The Weiss conjecture and Katos method for the Navier-Stokes equations

Bernhard H. Haak and Peer Chr. Kunstmann

***Abstract*— We investigate Kato's method for parabolic equations with a quadratic non-linearity in an abstract form. We extract several properties known from linear systems theory which turn out to be the essential ingredients for the method. We give necessary and sufficient conditions for these conditions and provide new and more general proofs, based on real interpolation. In application to the Navier-Stokes equations, our approach unifies several results known in the literature, partly with different proofs.**

I. INTRODUCTION

This talk is based on a joint article with P. Kunstmann [5]. We discuss in this article the existence of solutions subject to a linear system

$$\left.\begin{array}{rcll} x'(t)+Ax(t) &=& Bu(t), & t>0,\\ x(0) &=& x_0, & \\ y(t) &=& Cx(t), & t>0\\ u(t) &=& F(y(t),y(t)) & t>0 \end{array}\right\} \quad (1)$$

under a quadratic feedback law $u(t)=F(y(t),y(t))$. To be more precise, we assume that we are given three Banach spaces X, Y, U and some real number $\tau \in (0,\infty]$. We suppose that $-A$ generates a (not necessarily strongly continuous) bounded analytic semigroup $T(\cdot)$ on X and that the (control and observation) operators B and C satisfy $B \in B(U,X_{-1})$ and $C : X \to Y$ be a closed linear operator that is bounded $X_1 \to Y$. Finally we suppose that $F : Y \times Y \to U$ be a bilinear map satisfying $\|F(y,\widetilde{y})\| \le K\|y\|\,\|\widetilde{y}\|$ for some $K>0$.

Before stating the main abstract result, we would like to stress the link to the motivating question, to find (mild) solutions for the Navier-Stokes equations in critical spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain, i.e. an open and connected subset. Consider the Navier–Stokes equation in the form

$$\left.\begin{array}{rcll} u_t-\Delta u+(u\cdot\nabla)u+\nabla p &=& f, & (t>0)\\ \nabla\cdot u &=& 0 & \\ u(0,\cdot) &=& v_0 & \\ u|_{\partial\Omega} &=& 0. & \end{array}\right\} \quad (2)$$

The equation (2) describes the motion of an incompressible fluid filling the region Ω under "no slip" boundary conditions, where $u=u(t,x)\in\mathbb{R}^n$ denotes the unknown velocity vector at time t and point x, $p=p(t,x)\in\mathbb{R}$ denotes the unknown pressure, and v_0 denotes the initial velocity field which is also assumed to be divergence–free, i.e. $\nabla\cdot v_0=0$. We remark that $\nabla\cdot u=0$ allows to rewrite $(u\cdot\nabla)u=\nabla\cdot(u\otimes u)$ where $u\otimes u$ is the matrix of all products u_iu_j of the coordinate functions of u.

II. MAIN RESULTS

The fixed point equation is obtained from (2) by first applying the Helmholtz projection $\mathbb{P}$ to get rid of the pressure term

$$\left.\begin{array}{rcll} u_t-\mathbb{P}\Delta u+\mathbb{P}\nabla\cdot(u\otimes u) &=& \mathbb{P}f, & (t>0)\\ \nabla\cdot u &=& 0 & \\ u(0,\cdot) &=& v_0 & \\ u|_{\partial\Omega} &=& 0. & \end{array}\right\} \quad (3)$$

The operator $-\mathbb{P}\Delta$ with Dirichlet boundary conditions is, basically, the *Stokes operator* A which – hopefully – is the negative generator of a bounded analytic semigroup $T(\cdot)$, the *Stokes semigroup*, in the divergence–free function space X under consideration. Then the solution to (3) is formally given by the variation-of-constants formula:

$$u=T(\cdot)v_0-T(\cdot)*\mathbb{P}\nabla\cdot(u\otimes u)+T(\cdot)*\mathbb{P}f. \quad (4)$$

This is essentially a fixed point equation for u. A *mild solution* to (2) is a solution to this fixed-point problem. The non-linearity is quadratic and may be rewritten using the bilinear map $F(u,v):=\mathbb{P}\nabla\cdot(u\otimes v)$.

We shall use the notation

$$L^p_\alpha((0,\tau),X):=\{f:(0,\tau)\to X:\ t^\alpha f(t)\in L^p((0,\tau),X)\}.$$

Going back into the abstract setting, we use the following notion of L^p-admissibility of type α:

Definition II.1. Given $p\in[1,\infty]$ and a bounded analytic semigroup $T(\cdot)$ on X,

1) C is called finite-time L^p-admissible of type $\alpha > -1/p$ if for any $\tau>0$ there is a constant $M_\tau>0$ such that for all $x\in X_k$ we have

$$\left\|t\mapsto CT(t)x\right\|_{L^p_\alpha([0,\tau],Y)} \le M_\tau\|x\|_X; \quad (5)$$

2) B is called finite-time L^p-admissible of type $\alpha < 1/p'$ if for any $\tau>0$ there is a constant $K_\tau>0$ such that for all $u\in L^p_\alpha((0,\tau),U)$ we have

$$\left\|T_{-k}B*u\right\|_{L^\infty((0,\tau),X)} \le K_\tau\|u\|_{L^p_\alpha((0,\tau),U)}. \quad (6)$$

If the above estimates hold with constants M_τ and K_τ that can be chosen independently of $\tau>0$, the operators B and C are called (infinite-time) L^p-admissible of type α.

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As explained above, we seek for mild solutions $x(\cdot)$ in the space $C([0,\tau),X)$, i.e. for functions x satisfying

$$x(t) = T(t)x_0 + \int_0^t T(t-s)BF(Cx(s),Cx(s))\,ds. \quad (7)$$

Theorem II.2. *Let $\tau \in (0,\infty]$ and $p \in (2,\infty]$. Let $\alpha \geq 0$ such that $\alpha + 1/p \in (0, 1/2)$. We assume*

[A1] *The observation C is L^p-admissible of type α from X to Y in time $(0,\tau)$.*

[A2] *The control B is $L^{p/2}$-admissible of type 2α from U to X in time $((0,\tau)$.*

[A3] *The input-output map $(CT_{-1}(\cdot)B)*$ is bounded from $L^{p/2}_{2\alpha}((0,\tau),U)$ to $L^p_\alpha((0,\tau),Y)$.*

Then, under the above assumptions on the operators B, C and F, for any initial value $x_0 \in X^\flat := \overline{\mathcal{D}(A)}$ (the closure being taken in X) there exits $\eta \in (0,\tau]$ such that the abstract problem (1) has a unique local mild solution x in $C([0,\eta),X^\flat)$ satisfying $Cx \in L^p_\alpha((0,\eta),Y)$. Moreover, if $\|x_0\|_X$ is sufficiently small, then the solution exists globally.

Sketch of the proof. The proof of the theorem essentially relies on a standard lemma on fixed point equations of the form $z = y + B(z,z)$ where B is a bilinear map on a Banach space E satisfying $\|B(u,v)\| \leq \eta\|u\|\|v\|$ and where y has to satisfy a smallness condition with respect to η. For the given initial value x_0 to the problem (7) we shall let $y = CT(\cdot)x_0$ on $E = L^p_\alpha((0,\tau),Y)$. The growth condition on x_0 (if present) and adjusting the time interval $(0,\tau)$ (if necessary) assures the smallness of y in the norm of E. The lemma assures a fixed point $z \in E$ and letting

$$x(t) := T(t)x_0 + \int_0^t T(t-s)BF(z(s),z(s))\,ds$$

it readily follows that x is a mild solution to (7). Uniqueness of the solution $x(\cdot)$ follows from the fixed-point equation for the observed solutions $y(\cdot) = Cx(\cdot)$ that must be satisfied by any solution; using again the assumptions yields first uniqueness on sufficiently small time intervals. ☐

The 'Weiss conjecture'. Admissibility of observation and control operators is in general difficult to check. In the most common case, $p = 2$ and $\alpha = 0$ it was a long time open question whether the conditions

$$\|Re(\lambda)CR(\lambda,A)\| \leq C \qquad \|Re(\lambda)R(\lambda,A)B\| \leq C \quad (8)$$

for all λ with positive real part would be sufficient to guarantee admissibility. This however turned out to be wrong, even on Hilbert spaces, even for rank one operators B and C, see [7] for a survey on admissibility. It is therefore quite astonishing that for bounded analytic semigroups on arbitrary Banach spaces that additionally satisfy certain 'square function estimates', both condition do characterise L^2-admissibility (of type 0), see [11]. In Hilbert spaces, semigroups with such square function estimates are in particular analytic semigroups that are similar to contraction semigroups.

In a former work [3] we could show that resolvent conditions of the type (8) characterise by a boundedness condition of C and B on certain real interpolation spaces of X and $D(A)$ (if A is invertible) and some homogeneous fractional domain space if A is only injective. Moreover, Le Merdy's result [11] (that discusses only observation operators) extends to L^p admissibility of type α for all $p \in [1,\infty]$, see [6] and [4] where also control operators are treated. In contrast with the proofs in these works that rely on the H^∞ functional calculus for semigroup generators, we present in [5] new proofs that work entirely with real interpolation arguments and (weakly) singular integrals. This does not only provide quicker and more transparent proofs but also extends e.g. to not necessarily densely defined operators and we shall make use of this in the following. It may be important to remark that in the case $\alpha \neq 0$ the notions of L^p-admissibility of type α are not 'dual' in the sense that for $\alpha > 0$, L^p-admissibility of an observation operator C *implies* $L^{p'}$ admissibility of C' on X' but that the converse is not true in general. This asymmetry (for $\alpha \neq 0$) expresses in a characterisation for the L^p admissibility of type α of an observation operator C whereas for control operators (and by the same argument also for the input-output map) we only know sufficient conditions so far.

The notation in terms of real interpolation spaces that guarantee (or characterise) [A1], [A2] and [A3] being a bit technical, we do not fully cite it here and ask the interested reader to take a look in the full article [5, Theorem 3.6].

III. Application to Navier-Stokes equations in $\mathbb{R}^n$

For the Navier-Stokes equations B and C are identity operators on suitably chosen spaces. So essentially we have to arrange the three spaces Y, U and X such that embeddings between certain homogeneous fractional domains of A on X and U and Y are satisfied. Our strategy is to actually choose the space Y and first to arrange U such that $\nabla\cdot(y\otimes y) \in U$ whenever $y \in Y$. Then we obtain X be real interpolation.

A. Lebesgue spaces

Choosing $Y = L^q(\mathbb{R}^n)$ naturally leads to $U = \dot{H}^{-1}_{q/2}(\mathbb{R}^n)$ and by real interpolation, we chose $X = \dot{B}^{-1+n/q}_{q,p}(\mathbb{R}^n)$

Theorem III.1. *Let $n \geq 2$, $q \in (n,\infty)$, and let $\alpha \geq 0$ and $p \in (2,\infty]$ such that*

$$\alpha + 1/p = \tfrac{1}{2} - \tfrac{n}{2q} \quad (9)$$

Then the Navier-Stokes equation (2) admits a time-local mild solution in $C([0,\tau),X)$ for every $u_0 \in X^\flat = \overline{\mathcal{D}(A)}$ satisfying $\nabla\cdot u_0 = 0$. The solution is unique in $C([0,\tau),X) \cap L^p_\alpha((0,\tau),L^q(\mathbb{R}^n))$. If the norm $\|u_0\|_X$ is sufficiently small, the solution exists globally.

Notice that $X^\flat = X$ in case $p < \infty$ whereas in case $p = \infty$, $X^\flat$ equals the (homogeneous) *little Besov space* $\dot{b}^{-1+n/q}_{q,\infty}$ or the (homogeneous) *little Nikolski space* $\dot{n}^{-1+n/q}_q$.

This result resembles [2, Theorem 3.3.4] when we consider the special case $n = 3$, $3 < q < 6$, and $p = \infty$. The general case $n \geq 2$, $q > n$ and $p = \infty$, is [1]. However, the latter result also covers the case of (sufficiently smooth) domains $\Omega \subset \mathbb{R}^n$ which are not considered here. Other results on $\mathbb{R}^n$ with somewhat different approaches are [8] and [9]. Finally, in [13] the existence of time–local mild solutions for the inhomogeneous space $X = B_{q,p}^{-1+n/q+\epsilon}$ where $q \in (n,\infty]$, $p \in [1,\infty]$ and $\epsilon \in (0,1]$ (thus $X = B^0_{\infty,\infty}$ is included). Observe that Theorem III.1 yields, for $q \in (n,\infty)$ and $p \in [\frac{2q}{q-n},\infty]$, local solutions in the space $X = \dot{B}_{q,p}^{-1+n/q}$ (i.e. for $\epsilon = 0$) and global solutions for small initial data (which is not covered by the result in [13]). Moreover, the proof in [13] relied on a Hölder type inequality for products of Besov space functions whereas our proof simply uses the Hölder inequality for the product of two L^q–functions.

B. Weak Lebesgue spaces

When choosing $Y = L^{q,\infty}$ and a weak Sobolev space $U = \dot{H}^{-1}_{q/2,\infty}$, we obtain

Theorem III.2. *Let $n \geq 2$, $q \in (n,\infty)$ and let $\alpha \geq 0$ and $p \in (2,\infty]$ such that (9) holds. Let $X := \dot{B}^{-1+n/q}_{(q,\infty),p}$. Then the Navier-Stokes equation* (2) *admits a time-local mild solution in $C([0,\tau),X)$ for every $u_0 \in X^\flat = \overline{\mathcal{D}(A)}$ satisfying $\nabla\cdot u_0 = 0$. The solution is unique in $C([0,\tau),X) \cap L^p_\alpha((0,\tau),L^{q,\infty}(\mathbb{R}^n))$. If the norm of u_0 is sufficiently small, the solution exists globally.*

The limit case $q = n$, i.e $X = Y = L^{n,\infty}$ which is covered in [12, Theorem 18.2] can also be obtained by a variation of our proof for III.2.

C. Morrey spaces

Letting $Y = \mathcal{M}^{q,\lambda}$ and $U = \dot{\mathcal{M}}_{q/2,2\lambda-1}$ we obtain a result where X is a real interpolation space of the form $X = (\dot{\mathcal{M}}^{q,\lambda,-2},\mathcal{M}^{q,\lambda})_{\gamma,p}$. Notice that Morrey spaces do not form an interpolation scale, so this space cannot be easily identified with another Morrey space. We obtain

Theorem III.3. *Let $n \geq 2$, $\lambda \in (0,n/q)$, $q \in (n,\infty)$ and let $\alpha \geq 0$ and $p \in (2,\infty]$ such that $\alpha + \frac{1}{p} = \frac{1}{2} - \frac{\lambda}{2}$ holds. Then the Navier-Stokes equation* (2) *admits a time-local mild solution in $C([0,\tau),X)$ for every $u_0 \in X^\flat = \overline{\mathcal{D}(A)}$ satisfying $\nabla\cdot u_0 = 0$. The solution is unique in $C([0,\tau),X) \cap L^p_\alpha((0,\tau),\mathcal{M}^{q,\lambda}(\mathbb{R}^n))$. If the norm of u_0 in X is sufficiently small, the solution exists globally.*

Several results for Morrey spaces are discussed in the literature, see e.g. [12] for an overview. The closest result to our theorem is [10] who first introduced real interpolation spaces of Morrey spaces and of local Morrey spaces. It turns out that their result is generalised taking $p = \infty$ in the above theorem.

D. Hölder spaces

Finally consider $Y = (C^\epsilon)^n = (B^\epsilon_{\infty,\infty})^n$. We are led to $X = B^{-2(\alpha+1/p)+\epsilon}_{\infty,p}$ and

Theorem III.4. *Let $n \geq 2$, $p \in (2,\infty]$, and $\epsilon \in (0,1)$. Let $\alpha > 0$ be such that $\alpha + 1/p < 1/2$. Then the Navier-Stokes equation* (2) *admits a time-local mild solution in $C([0,\tau),X)$ for every divergence–free $u_0 \in X = B^{-2(\alpha+1/p)+\epsilon}_{\infty,p}(\mathbb{R}^n)$, which is unique in the space $C([0,\tau),X) \cap L^p_\alpha((0,\tau),C^\epsilon(\mathbb{R}^n))$.*

The result [13] also covers time–local solutions for initial values in spaces $B^{-1+\epsilon}_{\infty,p}$ for p up to ∞. However, the space for uniqueness does not involve $L^p_\alpha(C^\epsilon)$ but L^∞_β–spaces with values in certain Besov spaces. This is due to the fact that the key stone in [13] is a Hölder type inequality for products in (inhomogeneous) Besov spaces which is proved there by means of Littlewood-Paley decomposition and paraproducts. Our proof uses the simple product inequality in C^ϵ instead, and we obtain the second index p in X by taking L^p in time. So, in our proof, improvement comes from a better understanding of the *linear* ingredients for the problem whereas in [13] it comes from a new insight for the *nonlinearity*. We remark that [13] includes the case $\epsilon = 1$.

Remark III.5. The full article [5] contains also a section on existence of solutions in arbitrary domains in $\mathbb{R}^3$ but this goes far beyond the material we would be able to present in one conference talk; for this reason the material is not cited here.

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