An $\mathcal{H}_\infty$ calculus of admissible operators

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***Abstract*— Given a Hilbert space and the generator A of a strongly continuous, exponentially stable, semigroup on this Hilbert space. For any $g(-s) \in \mathcal{H}_\infty$ we show that there exists an infinite-time admissible output operator $g(A)$. If g is rational, then this operator is bounded, and equals the "normal" definition of $g(A)$. In particular, when $g(s) = 1/(s+\alpha)$, $\alpha \in \mathbb{C}_0^+$, then this admissible output operator equals $(\alpha I - A)^{-1}$.**

Although in general $g(A)$ may be unbounded, we always have that $g(A)$ multiplied by the semigroup is a bounded operator for every (strictly) positive time instant. Furthermore, when there exists an admissible output operator C such that (C, A) is exactly observable, then $g(A)$ is bounded for all g's with $g(-s) \in \mathcal{H}_\infty$.

I. Introduction

Functional calculus is a sub-field of mathematics with a long history. It started in the thirties of the last century with the work by von Neumann for self-adjoint operators [9], and was further extended by many researchers, see e.g. [8] and [3]. For an overview, see the book by Markus Haase, [7]. The basic idea behind functional calculus for the operator A is to construct a mapping from an algebra of (scalar) functions to the class of bounded operators, such that

- The function identically equals to one is mapped to the identity operator;
- If $f(s) = (s-a)^{-1}$, then $f(A) = (sI - A)^{-1}$;
- Furthermore, the operator associated to $f_1 \cdot f_2$ equals $f(A)f_2(A)$.

Before we explain the contribution of this paper, we introduce some notation. By X we denote separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and by A we denote an unbounded operator from its domain $D(A) \subset X$ to X. We assume that A generates an exponentially stable semigroup on X, which we denote by $(T(t))_{t \geq 0}$.

By $\mathcal{H}_\infty^-$ we denote the space of all bounded, analytic functions defined on the half-plane $\mathbb{C}^- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$. It is clear that this function class is an algebra under pointwise multiplication and addition. Hence this could serve as a class for which one could build a functional calculus. However, it is known that there exists a generator of exponential stable semigroup, which does not have a functional calculus with respect to $\mathcal{H}_\infty^-$. For a proof of this and many more we refer to [1], [7], and the references therein. Although a bounded functional calculus is not possible, an unbounded functional calculus is always possible.

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Theorem 1.1: Under the assumptions stated above, we have that for all $g \in \mathcal{H}_\infty^-$ there exists an operator $g(A)$ which is bounded from the domain of A to X, and which is admissible, i.e.,

$$\int_0^\infty \|g(A)T(t)x_0\|^2 dt \leq \gamma_A \|g\|_\infty^2 \|x_0\|^2, \qquad x_0 \in X.$$

The mapping $g \mapsto g(A)$ satisfies the conditions of a functional calculus. Furthermore, for all $t > 0$, we have that $g(A)T(t)$ can be extended to a bounded operator, and

$$\|g(A)T(t)\| \leq \frac{\gamma}{\sqrt{t}}.$$

Apart from proving this theorem, we shall also rediscover some classes of generators for which $g(A)$ is bounded for all $g \in \mathcal{H}_\infty^-$, i.e., for which there is a bounded functional calculus. For more results, we refer to [11].

For the proof of the above result, we need beside the Hardy space $\mathcal{H}_\infty^-$ also the Hardy spaces $\mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$.

$\mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$ denote the Laplace transform, $\mathcal{L}$, of functions in $L^2((0,\infty), X)$ and $L^2((-\infty), 0), X)$, respectively. It is known that this transformation is an isometry. Every function in $\mathcal{H}_\infty^-, \mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$ has a unique extension to the imaginary on which this functions are bounded, and square integrable, respectively. Furthermore, the norm of $g \in \mathcal{H}_\infty^-$ equals the (essential) supremum over the imaginary axis of the boundary function. Let $f(t)$ be a function in $L^2((0,\infty), X)$ with Laplace transform $F(s)$, and let $f_{\rm ext}(t)$ be the function in $L^2((-\infty,\infty), X)$ defined by

$$f_{\rm ext}(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then the Fourier transform $\hat{f}_{\rm ext}$ of $f_{\rm ext}(t)$ satisfies $\hat{f}_{\rm ext}(\omega) = F(i\omega)$, for almost all $\omega \in \mathbb{R}$. Here $F(i\cdot)$ denote the boundary function of the Laplace transform $F(s)$.

We define the following Toeplitz operator on $L^2(0,\infty); X)$

Definition 1.2: Let g be an element of $\mathcal{H}_\infty^-$. Associated to this function we define the mapping M_g as

$$M_g f = \mathcal{L}^{-1}\left(\Pi\left(gF\right)\right), \qquad f \in L^2((0,\infty), X), \qquad (1)$$

where F denotes the Laplace transform of f. Π denotes the projection onto $\mathcal{H}_2(X)$.

It is clear that this is a linear bounded map from $L^2((0,\infty); X)$ into itself, and

$$\|M_g\| \leq \|g\|_\infty. \qquad (2)$$

Furthermore, it follows easily from (1) that if K is a bounded mapping on X, then its commutes with M_g, i.e.,

$$KM_g = M_g K. \qquad (3)$$

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It is easy to see that $\mathcal{H}_\infty^-$ is an algebra under the multiplication and addition. In particular $g_1 g_2 \in \mathcal{H}_\infty^-$ whenever g_1, $g_2 \in \mathcal{H}_\infty^-$. Furthermore, we have the following result.

Lemma 1.3: Let g_1 and g_2 be elements of $\mathcal{H}_\infty^-$. Then

$$M_{g_1 g_2} = M_{g_1} M_{g_2}. \tag{4}$$

In particular, if g is invertible in $\mathcal{H}_\infty^-$, then M_g is (boundedly) invertible and $(M_g)^{-1} = M_{g^{-1}}$.

Proof We use the fact that any $g \in \mathcal{H}_\infty^-$ maps $\mathcal{H}_2^\perp$ into $\mathcal{H}_2^\perp$.

$$\begin{aligned} M_{g_1} M_{g_2} f &= \mathcal{L}^{-1}\left(\Pi g_1 \left(\Pi\left(g_2 F\right)\right)\right) \\ &= \mathcal{L}^{-1}\left(\Pi\left(g_1 g_2 F\right)\right) + \\ &\quad \mathcal{L}^{-1}\left(\Pi\left(g_1 (I-\Pi)\left(g_2 F\right)\right)\right) \\ &= \mathcal{L}^{-1}\left(\Pi\left(g_1 g_2 F\right)\right) + 0, \end{aligned}$$

where we have used the above mentioned fact that $g_1(I-\Pi)$ maps into $\mathcal{H}_2^\perp$, and so $\Pi g_1(I-\Pi) = 0$. Since by definition $\mathcal{L}^{-1}\left(\Pi\left(g_1 g_2 F\right)\right)$ equals $M_{g_1 g_2} f$, we have proved the first assertion.

The last assertion follows directly, since $M_1 = I$. □

By σ_τ we denote the shift with τ, i.e.,

$$\left(\sigma_\tau(f)\right)(t) = f(t+\tau), \qquad t \geq 0. \tag{5}$$

This is also a linear bounded map from $L^2((0,\infty);X)$ into itself. This mapping commutes with M_g as is shown next.

Lemma 1.4: For all $\tau > 0$ and all g in $\mathcal{H}_\infty^-$, we have that

$$\sigma_\tau\left(M_g f\right) = M_g\left(\sigma_\tau f\right), \qquad f \in L^2((0,\infty),X). \tag{6}$$

Proof We use the following well-known equality. If h is Fourier transformable, then the Fourier transform of $h(\cdot + \tau)$ equals $e^{i\omega\tau}\hat{h}(\omega)$, where $\hat{h}$ denotes the Fourier transform of h.

Let $h \in L^2(0,\infty);X)$, then

$$\begin{aligned} \mathcal{L}(\sigma_\tau h) &= \widehat{(\sigma_\tau h)_{\text{ext}}} = \widehat{\sigma_\tau h_{\text{ext}}} - \hat{q} \\ &= e^{i\omega\tau}\widehat{h_{\text{ext}}} - \hat{q} = e^{i\omega\tau}\mathcal{L}(h) - \hat{q}, \end{aligned} \tag{7}$$

with $q \in L^2((-\infty,0);X)$. In particular, we find for every $h \in L^2(0,\infty);X)$ that

$$\begin{aligned} \mathcal{L}(\sigma_\tau h) &= \Pi\left(\mathcal{L}(\sigma_\tau h)\right) \\ &= \Pi\left(e^{i\omega\tau}\mathcal{L}(h)\right) - 0 = \mathcal{L}\left(M_{e^{i\cdot\tau}} h\right), \end{aligned} \tag{8}$$

where we have used that $e^{i\omega\tau}$ is the boundary function corresponding to $e^{is\tau} \in \mathcal{H}_\infty^-$.

Using (7) we see that

$$\begin{aligned} M_g\left(\sigma_\tau f\right) &= \mathcal{L}^{-1}\left(\Pi\left(g e^{i\cdot\tau}\mathcal{L}(f)\right)\right) - \mathcal{L}^{-1}\left(\Pi\left(g\hat{q}\right)\right) \\ &= \mathcal{L}^{-1}\left(\Pi\left(g e^{i\cdot\tau}\mathcal{L}(f)\right)\right), \end{aligned} \tag{9}$$

since $\hat{q} \in \mathcal{H}_2^\perp(X)$, and since $g \in \mathcal{H}_\infty^-$. Using Lemma 1.3, we find that

$$\begin{aligned} M_g\left(\sigma_\tau f\right) &= \mathcal{L}^{-1}\left(\Pi\left(g e^{i\cdot\tau}\mathcal{L}(f)\right)\right) \\ &= M_{e^{i\cdot\tau} g} f = M_{e^{i\cdot\tau}} M_g f. \end{aligned} \tag{10}$$

Now using (8), we see that

$$M_g\left(\sigma_\tau f\right) = \sigma_\tau\left(M_g f\right). \tag{11}$$

□

II. Output maps and admissible output operators

In this section we study admissible operators which commute with the semigroup. We begin by defining well-posed output maps.

Definition 2.1: Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Hilbert space X, and let Y be another Hilbert space. We say that the mapping $\mathcal{O}$ is a well-posed (infinite-time) output map if

- $\mathcal{O}$ is a bounded linear mapping from X into $L^2((0,\infty);Y)$, and
- For all $\tau \geq 0$ and all $x_0 \in X$, we have that $\sigma_\tau \mathcal{O} x_0 = \mathcal{O}\left(T(\tau)x_0\right)$.

Closely related to well-posed output mapping are admissible operators, which are defined next.

Definition 2.2: Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Hilbert space X. Let $D(A)$ be the domain of its generator A. A linear mapping C from $D(A)$ to Y, another Hilbert space, is said to be an (infinite-time) admissible output operator for $(T(t))_{t\geq 0}$ if $CT(\cdot)x_0 \in L^2((0,\infty),Y)$ for all $x_0 \in D(A)$ and there exists an m independent of x_0 such that

$$\int_0^\infty \|CT(t)x_0\|_Y^2 dt \leq m\|x_0\|_X^2. \tag{12}$$

If C is (infinite-time) admissible, then for all $x_0 \in X$ we can uniquely define an $L^2((0,\infty),X)$-function. We denote this function by $CT(\cdot)x_0$. Hence $\mathcal{O} : X \to L^2((0,\infty);Y)$ defined by $\mathcal{O}x_0 = CT(\cdot)x_0$ is a well-posed output map. From [10] we know that the converse holds as well.

Lemma 2.3: If $\mathcal{O}$ is a well-posed output mapping, then there exists a (unique) linear bounded mapping from $D(A)$ to Y, C, such that $\mathcal{O}x_0 = CT(\cdot)x_0$ for all x_0.

In the sequel of this section we concentrate on admissible output operators which commute with A, i.e., C a linear operator from $D(A)$ to X and

$$CA^{-1} = A^{-1}C \qquad \text{on } D(A). \tag{13}$$

For these operators we have the following results.

Lemma 2.4: Let C be the admissible output operator associated with the well-posed output map $\mathcal{O}$. Then (13) holds if and only if for all $t \geq 0$ there holds $\mathcal{O}T(t) = T(t)\mathcal{O}$.

Theorem 2.5: Let C be a bounded linear operator from $D(A)$ to X, which is admissible for the exponentially stable semigroup $(T(t))_{t\geq 0}$ and which commutes with A. Then the following holds

1) For all $x_0 \in D(A)$ and all $t \geq 0$, we have that $CT(t)x_0 = T(t)Cx_0$.
2) For all $t > 0$, the operator $CT(t) : D(A) \to X$ can be extended to a bounded operator on X. Furthermore, $\|CT(t)\| \leq \gamma t^{-1/2}$ for some γ independent of t.

Proof The first assertion follows easily from (13) by using Laplace transforms. We concentrate on the second assertion.

Let $x_0 \in D(A)$ and $x_1 \in X$, then for $t > 0$ we have that

$$\begin{aligned} t\langle x_1, CT(t)x_0\rangle &= \int_0^t \langle x_1, CT(t)x_0\rangle d\tau \\ &= \int_0^t \langle x_1, CT(\tau)T(t-\tau)x_0\rangle d\tau \\ &= \int_0^t \langle x_1, T(\tau)CT(t-\tau)x_0\rangle d\tau \\ &= \int_0^t \langle T(\tau)^* x_1, CT(t-\tau)x_0\rangle d\tau \\ &\leq \sqrt{\int_0^t \|T(\tau)^* x_1\|^2 d\tau} \cdot \\ &\quad \sqrt{\int_0^t \|CT(t-\tau)x_0\|^2 d\tau}. \end{aligned}$$

Using the fact that the semigroup, and hence its adjoint, are uniformly bounded, and the fact that C is (infinite-time) admissible, we find that

$$t\langle x_1, CT(t)x_0\rangle \leq \sqrt{t}M\|x_1\|m\|x_0\|.$$

Since this holds for all $x_1 \in X$, we conclude that

$$t\|CT(t)x_0\| \leq \sqrt{t}mM\|x_0\|.$$

This inequality holds for all $x_0 \in D(A)$. The domain of a generator is dense, and hence we have proved the second assertion. □

From the above it is clear that if the semigroup is surjective, then any admissible C which commutes with the generator is bounded.

The Lebesgue extension of an admissible operator is defined by

$$C_L x = \lim_{t\to 0} \frac{1}{t} C \int_0^t T(\tau)x d\tau,$$

where

$$D(C_L) = \{x \in X \mid \text{limit exists}\}.$$

Lemma 2.6: Let C be an admissible operator which commutes with the generator, then the same holds for its Lebesgue extension. Furthermore, this Lebesgue extension is a closed operator.

Proof Let x_n be a sequence in $D(C_L)$ which converges to $x \in X$, such that $C_L x_n$ converges to $z \in X$. For $\tau \geq 0$, we have that

$$C_L T(\tau)x_n = T(\tau)C_L x_n \to T(\tau)z.$$

Hence

$$C_L \int_0^t T(\tau)x_n d\tau \to \int_0^t T(\tau)z d\tau$$

The expression on the left-hand side equals

$$\begin{aligned} C_L \int_0^t T(\tau)x_n d\tau &= C_L A^{-1}[T(t)x_n - x_n] \\ &= CA^{-1}[T(t)x_n - x_n] \\ &\to CA^{-1}[T(t)x - x] \\ &= C\int_0^t T(\tau)x d\tau, \end{aligned}$$

where we have used the fact that CA^{-1} is bounded. Hence we have that

$$\int_0^t T(\tau)z d\tau = C\int_0^t T(\tau)x d\tau.$$

Since $t^{-1}\int_0^t T(\tau)z d\tau$ converges to z for $t \downarrow 0$, we conclude from the above equality that $x \in D(C_L)$ and $C_L x = z$. □

III. $\mathcal{H}_\infty$-CALCULUS

For $g \in \mathcal{H}_\infty^-$ we define the following mapping from X to $L^2((0,\infty); X)$

$$\mathfrak{O}_g x_0 = M_g\left(T(t)x_0\right). \tag{14}$$

Hence we have taken in Definition 1.2 $f(t) = T(t)x_0$.

It is clear that $\mathfrak{O}_g$ is a linear bounded operator from X into $L^2((0,\infty); X)$. Furthermore, from (6) we have that

$$\sigma_\tau\left(\mathfrak{O}_g x_0\right) = M_g\left(\sigma_\tau\left(T(t)x_0\right)\right) = M_g T(t+\tau)x_0 = \mathfrak{O}_g\left(T(\tau)x_0\right), \tag{15}$$

where we have used the semigroup property. Hence $\mathfrak{O}_g$ is a well-posed output map, and so by Lemma 2.3 we conclude that $\mathfrak{O}_g$ can be written as

$$\mathfrak{O}_g x_0 = g(A)T(t)x_0 \tag{16}$$

for some infinite-time admissible operator $g(A)$ which is bounded from the domain of A to X.

Since $A^{-1}T(t) = T(t)A^{-1}$ we conclude from (14) and (3) that

$$\mathfrak{O}_g A^{-1} = A^{-1}\mathfrak{O}_g$$

Hence by (16), we see that $g(A)$ is an admissible operator which commutes with A^{-1}. Theorem 2.5 implies that for $t > 0$, $g(A)T(t)$ can be extended to a bounded operator and

$$\|g(A)T(t)\| \leq \frac{\gamma}{\sqrt{t}}. \tag{17}$$

Since we have written this admissible operator as the function g working on the operator A, there is likely to be a relation with functional calculus. This is shown next.

Lemma 3.1: If $g \in \mathcal{H}_\infty^-$ is the inverse Fourier transform of the function h, with $h \in L^1(-\infty,\infty)$ with support in $(-\infty, 0)$, then $g(A)$ is bounded

$$g(A)x_0 = \int_0^\infty T(t)h(-t)x_0 dt, \tag{18}$$

and so $g(A)$ corresponds to the classical definition of the function of an operator.

So if g is the Fourier transform of an absolutely integrable function, then $g(A)$ is bounded. We would like to know when it is bounded for every g. For this, we extend the definition of $\mathfrak{O}_g$.

Let C be an admissible output operator for the semigroup $(T(t))_{t\geq 0}$. By definition, we know that $CT(\cdot)x_0 \in L^2((0,\infty); Y)$ for all $x_0 \in X$. We define

$$\left(C \circ \mathfrak{O}_g\right)x_0 = M_g\left(CT(t)x_0\right) \tag{19}$$

It is clear that this is a bounded mapping from X to $L^2((0,\infty); Y)$.

As before we have that

$$\sigma_\tau\left(\left(C\circ\mathfrak{O}_g\right)(x_0)\right)=\left(C\circ\mathfrak{O}_g\right)\left(T(\tau)x_0\right). \qquad (20)$$

And so we can write $\left(C\circ\mathfrak{O}_g\right)x_0$ as $\tilde{C}_g T(\cdot)x_0$ for some infinite-time admissible $\tilde{C}_g$. We have that

Lemma 3.2: The infinite-time admissible operator $\tilde{C}_g$ satisfies

$$\tilde{C}_g x_0 = Cg(A)x_0, \qquad \text{for } x_0\in D(A^2). \qquad (21)$$

Proof For $x_0 \in D(A^2)$, we introduce $x_1 = Ax_0$. Then in $L^2((0,\infty);Y)$ there holds

$$\begin{aligned}\tilde{C}_g T(t)x_0 &= \left(C\circ\mathfrak{O}_g\right)x_0\\ &= M_g\left(CT(t)x_0\right)\\ &= M_g\left(CT(t)A^{-1}x_1\right)\\ &= M_g\left(CA^{-1}T(t)x_1\right)\\ &= CA^{-1}g(A)T(t)x_1\\ &= Cg(A)T(t)A^{-1}x_1 = Cg(A)T(t)x_0,\end{aligned}$$

where we have used (3). Since both functions are continuous, we find that (21) holds. □

Based on this result, we denote $\tilde{C}_g$ by $C\circ g(A)$.

Using this, we can prove the following theorems.

Theorem 3.3: The mapping $g \mapsto g(A)$ forms a (unbounded) $\mathcal{H}_\infty^-$-calculus.

Proof It only remains to show that $(g_1g_2)(A) = g_1(A)g_2(A)$. By Lemma 1.3 we have that

$$\mathfrak{O}_{g_1g_2}x_0 = M_{g_1g_2}\left(T(t)x_0\right) = M_{g_1}M_{g_2}\left(T(t)x_0\right).$$

For $x_0 \in D(A)$ the last expression equals $M_{g_1}\left(g_2(A)T(t)x_0\right)$, see (16). Since $g_2(A)$ commutes with the semigroup, we find that

$$\mathfrak{O}_{g_1g_2}x_0 = M_{g_1}\left(T(t)g_2(A)x_0\right).$$

Using (16) twice, we obtain

$$(g_1g_2)(A)T(t)x_0 = \mathfrak{O}_{g_1g_2}x_0 = g_1(A)T(t)g_2(A)x_0$$

This is an equality in $L^2((0,\infty);X)$. However, if we take $x_0 \in D(A^2)$, then this holds pointwise, and so for $x_0 \in D(A^2)$.

$$(g_1g_2)(A)x_0 = g_1(A)g_2(A)x_0$$

This concludes the proof. □

Theorem 3.4: If there exists an admissible C such that (C,A) is exactly observable, i.e., these exists an $m_1 > 0$ such that for all $x_0 \in X$ there holds

$$\int_0^\infty \|CT(t)x_0\|^2 dt \geq m_1\|x_0\|^2$$

then $g(A)$ is bounded for every $g \in \mathcal{H}_\infty^-$. Furthermore, if m_2 is the admissibility constant, see equation (12), then

$$\|g(A)\| \leq \sqrt{m_1m_2}\|g\|_\infty. \qquad (22)$$

Proof Let $x_0 \in D(A^2)$

$$\begin{aligned}\|g(A)x_0\|^2 &\leq m_1\|CT(t)g(A)x_0\|_{L^2}^2\\ &= m_1\|Cg(A)T(t)x_0\|_{L^2}^2\\ &= m_1\|C\circ\mathfrak{O}_g x_0\|_{L^2}^2\\ &\leq m_1\|g\|_\infty^2\|CT(t)x_0\|_{L^2}^2\\ &\leq m_1m_2\|g\|_\infty^2\|x_0\|^2.\end{aligned}$$

Since $D(A^2)$ is dense, we obtain the result. □

As a corollary we obtain the well-known von Neumann inequality.

Corollary 3.5: If A is a strict contraction, then A has a bounded $\mathcal{H}_\infty^-$ calculus and for all $g \in \mathcal{H}_\infty^-$

$$\|g(A)\| \leq \|g\|_\infty. \qquad (23)$$

Proof Since A is a strict contraction, we have that

$$\langle Ax, x\rangle + \langle x, Ax\rangle = -\langle x, Qx\rangle, \qquad x\in D(A) \qquad (24)$$

with Q self-adjoint and $Q > 0$. Define $C = \sqrt{Q}$, then (24) together with the exponential stability implies that

$$\int_0^\infty \|CT(t)x_0\|^2 dt = \|x_0\|^2. \qquad (25)$$

Thus we see that the constants m_1 and m_2 in Theorem 3.4 can be chosen to be one, and so (22) gives the results. □

If A generates an exponentially stable semigroup and if there exists an admissible C for which (C,A) is exactly observable, then it is not hard to show that the semigroup is similar to a contraction semigroup. Using this, one can also obtain the above result by Theorem G of [1]. The following result has been proved by Le Merdy.

Theorem 3.6: If A generates an analytic semigroup, and $A^{\frac{1}{2}}$ is admissible, then $g(A)$ is bounded for every $g \in \mathcal{H}_\infty^-$. Since $A^{1/2}$ is admissible, Lemma 3.2 gives that $A^{1/2}\circ g(A)$ is also admissible. For any admissible operator S there holds that $\|S(\lambda I - A)^{-1}\|^2 \leq m_2/\lambda$, $\lambda > 0$. Since A generates an analytic semigroup this implies that $SA^{-1/2}$ is bounded.

Hence the operator $\left(A^{1/2}\circ g(A)\right)A^{-1/2}$ is bounded. On the dense set $D(A^3)$ this equals $g(A)$, and so the result is shown. □

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