \tag{6}$$

for $\theta \in [0,\pi]$. By direct inverse Fourier transformation of the so defined $\breve{\phi}_n$, $\breve{\psi}_n$ one obtains a series of modulated sinc sampling and hold functions

$$\begin{aligned} h\psi_1(t) &= \phi_1(t) = \mathrm{sinc}_{2h}(t)\cos(\tfrac{1}{2}\omega_N t) = \mathrm{sinc}_h(t)\\ h\psi_2(t) &= \phi_2(t) = \mathrm{sinc}_{2h}(t)\cos(\tfrac{3}{2}\omega_N t)\\ h\psi_3(t) &= \phi_3(t) = \mathrm{sinc}_{2h}(t)\cos(\tfrac{5}{2}\omega_N t)\\ &\vdots \end{aligned}$$

Many other splittings of $\breve{f}(e^{i\theta})$ exist. Indeed (6) clearly holds true for

$$\begin{aligned}\begin{bmatrix} \breve{\phi}_1(e^{i\theta};\tau) & \breve{\phi}_2(e^{i\theta};\tau) & \cdots \end{bmatrix} &:= \begin{bmatrix} e^{i\omega_0\tau} & e^{i\omega_{-1}\tau} & \cdots \end{bmatrix}\bar{A}^{-1}(\theta)\\ \begin{bmatrix} \breve{\psi}_1(e^{i\theta};\sigma) \\ \breve{\psi}_2(e^{i\theta};\sigma) \\ \vdots \end{bmatrix} &:= \frac{1}{h}\bar{A}(\theta)\begin{bmatrix} e^{i\omega_0\sigma} \\ e^{i\omega_{-1}\sigma} \\ \vdots \end{bmatrix}\end{aligned} \tag{7}$$

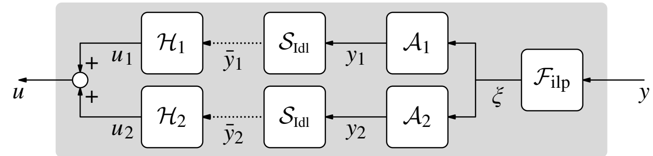


Fig. 4. An implementation of optimal $\mathcal{H}\mathcal{S}$ (for $p = 2$)

for any $p \times p$ "mixing matrix" $\bar{A}(\theta)$ that is boundedly invertible. This way the separate channels $\mathcal{H}_i\mathcal{S}_i$ could be time-varying while we know that their sum $\mathcal{H}\mathcal{S}$ is LTI.

Now consider the alternative implementation of $\mathcal{H}\mathcal{S}$ as shown in Fig. 2, copied in Fig. 4 for ease of reference.

Lemma III.1. *If F_{ilp} is the ideal low-pass filter with cut-off frequency $p \times \omega_N$ then the overall mapping from y to u in Fig. 4 is the ideal low-pass filter with cut-off frequency $p \times \omega_N$ if we choose the holds $\mathcal{H}_1, \dots, \mathcal{H}_p$ to have hold functions, in frequency domain, equal to*

$$\begin{bmatrix} \phi_1(e^{i\theta}) & \phi_2(e^{i\theta}) & \cdots \end{bmatrix} = \begin{bmatrix} e^{i\omega_0\tau} & e^{i\omega_{-1}\tau} & \cdots \end{bmatrix}\bar{A}^{-1}(\theta)$$

for $\theta \in [0,\pi]$, in which $\bar{A}(\theta)$ is the $p \times p$ matrix

$$\bar{A}(\theta) := \begin{bmatrix} A_1(i\omega_0) & A_1(i\omega_{-1}) & A_1(i\omega_{+1}) & \cdots \\ A_2(i\omega_0) & A_2(i\omega_{-1}) & A_1(i\omega_{+1}) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ A_p(i\omega_0) & A_p(i\omega_{-1}) & A_p(i\omega_{+1}) & \cdots \end{bmatrix} \tag{8}$$

for $\theta \in [0,\pi]$. The hold functions are well defined if $\bar{A}(\theta)$ is boundedly invertible.

Proof. For ease of exposition we prove it for $p = 2$. The mapping from y to $\bar{y}_1$ in Fig. 4 is a sampler $\mathcal{S}_{\text{idl}}\mathcal{A}_1\mathcal{F}_{\text{ilp}}$. The sampling function of this sampler is the impulse response of $\mathcal{A}_1\mathcal{F}_{\text{ilp}}$. Its frequency response according to the Key Lifting Formula (4b) is $\breve{\psi}_1(e^{i\theta};\sigma) = \frac{1}{h}\sum_{k\in\mathbb{Z}} A_1(i\omega_k)F_{\text{ilp}}(i\omega_k)e^{i\omega_k\sigma}$, which for $\theta \in [0,\pi]$ and by the bandlimitness of the ideal low-pass filter becomes the finite sum

$$\begin{aligned}\breve{\psi}_1(e^{i\theta};\sigma) &= \frac{1}{h}\sum_{k\in\mathbb{Z}} A_1(i\omega_k)F_{\text{ilp}}(i\omega_k)e^{i\omega_k\sigma}\\ &= \frac{1}{h}\left[A_1(i\omega_0)e^{i\omega_0\sigma} + A_1(i\omega_{-1})e^{i\omega_{-1}\sigma}\right]\\ &= \frac{1}{h}\begin{bmatrix} A_1(i\omega_0) & A_1(i\omega_{-1}) \end{bmatrix}\begin{bmatrix} e^{i\omega_0\sigma} \\ e^{i\omega_{-1}\sigma} \end{bmatrix}.\end{aligned}$$

For the second loop, the A_1 has to be replaced with A_2, et cetera. This shows that the samplers $\mathcal{S}_i$ that map y to $\bar{y}_i$ in Fig. 4 have sampling functions that satisfy (7). ☐

IV. Samples and derivatives

Assume that $p = 2$. If $\mathcal{A}_1$ is the identity and $\mathcal{A}_2$ the differentiator then the mixing matrix (8) is

$$\bar{A}(\theta) = \begin{bmatrix} 1 & 1 \\ i\omega_0 & i\omega_{-1} \end{bmatrix}.$$

This matrix has constant nonzero determinant $-i2\pi/h$. The hold functions now become, in frequency domain,

$$\begin{aligned}&\begin{bmatrix}\phi_1(e^{i\theta}) & \phi_2(e^{i\theta})\end{bmatrix}\\ &= \begin{bmatrix}e^{i\omega_0\tau} & e^{i\omega_{-1}\tau}\end{bmatrix}\bar{A}^{-1}(\theta)\\ &= \begin{bmatrix}\dfrac{e^{i\omega_0\tau}i\omega_{-1} - e^{i\omega_{-1}\tau}i\omega_0}{-i2\pi/h} & \dfrac{-e^{i\omega_0\tau}+e^{i\omega_{-1}\tau}}{-i2\pi/h}\end{bmatrix}.\end{aligned}$$

Inverse Fourier transformation subsequently yields the two hold functions

$$\phi_1(t) = \operatorname{sinc}_h^2(t), \qquad \phi_2(t) = t\operatorname{sinc}_h^2(t).$$

Since $\mathcal{HS} = I$ on the space of $p \times \omega_N$-bandlimited signals, we get that

$$f(t) = \sum_{k\in\mathbb{Z}} \phi_1(t-kh)f(kh) + \phi_2(t-kh)f'(kh)$$

for any $f(t)$ that is $2\omega_N$-bandlimited. This is a well known reconstruction formula [4]. The results bears close resemblance with the generalized sampling theorems of [9].

For two channels the mixing matrix $\bar{A}(\theta)$ is 2×2. It is in theory straightforward to extend the ideas to more than two channels. For instance when p derivative samples, $y^{(i)}(kh)$ for $i = 0,\dots,p-1$, are available etcetera. The formulae are unwieldy though.

V. Recurring non-uniform sampling

Suppose that at the kth sampling interval we have p samples of the signal available at times

$$\begin{aligned}&kh + T_1,\\ &kh + T_2,\\ &\quad\vdots\\ &kh + T_p.\end{aligned}$$

This is called recurring non-uniform sampling because the $T_1,\dots,T_p$ are arbitrary (non-uniform) but this group of p repeats every h time units. Clearly this situation may be modeled as in Fig. 4 with the systems $\mathcal{A}_1,\dots,\mathcal{A}_p$ equal to the delay/advance operators,

$$\begin{aligned}A_1(s) &= e^{-sT_1},\\ A_2(s) &= e^{-sT_2},\\ &\ \ \vdots\\ A_p(s) &= e^{-sT_p}.\end{aligned}$$

Now the mixing matrix (8) is a Vandermonde-like matrix

$$\bar{A}(\theta) = \begin{bmatrix} e^{-iT_1\omega_0} & e^{-iT_1\omega_{-1}} & e^{-iT_1\omega_{+1}} & \cdots\\ e^{-iT_2\omega_0} & e^{-iT_2\omega_{-1}} & e^{-iT_2\omega_{+1}} & \cdots\\ e^{-iT_3\omega_0} & e^{-iT_3\omega_{-1}} & e^{-iT_3\omega_{+1}} & \cdots\\ \vdots & \vdots & \vdots & \cdots\end{bmatrix} \tag{9}$$

for $\theta \in [0,\pi]$.

Example V.1 (Recurring non-uniform delayed sampling for $p = 2$ [8])**.** If $\mathcal{A}_1$ is the identity and $\mathcal{A}_2$ the T-delay operator $A_2(i\omega) = e^{-iT\omega}$ then (9) becomes the Vandermonde-like matrix

$$\bar{A}(\theta) = \begin{bmatrix}1 & 1\\ e^{-iT\omega_0} & e^{-iT\omega_{-1}}\end{bmatrix}$$

for $\theta \in [0,\pi]$. It is invertible iff the delay T is not a multiple of the sampling period h, in which case

$$\bar{A}^{-1}(\theta) = \frac{1}{e^{-iT\omega_{-1}} - e^{-iT\omega_0}}\begin{bmatrix}e^{-iT\omega_{-1}} & -1\\ -e^{-iT\omega_0} & 1\end{bmatrix}.$$

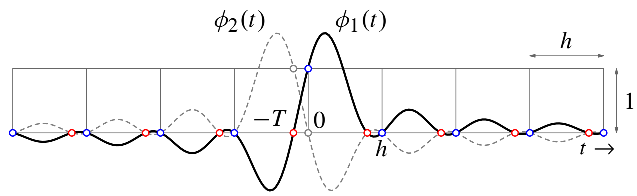
Direct inverse Fourier transformation of

$$\begin{bmatrix}\phi_1(e^{i\theta}) & \phi_2(e^{i\theta})\end{bmatrix} = \begin{bmatrix}e^{i\omega_0\tau} & e^{i\omega_{-1}\tau}\end{bmatrix}\bar{A}^{-1}(\theta)$$

now yields the optimal hold functions

$$\begin{aligned}\phi_1(t) &= \operatorname{sinc}_h(t)\frac{\sin(\omega_N(t+T))}{\sin(\omega_N T)},\\ \phi_2(t) &= \phi_1(-t-T)\end{aligned}$$

For $T = 0.2h$ the two hold functions are



▽

Notice that in the above example the $\phi_1(kh)$ at multiples of the sampling period equals the Kronecker delta $\bar{\delta}_k$ and that it is zero at every $-T+kh$. In fact as shown below this $\phi_1(t)$ is the unique $2\omega_N$-bandlimited signal that satisfies these interpolation conditions. By symmetry $\phi_2(t) = \phi_1(-t-T)$ has comparable interpolation properties.

Lemma V.2 ([12])**.** *If we have p samples every $[hk, hk+h)$ at*

$$t = hk + T_1, \quad t = hk+T_2, \quad \dots \quad t = hk + T_p$$

with no two differences $T_n - T_{i\neq n}$ a multiple of h. Then the p optimal hold functions $\phi_1,\dots,\phi_p$ are

$$\phi_n(t) = \operatorname{sinc}_h(t+T_n)\prod_{i\neq n}\frac{\sin(\omega_N(t+T_i))}{\sin(\omega_N(-T_n+T_i))}. \tag{10}$$

They are the unique functions that are $p\times\omega_N$-bandlimited and satisfy the interpolation conditions that $\phi_n(-T_n+kh) = \bar{\delta}_k$ and $\phi(t) = 0$ at every other $t = -T_{i\neq n} + kh$.

Proof. Since the optimal $\mathcal{HS}$ is the ideal low-pass filter with cut-off frequency $p\times\omega_N$ it reconstructs any $p\times\omega_N$-bandlimited signal y error free. Now take $y = \phi_n$ as defined in (10). Clearly these are $p\times\omega_N$-bandlimited because they are p products of ω_N-bandlimited signals. By construction the output $\bar{y}_i$ of the ith sampler $\mathcal{S}_i := \mathcal{S}_{\text{idl}}\mathcal{A}_i$ is zero for this input $y = \phi_n$ if $i \neq n$. So for this y only the nth channel in Fig. 4 is active and so we have $u = \mathcal{H}_n\bar{y}_n = \mathcal{H}_n\bar{\delta}$ and this has to equal $y = \phi_n$. Thus ϕ_n is the hold function of $\mathcal{H}_n$.

The ϕ_n are uniquely determined by the two mentioned conditions, because if two $p\times\omega_N$-bandlimited signals ζ_n, η_n

satisfy the same interpolation conditions then $\zeta_n = \mathcal{H}\mathcal{S}\zeta_n = \mathcal{H}(\mathcal{S}\zeta_n) = \mathcal{H}(\mathcal{S}\eta_n) = \mathcal{H}\mathcal{S}\eta_n = \eta_n$ i.e., then they are the same. □

This recovers Yen's original work [12] and in contrast to the previous section the formulae are now manageable for any p. Figure 5 shows a possible set of optimal hold functions for $p = 3$.

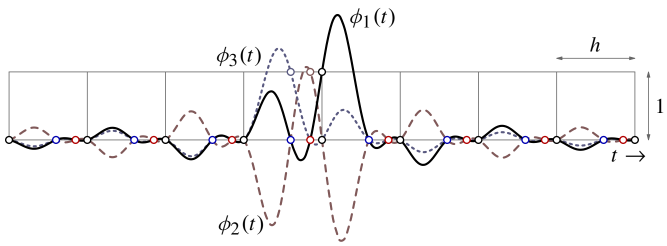


Fig. 5. Three optimal hold functions for $p = 3$

VI. Concluding remarks

Besides [12], [9], [4] the results are also closely related to [11]. They treat the same problem but then aim at consistent rather than norm-optimal holds and samplers. This, however, is closely related to norm-optimality because consistency is they defined it is an interpolation condition and as we saw in Lemma V.2 norm-optimality for this case is equivalent to an interpolation condition.

References

[1] T. Chen and B. A. Francis. *Optimal Sampled-Data Control Systems*. Springer-Verlag, London, 1995.
[2] G. E. Dullerud. *Control of Uncertain Sampled-Data Systems*. Birkhäuser, Boston, 1996.
[3] A.J. Janssen. The zak transform: A signal transform for sampled time-continuous signals. *Philips J. Research*, 43:23–69, 1988.
[4] A. J. Jerri. The Shannon sampling theorem—its various extensions and applications: A tutorial review. *Proc. IEEE*, 65(11):1565–1596, 1977.
[5] E. I. Jury. *Sampled-Data Control Systems*. John Wiley & Sons, New York, 1958.
[6] G. Meinsma and L. Mirkin. System theoretic perspectives of the Sampling Theorem. In *Proc. 17th MTNS Symposium*, pages 730–741, Kyoto, Japan, 2006.
[7] G. Meinsma and L. Mirkin. Sampling from a system-theoretic viewpoint: Part I—concepts and tools. *IEEE Trans. Signal Processing*, 58, 2010. (to appear).
[8] G. Meinsma and L. Mirkin. Sampling from a system-theoretic viewpoint: Part II—non-causal solutions. *IEEE Trans. Signal Processing*, 58, 2010. (to appear).
[9] A. Papoulis. Generalized sampling expansion. *IEEE Trans. Circuits and Systems*, 24(5):652–654, 1977.
[10] M. Unser. On the optimality of ideal filters for pyramid and wavelet signal approximation. *IEEE Trans. Signal Processing*, 41(12):3591–3596, 1993.
[11] M. Unser and J. Zerubia. A generalized sampling theorem without band-limiting constraints. *IEEE Trans. Circuits Syst. I*, 45(8):959–969, 1998.
[12] J. L. Yen. On nonuniform sampling of bandwidth-limited signals. *IRE Trans. Circuit Theory*, CT-3:251–257, 1956.