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# Parametrization of Stabilizing Controllers with Fixed Precompensators

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***Abstract*— In the framework of the factorization approach, we give a parameterization of a class of stabilizing controllers. This class is characterized by some fixed strictly causal precompensators. As applications, we present the parameterization of all causal stabilizing controllers including the some fixed number or more integrators, and the parameterization of all strictly causal stabilizing controllers which has the some fixed number or more delay operators.**

## I. Introduction

Since stabilizing controllers of a plant are not generally unique, the choice of the stabilizing controllers is important for the resulting closed-loop. In the classical case, that is, in the case where the given plant admits coprime factorizations, the stabilizing controllers can be parametrized by the so-called Youla-Kučera-parametrization [1], [2], [3], [4], [5], [6]. However, this parametrization may include stabilizing controllers which are not causal and, in the case of a discrete-time system, may result in a closed-loop that does not contain even one-step delay, which is not physically realizable. There are models such that some stabilizable plants do not admit coprime factorizations [7]. A parametrization that can be applied even to stabilizable plants that do not admit doubly coprime factorizations is given in [8], [9], which may also include stabilizing controllers that are not causal and closed-loop systems that are causal but not strictly causal.

In this paper, we give a parametrization method of a class of stabilizing controllers. For the classical continuous-time system model, the parametriztaion can give a parameterization of all causal stabilizing controllers which has the some integrators. For the classical discrete-time system model, it can give a parameterization of all causal stabilizing controllers which has at least a fixed number of delays.

## II. Preliminaries

We employ the factorization approach [1], [3], [4], [5] and the symbols used in [8] and [10]. The reader is referred to Appendix A of [4] for algebraic preliminaries if necessary.

Denote by $\mathcal{A}$ a commutative ring that is the set of stable causal transfer functions. Because in many attractive applications, the commutative ring $\mathcal{A}$ is a unique factorization domain, we restrict, in this paper, ourselves to consider that $\mathcal{A}$ is a unique factorization domain rather than a general commutative ring. The total field of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d \neq 0\}$. This $\mathcal{F}$ is considered to be the set of all possible transfer functions.

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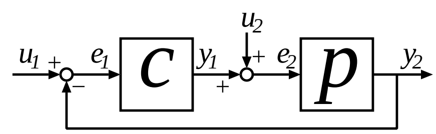


Figure 1. Feedback system Σ.

Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$ with $\mathcal{Z} \neq \mathcal{A}$. Further, let $\mathcal{P} = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{A},\ b \in \mathcal{A} - \mathcal{Z}\}$ and $\mathcal{P}_{\rm s} = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{Z},\ b \in \mathcal{A} - \mathcal{Z}\}$. A transfer function $f$ is said to be *causal* (*strictly causal*) if and only if $f$ is in $\mathcal{P}$ ($\mathcal{P}_{\rm s}$).

Throughout the paper, the plant we consider is causal, single-input and single-output, and its transfer function, which is also called a *plant* itself, is denoted by $p$ and belongs to $\mathcal{P}$. We consider the feedback system $\Sigma$ [4, Ch.5, Figure 5.1] shown in Figure 1. In the figure, $c$ denotes a *controller* and is a transfer function of $\mathcal{F}$. The stabilization problem, considered in this paper, follows the one developed in [1], [4], [5]. For details, the reader is referred to [4], [11], [10], [12].

Let $H(p, c)$ denote the transfer matrix from $[\,u_1^t \quad u_2^t\,]^t$ to $[\,e_1^t \quad e_2^t\,]^t$ of the feedback system $\Sigma$, that is,

$$H(p,c) := \begin{bmatrix} (1+pc)^{-1} & -p(1+pc)^{-1} \\ c(1+pc)^{-1} & (1+pc)^{-1} \end{bmatrix} \in \mathcal{F}^{2\times 2} \quad (1)$$

provided that $1+pc$ is nonzero. We say that the plant $p$ is *stabilizable*, $p$ is *stabilized* by $c$, and $c$ is a *stabilizing controller* of $p$ if and only if $1+pc$ is nonzero and $H(p,c) \in \mathcal{A}^{2\times 2}$. In the definition above, we do not mention the causality of the stabilizing controller. However, it is known that if a causal plant is stabilizable, there always exists a causal stabilizing controller of the plant [11].

A pair $a$ and $b$ of $\mathcal{A}$ are said to be *coprime* (over $\mathcal{A}$) if and only if there exist $x$ and $y$ of $\mathcal{A}$ such that $xa + yb = 1$ holds. An ordered pair $n$ and $d$ of $\mathcal{A}$ are said to be a *coprime factorization* of $p$ if and only if (i) $d$ is nonzero, (ii) $p = n/d$ over $\mathcal{F}$, and (iii) $n$ and $d$ are coprime [1], [4], [5].

A pair $a$ and $b$ of $\mathcal{A}$ are said to be *factor coprime* (over $\mathcal{A}$) if and only if the following holds: for any $x$ of $\mathcal{A}$, if $x$ divides both $a$ and $b$, then $x$ is a unit of $\mathcal{A}$. Especially, if $a$ and/or $b$ are units of $\mathcal{A}$, then $a$ and $b$ are always factor coprime. On the other hand, if (i) both $a$ and $b$ are nonunits of $\mathcal{A}$ and (ii) $a$ and/or $b$ are equal to $0$, then $a$ and $b$ never be factor coprime. A pair $a$ and $b$ of $\mathcal{F}$ are said to be *rationally factor coprime* (over $\mathcal{F}$) if and only if there exist $x_1$, $y_1$, $x_2$, $y_2$ of $\mathcal{A}$ such that (i) $a = y_1/x_1$ and $b = y_2/x_2$, (ii)

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$y_1$ and $x_2$ are factor coprime, and (iii) $y_2$ and $x_1$ are factor coprime. Roughly speaking, $a$ and $b$ being rationally factor coprimeness implies that they have no pole-zero cancellation.

For the notion of rational factor coprimeness, we have the following propositions.

*Proposition 1:* Let $p$ be in $\mathcal{P}$ and $c$ in $\mathcal{F}$. Then if $c$ is a stabilizing controller of $p$, then $p$ and $c$ are rationally factor coprime.

*Proof:* Suppose that $c$ is a stabilizing controller of $p$. Then, by Corollary 2.1.5 of [12], there exist $n$, $d$, $y$, $x$ of $\mathcal{A}$ with $p = n/d$, $c = y/x$ such that $ny + dx = u$, where $u$ is a unit of $\mathcal{A}$. These $n$, $d$, $y$, and $x$ satisfy the three conditions (i) to (iii) above. □

*Proposition 2:* Let $a$ and $b$ be elements of $\mathcal{F}$. Suppose that $a$ and $b$ are rationally factor coprime. Then $ab$ is in $\mathcal{A}$ if and only if both $a$ and $b$ are in $\mathcal{A}$.

*Proof:* "If" part is obvious. Hence we show "Only if" part only.

Let $a_n$, $a_d$, $b_n$, $b_d$ be in $\mathcal{A}$ with $a = a_n/a_d$ and $b = b_n/b_d$ such that each of pairs $(a_n, a_d)$, $(b_n, b_d)$, $(a_n, b_d)$ and $(b_n, a_d)$ is factor coprime. Suppose now that $ab$ is in $\mathcal{A}$. Then $a_d$ is a unit of $\mathcal{A}$ because the pairs $(a_n, a_d)$ and $(b_n, a_d)$ are factor coprime. This means that $a$ is in $\mathcal{A}$. Analogously $b$ is also in $\mathcal{A}$. □

*Proposition 3:* Let $a$, $b$, $c$ be in $\mathcal{F}$. Assume that $a$ and $b$ are rationally factor coprime. Then $ab$ and $c$ are rationally factor coprime if and only if (i) $a$ and $c$ are rationally factor coprime and (ii) $b$ and $c$ are rationally factor coprime. ▪

To prove this proposition, we provide one lemma.

*Lemma 1:* Let $a$ be in $\mathcal{F}$. Then $a$ and $0$ are rationally factor coprime if and only if $a$ is in $\mathcal{A}$.

*Proof:* This is a special case of Proposition 2. So the proof is omitted. □

*Proof of Proposition 3:* "If" part is obvious. Further, if all $a$, $b$ and $c$ are nonzero, the proposition is obvious. Thus, in the following, we prove the following cases of "Only if" part only: (i) $c = 0$ and (ii) $ab = 0$.

(i) $c = 0$. Then by Lemma 1, $ab$ is in $\mathcal{A}$. Because $a$ and $b$ are rationally factor coprime, both $a$ and $b$ are in $\mathcal{A}$ by Proposition 2. Hence $a$ and $c$ ($b$ and $c$) are rationally factor coprime.

(ii) $ab = 0$. By Lemma 1, $c$ is in $\mathcal{A}$. We assume, without loss of generality, that $a = 0$. Then by Lemma 1, $b$ is in $\mathcal{A}$. Now we have $a = 0$, $b, c \in \mathcal{A}$. Hence $a$ and $c$ ($b$ and $c$) are rationally factor coprime. □

Because we investigate the set of some kind of stabilizing controllers, we introduce some notations as follows ($p \in \mathcal{P}$):

$$\begin{aligned} \mathcal{S}(p) &:= \{c \in \mathcal{F} \mid H(p,c) \in \mathcal{A}^{2\times 2}\} && (2)\\ &= \text{“the set of all stabilizing controllers of } p\text{”}, \\ \mathcal{SP}(p) &:= \mathcal{S}(p) \cap \mathcal{P} && (3)\\ &= \text{“the set of all causal stabilizing controllers of } p\text{”}, \\ \mathcal{SP}_{\mathrm{s}}(p) &:= \mathcal{S}(p) \cap \mathcal{P}_{\mathrm{s}} && (4)\\ &= \text{“the set of all strictly causal stabilizing controllers of } p\text{”}. \end{aligned}$$

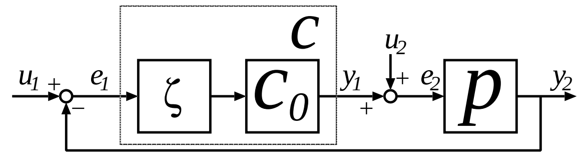


Figure 2. Feedback system with a precompensator.

For strictly causal plants, we know the following proposition.

*Proposition 4:* (cf. Proposition 1 of [9]) Let $p$ be a strictly causal plant. Then any stabilizing controller of $p$ is causal, that is, $\mathcal{SP}(p) = \mathcal{S}(p)$. ▪

In this paper, we consider a fixed causal *precompensator* $\zeta$ ($\in \mathcal{P}$) as a part of a controller $c$ as shown in Figure 2 ($c = c_0\zeta$). We assume that $c_0$ and $\zeta$ must be rationally factor coprime.

We further introduce the set of all causal stabilizing controllers of $p$ including a precompensator as follows ($p, \zeta \in \mathcal{P}$):

$$\begin{aligned} \mathcal{SP}(p;\zeta) &:= \{c_0\zeta \mid c_0 \in \mathcal{P},\ c_0\zeta \in \mathcal{SP}(p),\ c_0 \text{ and } \zeta \text{ are rationally factor coprime}\} && (5)\\ &= \text{“the set of all causal stabilizing controllers } c \text{ of } p \text{ in the form of } c = c_0\zeta \text{ with some } c_0 \in \mathcal{P} \text{ such that } c_0 \text{ and } \zeta \text{ are rationally factor coprime”}. \end{aligned}$$

*Note 1:* Before finishing this section, we consider the case where $c_0$ and $\zeta$ may not be rationally factor coprime. Let $\zeta_{n1}$ and $\zeta_d$ be in $\mathcal{A} - \mathcal{Z}$ and $\zeta_{n2}$ in $\mathcal{Z}$ such that (i) $\zeta = \zeta_{n1}\zeta_{n2}/\zeta_d$ and (ii) any nonunit factor of $\zeta_{n2}$ is in $\mathcal{Z}$. Then any stabilizing controller of $p$ in the form of $c'_0\zeta_{n2}$ can be rewritten as $c_0\zeta$ with $c_0 = c'_0\zeta_d/\zeta_{n1}$. This cancel $\zeta_{n1}$ and $\zeta_d$ of $c_0$ and $\zeta$. By assuming that $c_0$ and $\zeta$ must be rationally factor coprime, we avoid this type of cancel. ▪

## III. Stabilizing Controllers with a Precompensator

In this section, we investigate the set of stabilizing controllers with a precompensator $\zeta$.

*Theorem 1:* Let $p$, $\zeta$ and $c$ be elements of $\mathcal{P}$. Then the following (i) and (ii) are equivalent.

(i) (a) $p$ and $\zeta$ are rationally factor coprime and (b) $c$ is a causal stabilizing controller of $p\zeta$ ($c \in \mathcal{SP}(p\zeta)$).

(ii) (a) $c$ and $\zeta$ are rationally factor coprime and (b) $c\zeta$ is a ausal stabilizing controller of $p$ ($c\zeta \in \mathcal{SP}(p)$).

*Corollary 1:* Let $p$, $\zeta$ and $c$ be elements of $\mathcal{P}$. Suppose that each two of $p$, $\zeta$ and $c$ are rationally factor coprime. Then $c$ is a stabilizing controller of $p\zeta$ if and only if $c\zeta$ is a stabilizing controller of $p$ (or equivalently $H(p\zeta, c)$ is over $\mathcal{A}$ if and only if $H(p, c\zeta)$ is over $\mathcal{A}$).

*Proof of Theorem 1:*

We prove (i)→(ii) only. The opposite can be proved analogously.

Suppose that $H(p\zeta, c)$ is over $\mathcal{A}$. Let $n_0$, $d_0$, $y$ and $x$ be elements in $\mathcal{A}$ with $p\zeta = n_0/d_0$ and $c = y/x$ such that

$$n_0 y + d_0 x = 1 \tag{6}$$

(This Bézout identity exists from Corollary 2.1.5 of [12]). Because $p$ and $\zeta$ are rationally factor coprime, there exist $n$, $d$, $\zeta_n$, and $\zeta_d$ in $\mathcal{A}$ such that $p = n/d$, $\zeta = \zeta_n/\zeta_d$, $n_0 = n\zeta_n$, $d_0 = d\zeta_d$, and $(n, \zeta_d)$ and $(d, \zeta_n)$ are factor coprime. Then from (6) we have

$$ny\zeta_n + dx\zeta_d = 1$$

and

$$\begin{aligned} H(p, c\zeta) &= \begin{bmatrix} (1+pc\zeta)^{-1} & -p(1+pc\zeta)^{-1} \\ c\zeta(1+pc\zeta)^{-1} & (1+pc\zeta)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} dx\zeta_d & nx\zeta_d \\ dy\zeta_n & dx\zeta_d \end{bmatrix}, \end{aligned}$$

which is over $\mathcal{A}$. □

The proof of Corollary 1 is omitted.

*Theorem 2:* Let $p$ be an element of $\mathcal{P}$ and $\zeta$ an element of $\mathcal{P}_{\rm s}$. Suppose that $p$ and $\zeta$ are rationally factor coprime. Then the following (i) and (ii) hold.

(i)

$$\mathcal{SP}(p;\zeta) = \{c_0\zeta \,|\, c_0 \in \mathcal{SP}(p\zeta)\}. \tag{7}$$

(ii) Let $n$, $d$, $y$, $x$ be elements in $\mathcal{A}$ with $p\zeta = n/d$, $c = y/x$ such that $ny + dx = u$, where $u$ is a unit of $\mathcal{A}$. Then

$$\mathcal{SP}(p;\zeta) = \{\zeta\frac{y+rd}{x-rn} \,|\, r \in \mathcal{A}\}. \tag{8}$$

*Proof:* (i) We prove (7) by showing "⊃" and "⊂."

⊃: Let $c_0 \in \mathcal{SP}(p\zeta)$. By Theorem 1, $c_0\zeta \in \mathcal{SP}(p)$ and $c_0$ and $\zeta$ are rationally factor coprime. Hence $c_0\zeta \in \mathcal{SP}(p;\zeta)$.

⊂: Let $c \in \mathcal{SP}(p;\zeta)$ and $c_0 \in \mathcal{P}$ with $c = c_0\zeta$ such that $c_0$ and $\zeta$ are rationally factor coprime. By Theorem 1, $c_0$ is a stabilizing controller of $p\zeta$. Because $c_0$ is causal, $c_0$ is in $\mathcal{SP}(p\zeta)$.

(ii) The Bézout identity $ny + dx = u$ exists from Corollary 2.1.5 of [12]. Hence the Youla-Kučera-parameterization can be applied. We have

$$\mathcal{S}(p\zeta) = \{\frac{y+rd}{x-rn} \,|\, r \in \mathcal{A}\} \tag{9}$$

(Note, because of $p\zeta \in \mathcal{P}_{\rm s}$, we do not need the condition $x - rn \neq 0$). By Proposition 4, $\mathcal{S}(p\zeta) = \mathcal{SP}(p\zeta)$, that is,

$$\mathcal{SP}(p\zeta) = \{\frac{y+rd}{x-rn} \,|\, r \in \mathcal{A}\}. \tag{10}$$

Applying (i) to (10), we have (8). □

## IV. Application I: Multiple Delay Precompensator

Consider the classical discrete-time systems. In this case, the set $\mathcal{A}$ of stable causal transfer functions is given as

$$\mathcal{A} = \{f(z) \in \mathbb{R}(z) \,|\, f(z) \text{ has no poles in the closed unit disc of } \mathbb{C}\} \tag{11}$$

($z$ denotes the delay operator). It is also known that this $\mathcal{A}$ is a Euclidean domain with the degree function $\delta : (\mathcal{A}-\{0\}) \to \mathbb{Z}_+$:

$$\delta(f) = \text{“number of zeros of } f \text{ inside the close unit circle”}$$

(See again Chapter 2 of [4]).

The set $U_\mathcal{A}$ of all units of $\mathcal{A}$ is

$$U_\mathcal{A} = \{f \in \mathcal{A} \,|\, f^{-1} \in \mathcal{A}\} = \{f \in \mathcal{A} \,|\, \delta(f) = 0\}.$$

The ideal $\mathcal{Z}$ for the definition of the causality is given as

$$\mathcal{Z} = \{f \in \mathcal{A} \,|\, f = zf_0,\ f_0 \in \mathcal{A}\},$$

which is obviously a prime and principal ideal. In fact, for $f$ in $\mathcal{A}$, the ideal $(f)$ is equal to $\mathcal{Z}$ if and only if $f = zf_0$, where $f_0$ is a unit of $\mathcal{A}$. The generator of $\mathcal{Z}$ can be, for example, $z$, $(z^2 + 2z)/(z + 3)$, and so on.

*Example 1:* Let us consider the following plant:

$$p = \frac{z}{(1+3z)(1-2z)}.$$

Consider to obtain all stabilizing controllers that have *three or more delays*. That is, we obtain all stabilizing controllers such that every term of their numerator contains $z^3$ as a factor. Let $\zeta$ be $z^3$.

Then we have $ny + dx = 1$, where $p\zeta = n/d$ and

$$\begin{aligned} n &= -\frac{169}{216}z^4, \quad d = \frac{169}{216}(-1 - z + 6z^2), \\ y &= \frac{216}{169}(-55 + 78z), \\ x &= \frac{216}{169}(-1 + z - 7z^2 + 13z^3). \end{aligned}$$

Now $\mathcal{SP}(p;\zeta)$ is given as in (8). For example, letting $r = 7/(z + 2)$, we obtain the following stabilizing controller:

$$\frac{z^3(-5332087 + 4512329z + 4838730z^2)}{-93312 + 46656z - 606528z^2 + 886464z^3 + 806455z^4}$$

in which the numerator has a factor $z^3$ as wanted. ■

## V. Application II: Multiple Integrator Precompensator

Consider the classical continuous-time systems. Let $C_+$ denote the closed right half-plane $\{s \,|\, \Re s \geq 0\}$ and $C_{+e}$ denote the extended right half-plane, that is, $C_+$ together with the point at infinity. Then the set $\mathcal{A}$ of stable causal transfer functions is given by

$$\mathcal{A} = \{f(s) \in \mathbb{R}(s) \,|\, \sup_{s \in C_{+e}} |f(s)| < \infty\}.$$

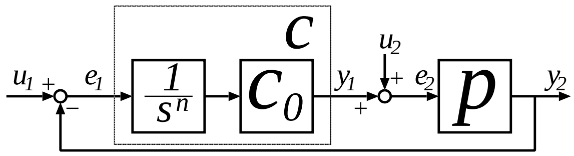


Figure 3. Feedback system with an integrator as a precompensator.

It is known that this $\mathcal{A}$ is a Euclidean domain with the degree function $\delta : (\mathcal{A} - \{0\}) \to \mathbb{Z}_+$:

$$\delta(f) = \text{“number of zeros of } f \text{ in } C_{+e}\text{”}$$

(See Chapter 2 of [4]). The set $U_\mathcal{A}$ of all units of $\mathcal{A}$ is

$$U_\mathcal{A} = \{f \in \mathcal{A} \,|\, f^{-1} \in \mathcal{A}\} = \{f \in \mathcal{A} \,|\, \delta(f) = 0\}.$$

The ideal $\mathcal{Z}$ for the definition of the causality is given as

$$\mathcal{Z} = \{f \in \mathcal{A} \,|\, f = n/d,\ n, d \in \mathbb{R}[s],\ \deg(n) < \deg(d)\},$$

which is a prime and principal ideal. In fact, for $f$ in $\mathcal{A}$, the ideal $(f)$ is equal to $\mathcal{Z}$ if and only if $\delta(f) = 1$ and $\deg(n) < \deg(d)$, where $n$ and $d$ are polynomials of $s$ over $\mathbb{R}$ with $f = n/d$. The generator of $\mathcal{Z}$ can be, for example,

$$\frac{1}{s+1},\quad \frac{-1}{s+2},\quad \frac{s+3}{(s+1)(s+2)},\quad \frac{s+5}{s^2+2s+2}, \tag{12}$$

and so on. Note that this means that

$$\mathcal{Z} = (\frac{1}{s+1}) = (\frac{-1}{s+2}) = (\frac{s+3}{(s+1)(s+2)}) = (\frac{s+5}{s^2+2s+2}) = \dots$$

hold.

Let us consider to parametrize all stabilizing controllers that are including the some integrators. The feedback system is as in Figure 3. In this case, the precompensator $\zeta$ in $\mathcal{P}_\mathrm{s}$ is the power of the integrator (that is, $\zeta = 1/s^n$).

In order to have integrators in a stabiziling controller, its plant cannot have a factor $s$ in the numerator by Proposition 1. Thus, in this section, we assume, without loss of generality that plants do not have a factor $s$ in the numerators.

Let us now consider $\mathcal{Z} = (\frac{1}{s+1})$. Then, for example, $\frac{s-5}{s^2+2s+2}$ is in $\mathcal{Z}$ because

$$\frac{s-5}{s^2+2s+2} = \frac{1}{s+1} \cdot \frac{(s+1)(s-5)}{s^2+2s+2}$$

holds and $\frac{(s+1)(s-5)}{s^2+2s+2} \in \mathcal{A}$.

*Example 2:* Let us consider the following plant $p$:

$$p = \frac{2s+1}{(s+1)(s-1)},$$

which appears in [13, p.215]. Let us consider to parametrize all its stabilizing controllers that are including *three or more* integrators. Then we have $ny + dx = 1$, where $p\zeta = n/d$ and

$$n = \frac{363(1+2s)}{256(1+s)^5},\quad d = \frac{363(-1+s)s^3}{256(1+s)^4},$$
$$y = \frac{256(3+18s+48s^2+187s^3)}{1089(1+s)^3},$$
$$x = \frac{256(115+279s+111s^2+27s^3+3s^4)}{1089(1+s)^4}.$$

Now we obtain the parametrization of $\mathcal{SP}(p;\zeta)$ as in (8). For example, letting $r = 7/(s+2)$ again, we obtain a stabilizing controller $c = c_n/c_d$ including three integrators, where

$$\begin{aligned} c_n &= (1+s)(393216 + 2949120s + 10027008s^2 \\ &\quad + 32360147s^3 + 42678573s^4 + 12255232s^5), \\ c_d &= s^3(12306131 + 53644710s + 76939264s^2 \\ &\quad + 43646976s^3 + 12976128s^4 + 2359296s^5 \\ &\quad + 196608s^6). \end{aligned}$$

■

## VI. Conclusion and Future Works

In this paper, we have given a parameterization method of all stabilizing controllers with some fixed precompensators. This includes (i) the parameterization of all causal stabilizing controllers including the some fixed number or more integrators, and (ii) the parameterization of all strictly causal stabilizing controllers which has the some fixed number or more delay operators.

We have considered, in this paper, single-input and single-output systems. Further, the author will report the further results for multi-input and multi-output systems. Also he will report the parameterization of all strictly causal stabilizing controllers for the multidimensional systems.

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