Controlled invariant varieties of polynomial control systems

Eva Zerz, Sebastian Walcher, and Fadime Güçlü

***Abstract*— We study input-affine control systems with polynomial nonlinearity. A variety V is said to be controlled invariant if there exists a feedback law of polynomial type that causes the closed loop system to have V as an invariant variety, which means that any trajectory starting in V will remain there for all times. Using the theory of Gröbner bases, we show (under certain conditions on the given representation of V) how to constructively decide whether a given variety is controlled invariant for a given system, and if so, how to determine all feedback laws achieving the task.**

I. Introduction

Consider an input-affine control system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{1}$$

where f, g are polynomials, that is, $f \in \mathbb{R}[X_1, \ldots, X_n]^n$ and $g \in \mathbb{R}[X_1, \ldots, X_n]^{n\times m}$. Let a variety

$$\begin{aligned} V &:= \mathcal{V}(p_1, \ldots, p_k) \\ &:= \{v \in \mathbb{R}^n \mid p_i(v) = 0 \text{ for all } 1 \leq i \leq k\} \subseteq \mathbb{R}^n \end{aligned}$$

with $p_i \in \mathbb{R}[X_1, \ldots, X_n]$ for $1 \leq i \leq k$ be given, i.e., V is the set of common zeros of $p_1, \ldots, p_k$. In this paper, we shall address the following question: Does there exist an $\alpha \in \mathbb{R}[X_1, \ldots, X_n]^m$ such that the state feedback law

$$u(t) = \alpha(x(t))$$

will yield a closed loop system

$$\dot{x}(t) = f(x(t)) + g(x(t))\alpha(x(t)) = (f + g\alpha)(x(t))$$

that has V as an invariant variety? Setting $F := f + g\alpha$, this signifies that any solution $x(\cdot)$ to $\dot{x}(t) = F(x(t))$ with $x(0) = x_0 \in V$ will remain within V, that is, $x(t) \in V$ for all t in the maximal existence interval of $x(\cdot)$. If the answer is positive, V will be called a *controlled invariant variety* of (1). This paper shows (under certain assumptions on the given representation of V) how to determine whether this is the case, and if it is, how to compute all α with the desired property.

Eva Zerz and Fadime Güçlü are with Lehrstuhl D für Mathematik, RWTH Aachen University, Templergraben 64, 52062 Aachen, Germany eva.zerz@math.rwth-aachen.de, fadime.gueclue@rwth-aachen.de

Sebastian Walcher is with Lehrstuhl A für Mathematik, RWTH Aachen University, Templergraben 55, 52062 Aachen, Germany walcher@matha.rwth-aachen.de

II. Invariant Varieties of Autonomous Systems

Consider $\dot{x}(t) = F(x(t))$, where $F \in \mathbb{R}[X_1, \ldots, X_n]^n$. Let $\varphi(t, x_0)$ denote the solution of the initial value problem

$$\begin{aligned} \dot{x}(t) &= F(x(t)) \\ x(0) &= x_0 \end{aligned}$$

at time t. Here, $x_0 \in \mathbb{R}^n$ is arbitrary and t belongs to the maximal existence interval $J(x_0) \subseteq \mathbb{R}$ of $\varphi(\cdot, x_0)$. We say that $V \subseteq \mathbb{R}^n$ is *invariant* for F if $x_0 \in V$ implies that $\varphi(t, x_0) \in V$ for all $t \in J(x_0)$.

Let $V \subseteq \mathbb{R}^n$ be a variety, that is, $V = \mathcal{V}(p_1, \ldots, p_k)$ for some $p_i \in \mathbb{R}[X_1, \ldots, X_n]$. Define

$$\mathcal{J}(V) := \{p \in \mathbb{R}[X_1, \ldots, X_n] \mid p(v) = 0 \text{ for all } v \in V\}.$$

By $I := \langle p_1, \ldots, p_k \rangle$, we denote the ideal generated by the polynomials p_i. Then $V = \mathcal{V}(I)$ and we always have $I \subseteq \mathcal{J}(V)$, but in general, equality does not necessarily hold, even if I is a radical ideal. This is due to the fact that $\mathbb{R}$ is not algebraically closed, and thus, the determination of $\mathcal{J}(V)$ is a non-trivial problem in real algebraic geometry. However, $\mathcal{J}(V)$ is known for many common surfaces and curves, in particular, when parametric representations are available.

The following characterization is standard [5], [12], but we restate it for the sake of completeness, since it is crucial for our approach.

Theorem 1: Let F, V and p_i for $1 \leq i \leq k$ be as described above.

1) If we have

$$\partial_1 p_i \cdot F_1 + \ldots + \partial_n p_i \cdot F_n \in \langle p_1, \ldots, p_k \rangle \tag{2}$$

for all $1 \leq i \leq k$, then V is invariant for F.

2) If V is invariant for F, then we have

$$\partial_1 p_i \cdot F_1 + \ldots + \partial_n p_i \cdot F_n \in \mathcal{J}(V)$$

for all $1 \leq i \leq k$.

Thus, if $\mathcal{J}(V) = \langle p_1, \ldots, p_k \rangle$ holds, then condition (2) for $1 \leq i \leq k$ is necessary and sufficient for V being invariant for F.

Note that in principle, one could always reduce the problem to the case where $k = 1$, since $\mathcal{V}(p_1, \ldots, p_k) = \mathcal{V}(p_1^2 + \ldots + p_k^2)$ holds over the real numbers. However, it will turn out that it may be advantageous from the computational point of view to admit several polynomials of lower degree defining V instead of working with one single, but more complicated polynomial. Moreover, we have $\langle p_1^2 + \ldots + p_k^2 \rangle \subsetneq \langle p_1, \ldots, p_k \rangle$ except for trivial cases, and thus, requiring $k = 1$ would make the gap between the sufficient and the necessary condition for F-invariance of V larger.

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Proof: 1) Let $x_0 \in V$ be given. Consider the solution $\varphi(\cdot, x_0)$ of the corresponding initial value problem and set $y_i(t) := p_i(\varphi(t, x_0))$. We need to show that $y_i(t) = 0$ for all $1 \leq i \leq k$ and all $t \in J(x_0)$. Clearly, $y_i(0) = p_i(x_0) = 0$ for all $1 \leq i \leq k$. Moreover, each y_i is continuously differentiable with

$$\dot{y}_i(t) = (\partial_1 p_i \cdot F_1 + \ldots + \partial_n p_i \cdot F_n)(\varphi(t, x_0)).$$

By assumption, we have

$$\partial_1 p_i \cdot F_1 + \ldots + \partial_n p_i \cdot F_n = L_{i1} p_1 + \ldots + L_{ik} p_k$$

for some $L_{ij} \in \mathbb{R}[X_1, \ldots, X_n]$. Thus

$$\dot{y}_i(t) = (L_{i1} p_1 + \ldots + L_{ik} p_k)(\varphi(t, x_0)).$$

Setting $y := [y_1, \ldots, y_k]^T$ and $A_{ij}(t) := L_{ij}(\varphi(t, x_0))$, we obtain the linear time-varying system

$$\dot{y}(t) = A(t) y(t),$$

where $A(\cdot)$ is continuous on $J(x_0)$. Since $y(0) = 0$, we may conclude, by uniqueness, that $y \equiv 0$ on $J(x_0)$.

2) Let $x_0 \in V$ be given. By assumption, we have $\varphi(t, x_0) \in V$ for all t in an open neighborhood $J(x_0)$ of zero. This means that

$$p_i(\varphi(t, x_0)) = 0 \quad \text{for all } 1 \leq i \leq k, \ t \in J(x_0).$$

Taking the derivative, this implies that

$$(\partial_1 p_i \cdot F_1 + \ldots + \partial_n p_i \cdot F_n)(\varphi(t, x_0)) = 0$$

for all $1 \leq i \leq k$, $t \in J(x_0)$. Plugging in $t = 0$, we get

$$(\partial_1 p_i \cdot F_1 + \ldots + \partial_n p_i \cdot F_n)(x_0) = 0 \quad \text{for all } 1 \leq i \leq k.$$

Since $x_0 \in V$ was arbitrary, we have therefore shown that the k polynomials $\partial_1 p_i \cdot F_1 + \ldots + \partial_n p_i \cdot F_n$ vanish at every $v \in V$, that is, they belong to $\mathcal{J}(V)$. □

Define $\mathcal{P} := \mathbb{R}[X_1, \ldots, X_n]$ and let $p_1, \ldots, p_k \in \mathcal{P}$ be given with $V = \mathcal{V}(p_1, \ldots, p_k)$. We define the syzygy module

$$N_i := \ker \begin{bmatrix} \partial_1 p_i & \ldots & \partial_n p_i & p_1 & \ldots & p_k \end{bmatrix} \subseteq \mathcal{P}^{n+k}$$

and set $M_i := \pi(N_i)$, where π denotes the projection onto the first n components. Finally, let $M := \bigcap_{i=1}^k M_i \subseteq \mathcal{P}^n$.

Theorem 2: In the situation described above, we have

$$M = \{F \in \mathcal{P}^n \mid F \text{ satisfies (2) for all } 1 \leq i \leq k\}.$$

Thus if $F \in M$, then V is F-invariant. The converse holds as well provided that $\mathcal{J}(V) = \langle p_1, \ldots, p_k \rangle$.

Proof: Clearly, F satisfies (2) if and only if there exist $L_{ij} \in \mathcal{P}$ such that

$$\partial_1 p_i \cdot F_1 + \ldots + \partial_n p_i \cdot F_n = L_{i1} p_1 + \ldots + L_{ik} p_k,$$

which is equivalent to

$$[F_1, \ldots, F_n, -L_{i1}, \ldots, -L_{ik}]^T \in N_i.$$

Thus F satisfies (2) if and only if $F \in \pi(N_i)$. Finally, F satisfies (2) for all i if and only if $F \in \pi(N_i)$ for all i, that is, $F \in \bigcap_i \pi(N_i) = M$. The rest follows directly from Theorem 1. □

There is an obvious submodule $M^0 = \bigcap_{i=1}^k M_i^0$ of $M = \bigcap_{i=1}^k M_i$, which is given by

$$\begin{aligned} M_i^0 \quad := \quad & \ker \begin{bmatrix} \partial_1 p_i & \ldots & \partial_n p_i \end{bmatrix} + \\ & \langle p_l e_j \mid 1 \leq l \leq k, 1 \leq j \leq n \rangle \subseteq M_i. \end{aligned}$$

This corresponds to the *trivial vector fields* for which V is invariant, according to [5].

Proposition 3: We have $M_i = M_i^0$ if and only if

$$\begin{aligned} \langle p_1, \ldots, p_k \rangle \cap \langle \partial_1 p_i, \ldots, \partial_n p_i \rangle \quad = \\ \langle p_1, \ldots, p_k \rangle \cdot \langle \partial_1 p_i, \ldots, \partial_n p_i \rangle. \end{aligned} \tag{3}$$

Thus if (3) holds for all $1 \leq i \leq k$, then $M = M^0$.

Proof: Let $I := \langle p_1, \ldots, p_k \rangle$ and $J_i := \langle \partial_1 p_i, \ldots, \partial_n p_i \rangle$.

First, assume that $I \cap J_i = I \cdot J_i$. Consider $m \in M_i$, that is,

$$\sum_{j=1}^n \partial_j p_i \cdot m_j + \sum_{l=1}^k p_l r_l = 0$$

for some $r_l \in \mathcal{P}$ and thus

$$\sum_{j=1}^n \partial_j p_i \cdot m_j \in I \cap J_i = I \cdot J_i,$$

which implies that

$$\sum_{j=1}^n \partial_j p_i \cdot m_j = \sum_{l=1}^k \sum_{j=1}^n s_{lj} p_l \cdot \partial_j p_i$$

for some $s_{lj} \in \mathcal{P}$. Thus

$$\sum_{j=1}^n \partial_j p_i \cdot (m_j - \sum_{l=1}^k s_{lj} p_l) = 0,$$

which shows that $m - n \in M_i^0$, where $n = \sum_j \sum_l s_{lj} p_l e_j \in M_i^0$, and hence $m \in M_i^0$.

Conversely, suppose that $M_i = M_i^0$ and let $q \in I \cap J_i$ be given. Then $q = \sum_{l=1}^k r_l p_l = \sum_{j=1}^n s_j \cdot \partial_j p_i$ for some $r_l, s_j \in \mathcal{P}$ and hence $s := [s_1, \ldots, s_n]^T \in M_i = M_i^0$. Write

$$s = s_0 + \sum_{l=1}^k \sum_{j=1}^n s_{lj} p_l e_j$$

with $s_0 \in \ker[\partial_1 p_i, \ldots, \partial_n p_i]$, then

$$q = \sum_j \sum_l s_{lj} p_l \cdot \partial_j p_i \in I \cdot J_i$$

as desired. □

Example: Consider the curve given by the parametric representation $\gamma(t) = [t^2, t^3, t^4]^T$. Implicitization yields $p_1 = X_1^2 - X_3$ and $p_2 = X_2^2 - X_1 X_3$. For p_1, we get $J_1 = \mathcal{P} = \mathbb{R}[X_1, X_2, X_3]$ and hence $I \cap J_1 = I \cdot J_1$ and $M_1 = M_1^0$. For p_2, we have $J_2 = \langle X_1, X_2, X_3 \rangle$ and we find that $I \cap J_2 \supsetneq I \cdot J_2$ and $M_2 \supsetneq M_2^0$. We will see below that this is due to the fact that the hypersurface $\mathcal{V}(p_1)$ is smooth, whereas $\mathcal{V}(p_2)$ has a singularity at the origin. ◊

For the rest of this section, let $k = 1$ and $p := p_1 \neq 0$. Then condition (3) reads

$$\begin{aligned} \langle p\rangle \cap \langle \partial_1 p, \ldots, \partial_n p\rangle &= \langle p\rangle \cdot \langle \partial_1 p, \ldots, \partial_n p\rangle \\ &= \langle p \cdot (\partial_1 p), \ldots, p \cdot (\partial_n p)\rangle. \quad (4) \end{aligned}$$

Since $p \neq 0$, this is equivalent to

$$\langle \partial_1 p, \ldots, \partial_n p\rangle : \langle p\rangle = \langle \partial_1 p, \ldots, \partial_n p\rangle, \quad (5)$$

which is used in [5] to characterize $M = M^0$ in the case $n = 2$. This result carries over to arbitrary n.

Lemma 4: Assume that $p \neq 0$. There is a $\mathcal{P}$-module isomorphism

$$M/M^0 \cong (J : \langle p\rangle)/J,$$

where $J := \langle \partial_1 p, \ldots, \partial_n p\rangle$.

Proof: For $F \in M$, there exists a unique $L \in \mathcal{P}$ such that

$$F_1 \cdot \partial_1 p + \ldots + F_n \cdot \partial_n p = Lp. \quad (6)$$

For $[F] \in M/M^0$, define $\Phi([F]) := [L] \in (J : \langle p\rangle)/J$, where L is as in (6). It is easy to check that Φ is well-defined, $\mathcal{P}$-linear, and bijective. □

Since $I + J = \mathcal{P}$ implies $I \cap J = I \cdot J$ for any two ideals I, J in $\mathcal{P}$, we observe: If $\mathcal{V}(p)$ has no singular point in $\mathbb{C}^n$, that is, if we have $\langle p\rangle + \langle \partial_1 p, \ldots, \partial_n p\rangle = \mathcal{P}$, then (4), and hence (5), is satisfied.

Examples: For $p = X_1^2 + X_2^2 + X_3^2 - 1$, there are no singular points in $\mathbb{C}^3$ and we have $\langle X_1, X_2, X_3\rangle : \langle p\rangle = \langle X_1, X_2, X_3\rangle$.

For $p = X_1^2 + X_2^2 - X_3^2$, we have a singular point at the origin and we also have $\langle X_1, X_2, X_3\rangle : \langle p\rangle = \mathcal{P}$. ◊

These examples motivate the following partial converse of the observation from above.

Proposition 5: Suppose that $J = \langle \partial_1 p, \ldots, \partial_n p\rangle$ has dimension ≤ 0. Then M/M^0 is a finite-dimensional real vector space. We have $M = M^0$ if and only if $\mathcal{V}(p)$ has no singular points in $\mathbb{C}^n$.

Proof: The first statement follows directly from Lemma 4. For the second part, we show that (5) holds if and only if $\mathcal{V}(p)$ has no singular points in $\mathbb{C}^n$. The "if" part, and the case where $J = \langle \partial_1 p, \ldots, \partial_n p\rangle = \mathcal{P}$ are clear. It suffices to consider the case where J is zero-dimensional and

$$J : \langle p\rangle = J.$$

Let $J = \bigcap \mathfrak{q}_i$ be a primary decomposition of J over $\mathbb{C}$. Since J is zero-dimensional, each $\mathfrak{p}_i := \mathrm{rad}(\mathfrak{q}_i)$ must be maximal. The condition $J = J : \langle p\rangle$ implies that $J = J : \langle p\rangle^\infty$. We have

$$J : \langle p\rangle^\infty = \bigcap (\mathfrak{q}_i : \langle p\rangle^\infty)$$

and, since each $\mathfrak{q}_i$ is primary,

$$\mathfrak{q}_i : \langle p\rangle^\infty = \begin{cases} \mathcal{P} & \text{if } p \in \mathfrak{p}_i \\ \mathfrak{q}_i & \text{if } p \notin \mathfrak{p}_i. \end{cases}$$

We conclude that

$$J = \bigcap_{p \notin \mathfrak{p}_i} \mathfrak{q}_i.$$

This implies that $\mathcal{V}(J) = \mathcal{V}(\mathrm{rad}(J)) = \bigcup_{p \notin \mathfrak{p}_i} \mathcal{V}(\mathfrak{p}_i)$ in $\mathbb{C}^n$. Recalling that the $\mathfrak{p}_i$ are maximal, $\mathcal{V}(J)$ consists of finitely many points in $\mathbb{C}^n$ which are not contained in $\mathcal{V}(p)$. Hence $\mathcal{V}(J) \cap \mathcal{V}(p) = \emptyset$, which means that $\mathcal{V}(p)$ has no singular points in $\mathbb{C}^n$. □

In [5], it is shown for $n = 2$ that M/M^0 is finite-dimensional over $\mathbb{R}$ provided that p is square-free. This does not hold in higher dimensions, as is illustrated by the following example.

Example: For $p = X_1 X_2 X_3 (X_1 X_2 X_3 - 1)$, we have

$$M/M^0 \cong \mathcal{P}/\langle X_1 X_2, X_1 X_3, X_2 X_3\rangle,$$

which is *not* finite-dimensional over $\mathbb{R}$. Note that the assumption of Proposition 5 is not satisfied here, since the partial derivatives of p have a non-trivial common divisor and hence, they generate a two-dimensional ideal in $\mathcal{P} = \mathbb{R}[X_1, X_2, X_3]$. ◊

III. Controlled Invariant Varieties

Let $\mathcal{P} = \mathbb{R}[X_1, \ldots, X_n]$ and let $p_1, \ldots, p_k \in \mathcal{P}$ be given with $V = \mathcal{V}(p_1, \ldots, p_k)$. Suppose that $\mathcal{J}(V) = \langle p_1, \ldots, p_k\rangle$. Based on the result of the previous section, we may reformulate the problem posed in the Introduction as follows: Given $f \in \mathcal{P}^n$ and $g \in \mathcal{P}^{n\times m}$, does there exist $\alpha \in \mathcal{P}^m$ such that $F := f + g\alpha$ satisfies (2) for all $1 \leq i \leq k$? If we have $\mathcal{J}(V) \supsetneq \langle p_1, \ldots, p_k\rangle$, then the feedback laws α satisfying (2) for all i will still make V an invariant variety for the closed loop system, but we have no guarantee for obtaining all such α this way.

The following result is an immediate consequence of Theorem 2.

Corollary 6: Let N_i, M_i for $1 \leq i \leq k$ and M be defined as above. Let $f \in \mathcal{P}^n$ and $g \in \mathcal{P}^{n\times m}$ be given. If we have $f \in M + \mathrm{im}(g)$, that is, if there exists $\alpha \in \mathcal{P}^m$ such that $f + g\alpha \in M$, then V is a controlled invariant variety for (1). The converse holds as well provided that $\mathcal{J}(V) = \langle p_1, \ldots, p_k\rangle$.

Using Gröbner basis techniques (see textbooks such as [1], [3], or [10], [11] for systems theoretic applications), which are implemented in computer algebra systems such as Singular [9], a finite generating system of M can be computed, that is, one can constructively find a matrix $h \in \mathcal{P}^{n\times l}$ such that $M = \mathrm{im}(h)$.

Thus all we need to test is: Can f be written as a $\mathcal{P}$-linear combination of the columns of g and h? Also this question can be answered algorithmically, using the theory of Gröbner bases. If the answer is yes, then we have

$$f = h\beta - g\alpha$$

for some $\alpha \in \mathcal{P}^m$, $\beta \in \mathcal{P}^l$ and thus $f + g\alpha \in M$, that is, α is a feedback law that makes V invariant for (1).

Moreover, the non-uniqueness of α can be determined by checking that the existence of β_1, β_2 with $f = h\beta_1 - g\alpha_1 = h\beta_2 - g\alpha_2$ amounts to $g(\alpha_1 - \alpha_2) \in \mathrm{im}(h)$. Thus the set of all admissible α is given by $\pi(\ker[g, h])$, where π denotes the projection onto the first m components this time.

Examples: 1) Let $n = 3$, $m = 1$, $k = 1$,

$$V = \mathcal{V}(X_1^2 + X_2^2 + X_3^2 - 1)$$

and consider

$$\begin{aligned} \dot{x}_1 &= x_3 + x_1 u \\ \dot{x}_2 &= x_2 + x_2 u \\ \dot{x}_3 &= x_1 + x_3 u. \end{aligned} \tag{7}$$

Using SINGULAR, we type

```
ring r=0,(x1,x2,x3),lp;
LIB "matrix.lib";
ideal l=2*x1,2*x2,2*x3,x1^2+x2^2+x3^2-1;
module n=syz(l);
module m=submat(n,1..3,1..ncols(n));
vector f=[x3,x2,x1];
vector g=[x1,x2,x3];
module k=m,g;
reduce(f,std(k));
```

SINGULAR returns 0 from which we may conclude that the problem is solvable. Next, we construct a solution α:

```
lift(k,f);
```

This computes a representation of f as a linear combination of the generators of k. The last component of the output is the coefficient of g. It equals $-\alpha = 2X_1X_3 + X_2^2$. Finally, we wish to determine the set of all solutions:

```
syz(k);
```

Considering the last component again, this shows that the set of all solutions is given by $\alpha + \langle X_1^2 + X_2^2 + X_3^2 - 1\rangle$, that is, α is essentially unique. It turns out that in this example, the module $M + \operatorname{im}(g)$ is actually all of $\mathcal{P}^3$, and thus, the problem is solvable for any f.

Now consider the same V and

$$\begin{aligned} \dot{x}_1 &= -x_3 + x_1 u \\ \dot{x}_2 &= 2x_1x_2x_3 \\ \dot{x}_3 &= -x_1 + x_3 u. \end{aligned} \tag{8}$$

Proceeding analogously as before, we find that the problem is solvable with $\alpha = 2X_1X_3$. The set of all solutions is given by $\alpha + \langle X_1^2 + X_2^2 + X_3^2 - 1\rangle$. Here, we don't have a solution for any choice of f, as can be seen from the modified system

$$\begin{aligned} \dot{x}_1 &= -x_3 + x_1 u \\ \dot{x}_2 &= x_1x_2x_3 \\ \dot{x}_3 &= -x_1 + x_3 u, \end{aligned}$$

where V is not controlled invariant.

2) Let $n = 3$, $m = 1$, $k = 2$,

$$\begin{aligned} V &= \mathcal{V}(X_1^2 + X_2^2 + X_3^2 - 1, X_1^2 + X_2^2 - X_3^2) \\ &= \mathcal{V}(2X_1^2 + 2X_2^2 - 1, 2X_3^2 - 1). \end{aligned}$$

With system (7) from above, there exists no solution. Let us consider the following modification with $m = 2$:

$$\begin{aligned} \dot{x}_1 &= x_3 + x_1 u_1 \\ \dot{x}_2 &= x_2 + x_2 u_2 \\ \dot{x}_3 &= x_1 + x_3 u_1. \end{aligned}$$

Now the problem has $\alpha_1 = -2X_1X_3$, $\alpha_2 = -(2X_1X_3 + 1)$ as a solution.

Considering (8), we find that $\alpha = 2X_1X_3$ from above is still a solution, but the set of all solutions is now given by $\alpha + \langle X_1^2 + X_2^2 + X_3^2 - 1, X_1^2 + X_2^2 - X_3^2\rangle$. ◊

IV. Linear Case

We conclude by showing that the well-known linear case of controlled invariant subspaces is a special case of the theory developed here. Let $f(x) = Ax$, $g(x) = B$ for some $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Let V be a subspace of $\mathbb{R}^n$, that is, let $p_i = p_{i1}X_1 + \ldots + p_{in}X_n$ for some $p_{ij} \in \mathbb{R}$. Without loss of generality, let the k coefficient vectors $[p_{i1}, \ldots, p_{in}]$ of the linear equations defining V be linearly independent. Note that then we have $\langle \partial_1 p_i, \ldots, \partial_n p_i\rangle = \mathcal{P}$ for all i, that is, the condition of Proposition 3 is satisfied and we have $M = M^0$. Moreover, the necessary and sufficient conditions from Theorem 1 coincide in the linear case.

Adapting the notion of invariance, it is easy to see that V is invariant for $\dot{x}(t) := \tilde{A}x(t)$ if and only if $x_0 \in V$ implies that $e^{t\tilde{A}}x_0 \in V$ for all $t \in \mathbb{R}$, which is equivalent to V being an $\tilde{A}$-invariant subspace in the classical sense, that is, $\tilde{A}V \subseteq V$. Thus, one says that V is controlled invariant for $\dot{x} = Ax + Bu$ if there exists a feedback matrix $F \in \mathbb{R}^{m\times n}$ such that V is $(A + BF)$-invariant. Such an F is called a *friend* of V. The following characterization is well-known.

Proposition 7: [2] The following are equivalent:

1) V is controlled invariant, i.e., there exists F such that V is $(A + BF)$-invariant.
2) $AV \subseteq V + \operatorname{im}(B)$.

Moreover, if the equivalent conditions are satisfied, a feedback matrix F can be found as follows: Let $v_1, \ldots, v_d$ be a basis of V and set $Q := [v_1, \ldots, v_d] \in \mathbb{R}^{n\times d}$. Then there exists $Q' \in \mathbb{R}^{d\times n}$ such that $Q'Q = I_d$ and by condition 2, we have $AQ = QA_1 - BF_1 = QA_1 - BF_1Q'Q$ for some A_1, F_1, hence $(A + BF_1Q')Q = QA_1$ and we set $F := F_1Q'$.

The approach of the previous section computes

$$N_i = \ker \begin{bmatrix} p_{i1} & \ldots & p_{in} & p_1 & \ldots & p_k \end{bmatrix} \subseteq \mathcal{P}^{n+k},$$

$M_i = \pi(N_i) \subseteq \mathcal{P}^n$ and $M = \bigcap_{i=1}^k M_i$. Then we have

$$M = \{y \in \mathcal{P}^n \mid \exists Z \in \mathcal{P}^{k\times k} : Py = ZPx\},$$

where $P \in \mathbb{R}^{k\times n}$ has the coefficients p_{ij} as entries and $x = [X_1, \ldots, X_n]^T \in \mathcal{P}^n$, that is, $Px = [p_1, \ldots, p_k]^T$.

Consider an element of M of total degree at most 1, that is, $y = y_0 + Yx$ for some $y_0 \in \mathbb{R}^n$, $Y \in \mathbb{R}^{n\times n}$ and $Py = Py_0 + PYx = ZPx$ for some Z. This implies that $y_0 \in \ker(P) = V$ and $PY = ZP$. Writing $Z = \sum_{\nu\in\mathbb{N}^n} Z_\nu X^\nu$ with $Z_\nu \in \mathbb{R}^{k\times k}$ and $X^\nu := X_1^{\nu_1}\cdots X_n^{\nu_n}$, we can see that $Z_\nu P = 0$ for all $\nu \neq 0$, because PY is constant.

Since P has full row rank by assumption, we may conclude that $Z_\nu = 0$ for all $\nu \neq 0$, that is, Z is constant.

To find out whether V is controlled invariant, we need to test if $f \in M + \operatorname{im}(g)$. This is equivalent to $Ax = y + Bz$ with $y \in M$ and $z \in \mathcal{P}^m$. Suppose that y has total degree at most 1. Then we have $PAx = Py + PBz = ZPx + PBz$ for some constant matrix Z. Similarly as above, we write $z = \sum_{\nu \in \mathbb{N}^n} z_\nu X^\nu$ with $z_\nu \in \mathbb{R}^m$. Comparing coefficients, we find that $PBz_\nu = 0$ for all ν with $\nu_1 + \ldots + \nu_n \neq 1$, that is, $PBz = -PBFx$ for some $F \in \mathbb{R}^{m \times n}$. Then we have $PAx = ZPx - PBFx$ and thus $P(A + BF) = ZP$ which is tantamount to $V = \ker(P)$ being $(A + BF)$-invariant.

Thus, the proposed polynomial strategy will always find the desired linear solutions $\alpha(x) = Fx$ if they exist. In particular, we can determine the set of all friends of a given controlled invariant subspace, see e.g. [7], [8].

Example: [7] Let

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $V = \operatorname{span}\{e_2, e_4\}$. Then V is not A-invariant, but controlled invariant. Some linear algebra reveals that the set of all friends of V is given by

$$F = \begin{bmatrix} * & 0 & * & -1 \\ * & * & * & * \end{bmatrix},$$

where the entries marked by stars can be chosen arbitrarily.

The proposed polynomial strategy yields the particular solution $\alpha_1 = -X_4$ and $\alpha_2 = X_3$ which translates to

$$F_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Computing the set of all solutions, we obtain

$$F_0 + \begin{bmatrix} * & 0 & * & 0 \\ * & * & * & * \end{bmatrix}$$

as above. ◊

Concluding Remarks

The gap between the sufficient and the necessary condition in Theorem 1 can be overcome using tools from real algebraic geometry such as the real Nullstellensatz [4], which involves the so-called real radical and sums of squares [6].

Experiments with computer algebra systems suggest that a generalization of Proposition 5 is possible; this remains an open question to be investigated.

The problem of determining all controlled invariant varieties of a given control system (1) has not been touched here. It is much more difficult than the problem we tackled. However, a first step into this direction is the study of parameter-dependent varieties with the goal of identifying those parameter constellations for which the variety becomes controlled invariant. This is another topic for future research.

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