

# Algebraic properties of Riccati equations

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**Abstract**—In this paper we examine the question when the LQR Riccati equation for matrices with components in a subalgebra  $\mathfrak{A}$  of  $\mathcal{L}(H)$ , where  $H$  is a Hilbert space, will have a unique nonnegative exponentially stabilizing solution with components in  $\mathfrak{A}$ . We give counterexamples to results claimed in the literature and some positive results for  $2 \times 2$  matrices. In addition, we pose a conjecture and an algebraic problem.

## I. INTRODUCTION

Over the years there has been considerable interest in the algebraic properties of the following Riccati equation

$$A \sim X + XA - XBB \sim X + C^*C = 0, \quad (1)$$

where  $A, B, C \in \mathfrak{A}$ , a Banach algebra with identity, and the involution operation  $\sim$ . Conditions are sought to ensure that (1) has a stabilizing solution in  $\mathfrak{A}$ : i.e.,  $X \in \mathfrak{A}$  satisfies (1) and  $A - BB \sim X$  is stable with respect to  $\mathfrak{A}$ . In the special case  $\mathfrak{A} = \mathcal{L}(H)$  and  $H$  is a Hilbert space with  $\sim = *$ , the adjoint operation, when  $(A, B)$  is exponentially stabilizable and  $(A, C)$  is exponentially detectable, it is well known that there exists a unique stabilizing solution in  $\mathcal{L}(H)$  (see [4]). However, it is easy to construct examples when there will exist no solution in the algebra, see [2], [5].

In this paper we investigate this problem for matrices with components in subalgebras  $\mathfrak{A}$  of  $\mathcal{L}(H)$ , where  $H$  is a Hilbert space, and  $\mathfrak{A}$  satisfy the following properties.

**Property I.1.** Suppose that  $H$  is a Hilbert space and  $\mathfrak{A}$  is a Banach subalgebra of  $\mathcal{L}(H)$  with involution equal to the adjoint on  $\mathcal{L}(H)$  and the following properties.

- 1) If  $a \in \mathfrak{A}$ , then  $a^* \in \mathfrak{A}$ .
- 2) If  $a \in \mathfrak{A}$  is invertible in  $\mathcal{L}(H)$ , then  $a^{-1} \in \mathfrak{A}$ .
- 3) The spectrum of  $a \in \mathfrak{A}$  equals its spectrum wrt  $\mathcal{L}(H)$ .
- 4) If  $a \in \mathfrak{A}$  and  $a > 0$  in  $\mathcal{L}(H)$ , then  $a^\alpha \in \mathfrak{A}$  for  $\alpha > 0$ .

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Matrix algebras with components in  $\mathfrak{A}$  inherit the analogues of properties 1-3.

**Property I.2.** Let  $\mathfrak{A}^{n \times n}$  be the set of all  $n \times n$  matrices with components in  $\mathfrak{A}$ , satisfying Properties I.1 Then

- 1)  $\mathfrak{A}^{n \times n}$  is a unital subalgebra of  $\mathcal{L}(H^n)$ .
- 2) If  $A \in \mathfrak{A}^{n \times n}$ , then  $A^* \in \mathfrak{A}^{n \times n}$ .
- 3) If  $A \in \mathfrak{A}^{n \times n}$  is invertible in  $\mathcal{L}(H^n)$  then  $A^{-1} \in \mathfrak{A}^{n \times n}$ .
- 4) The spectrum of  $A \in \mathfrak{A}^{n \times n}$  is the same wrt  $\mathcal{L}(H^n)$  and  $\mathfrak{A}^{n \times n}$ .

Banach algebras with the above properties have been studied in [11], where  $H = \ell_2(\mathbb{Z}^d; \mathbb{C})$ . An important special case is when  $d = 1$  and  $H = \ell_2(\mathbb{C})$  and  $\mathfrak{A} = \ell_1(\mathbb{C})$  with the weighted norm

$$\|X\|_\beta = \sum_{k \in \mathbb{Z}} \beta_k |x_k|,$$

where the weights  $\beta_k$  are submultiplicative ( $1 \leq \beta_{k+j} \leq \beta_k \beta_j$ ), and satisfy the Gelfand-Raikov-Shilov condition  $\lim_{k \rightarrow \infty} |\beta_k|^{1/k} = 1$ .

The weighted Wiener algebras  $\mathfrak{W}_\beta$  are obtained by taking Fourier transforms of elements in  $\ell_1(\mathbb{C})$  and  $\tilde{X} \in \mathfrak{W}_\beta$  has the representation

$$\tilde{x}(z) = \sum_{k=-\infty}^{\infty} x_k z^k, \quad z \in \mathbb{T},$$

$$\|\tilde{x}\|_\beta = \sum_{k \in \mathbb{Z}} \beta_k |x_k|.$$

These are commutative Banach algebras under the single assumption that  $\beta_k$  are multiplicative. The Gelfand representation of  $\tilde{X}$  is given by

$$\Gamma \tilde{x}(z) = \sum_{k=-\infty}^{\infty} x_k z^k, \quad z \in \Lambda_\beta,$$

where the character set  $\Lambda_\beta$  is the annulus  $\{z : \rho_1 \leq |z| \leq \rho_2\}$ , and

$$\rho_1 = \lim_{k \rightarrow -\infty} \beta_k^{1/k}, \quad \rho_2 = \lim_{k \rightarrow \infty} \beta_k^{1/k}.$$

The involution is defined by  $(\Gamma\check{a})^\sim(z) = (\Gamma\check{a})(\overline{1/z})^*$  for  $z \in \Lambda_\beta$ . There holds  $\|\check{x}\|_\beta \leq \|\Gamma\check{x}\|$  and  $\Gamma\mathfrak{W}_\beta$  is a dense subspace of the continuous functions on  $\Lambda_\beta$ . The functions  $\Gamma\check{x}$  are analytic in the interior of  $\Lambda_\beta$ . (see [10, p.824]). Since  $\Gamma\check{x}$  is essentially an extension of  $\check{x}$  to the annulus  $\Lambda_\beta$ , for simplicity we write  $\check{x}$ , omitting the  $\Gamma$ -notation. Under the Gelfand-Raikov-Shilov condition the character set reduces to the unit circle,  $\mathbb{T}$  (see [10, p.790, 813, 824]).

In [2, Theorem 2.2] the following is claimed:

**Theorem I.3.** *Suppose that  $A, B, C \in \mathfrak{A}^{n \times n}$  where  $\mathfrak{A}$  is a commutative Banach algebra. Denote their Gelfand representations by  $\Gamma A(z), \Gamma B(z), \Gamma C(z)$ , for  $z \in \Lambda$ . If there exists an integer  $k \geq 0$  such that  $\text{span}[B, AB, \dots, A^{k-1}B] = \mathfrak{A}^{n \times n}$  and  $(\Gamma A, \Gamma C(z))$  is observable for each  $z \in \Lambda$ , then the Riccati equation (1) will have a unique positive solution with components in in  $\mathfrak{A}$ .*

Unfortunately, this result is false as the following example illustrates.

**Example I.4.** Take  $\check{a}(z) = 2 + z, \check{b}(z) = 1, \check{c}(z) = 1 \in \mathfrak{W}_\beta$  for arbitrary weights  $\beta_k$ . Now  $\text{span}(\check{b}) = 1$  and  $\check{a}(z), \check{c}(z)$  is observable for all  $z \in \mathbb{C}$ . The solution to the scalar version of (1) is given by

$$2\check{x}(z) = \left( 4 + z + \frac{1}{z} + \sqrt{\left(4 + z + \frac{1}{z}\right)^2 + 4} \right).$$

The square root term becomes zero at

$$z = -1 \pm \sqrt{\frac{\sqrt{5}-1}{2}} - i \left( 1 \mp \sqrt{\frac{\sqrt{5}+1}{2}} \right).$$

Hence  $\check{x}$  is not analytic in this point and so  $\check{x}$  can only be in  $\mathfrak{W}_\beta$  for the annulus  $\Lambda_\beta = \{z : \rho \leq |z| \leq 1/\rho\}$ , where  $\rho = 2 + \sqrt{5} - \sqrt{2\sqrt{5}-2} - \sqrt{2\sqrt{5}+2}$ . ■

Note that for  $\check{a}(z) = 2 + \frac{1}{2}(z + 1/z) = \check{a}^\sim(z), \check{b}(z) = 1, \check{c}(z) = 1$  we would obtain the same answer. From this example it is clear that the result claimed in [2, Theorem 2.2] is false. In fact, the best one can hope for is the following result from [7].

**Proposition I.5.** *Suppose that for some  $\alpha > 0$*

$$H(z) = \begin{bmatrix} \check{A}(z) & -\check{B}(z)\check{B}^\sim(z) \\ -\check{C}^\sim(z)\check{C}(z) & -\check{A}^\sim(z) \end{bmatrix}$$

*is analytic in the annulus  $\mathfrak{A}(\alpha) = \{z : 1/\alpha < |z| \leq \alpha\}$ , where  $(\check{A}(z), \check{B}(z), \check{C}(z))$  is stabilizable and detectable for all  $z \in \mathbb{T}$ . Then there exists  $\beta, 0 < \beta \leq \alpha$ , such that for every  $z \in \mathfrak{A}(\beta)$ , there*

*exists a unique stabilizing solution  $\check{X}(z)$  of (1), and  $\check{X}(z)$  is analytic in  $\mathfrak{A}(\beta)$  with  $\check{X}^\sim(z) = \check{X}(z)$ .*

This result and the more detailed treatment of the analyticity properties in [7] (and the paper [8] in these proceedings) answers the spatially decaying question adequately. It suggests that a restricted version of the algebraic result claimed in [2, Theorem 2.2] may be true.

**Conjecture I.6.** *Let  $\mathfrak{W}_\beta$  be a weighted Wiener algebra satisfying the Gelfand-Raikov-Shilov condition. If  $\check{A}, \check{B}, \check{C} \in \mathfrak{W}_\beta^{n \times n}$ , and  $(\check{A}(z), \check{B}(z))$  is controllable and  $(\check{A}(z), \check{C}(z))$  is observable for all  $z \in \mathbb{T}$ , then there exists a unique positive solution  $\check{X} \in \mathfrak{W}_\beta^{n \times n}$  to the Riccati equation (1). Moreover,  $\check{A} - \check{B}\check{B}^\sim\check{X}$  generates an exponentially stable semigroup on  $\mathfrak{W}_\beta^{n \times n}$ .*

In this paper we examine the validity of this conjecture for systems  $\check{A}, \check{B}, \check{C} \in \mathfrak{W}_\beta^{n \times n}$ , where  $\check{B}$  has rank one and the coefficients of  $\check{A}, \check{C}$  are self-adjoint (or real). In Section II we obtain explicit solutions to the Riccati equation for canonical scalar and second order examples. In Section III we show how to reduce rank one systems  $\check{A}, \check{B}$  to a canonical controllable form. Hence we verify that Conjecture I.6 is true for systems in  $\mathfrak{W}_\beta^{2 \times 2}$ , provided that the components are real and  $\check{B}$  has rank one. In Section IV we analyze the Riccati equation for canonical controllable systems with real components and  $\check{B}$  of rank one in  $\mathfrak{W}_\beta^{n \times n}$  with  $n = 3, 4$ . While these calculations suggest that the conjecture is true in these cases, we have no proof as yet. The problem reduces to the following algebraic question: given that a polynomial equation with coefficients in  $\mathfrak{W}_\beta$  has a real solution in  $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C})$ , when will this solution be in  $\mathfrak{W}_\beta$ ? Finally, in Section V we discuss the algebraic problem for noncommutative algebras satisfying Properties I.1, and give some positive results that have applications to the spatially distributed systems studied in [12].

## II. RICCATI EQUATIONS OVER THE WIENER ALGEBRAS

First we recall a nice result from [6] for the case that  $\check{A}, \check{B}, \check{C} \in \mathcal{L}(\mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)) = \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$ .

**Theorem II.1.** *Suppose that  $\check{A}, \check{B}, \check{C} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$  are continuous on  $\mathbb{T}$ . If  $(\check{A}(z), \check{B}(z), \check{C}(z))$  is stabilizable and detectable for all  $z \in \mathbb{T}$ , then  $\check{A}, \check{B}, \check{C}$  is exponentially stabilizable and detectable wrt  $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$ . Moreover, the Riccati*

equation

$$\check{A}^* \check{X} + \check{X} \check{A} - \check{X} \check{B} \check{B}^* \check{X} + \check{C}^* \check{C} = 0 \quad (2)$$

has a unique nonnegative solution  $\check{X} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$  and  $\check{A} - \check{B} \check{B}^* \check{X}$  generates an exponentially stable semigroup on  $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^{n \times n})$ .  $\check{X}(z)$  is continuous for  $z \in \mathbb{T}$  and it has the form

$$\check{X}(z) = \sum_{k=-\infty}^{\infty} X_k z^k, \quad z \in \mathbb{T}, \quad X_k \in \mathbb{C}^{n \times n}.$$

According to [1], [12] for implementation of the LQR feedback it is important to know when the coefficients decay rapidly as  $k \rightarrow \infty$ , the so-called *spatially decaying property*. This question translates into the algebraic question of when the Riccati equation (2) has a solution in  $\mathfrak{W}_\beta^{n \times n}$ . We consider some simple cases.

**Example II.2.** Consider (2) for the stabilizable and detectable scalar system  $\check{a}, \check{b}, \check{c} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C})$ . Direct computation yields the unique stabilizing solution given by

$$2\check{b}\check{b}^{\sim} \check{x} = \check{a} + \check{a}^{\sim} + \sqrt{(\check{a} + \check{a}^{\sim})^2 + 4\check{b}\check{b}^{\sim} \check{c}^{\sim} \check{c}}.$$

Under the assumption that  $\mathfrak{W}_\beta$  satisfies the Gelfand-Raikov-Shilov condition, it has Properties I.1. Hence, if  $\check{b}\check{b}^{\sim}$  is invertible over  $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C})$ , then  $\check{x} \in \mathfrak{W}_\beta$  and it is exponentially stabilizing. ■

In our next two examples we solve the Riccati equation (2) for the stabilizing solution

$$\check{X} = \begin{bmatrix} \check{x}_1 & \check{x}_3^* \\ \check{x}_3 & \check{x}_2 \end{bmatrix}$$

for the second order system with the canonical controllable form

$$\check{A} = \begin{bmatrix} 0 & 1 \\ -\check{a}_1 & -\check{a}_2 \end{bmatrix}, \quad \check{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\check{C} = \text{diag}(\check{c}_1, \check{c}_2).$$

**Example II.3.** When  $\check{a}_1, \check{a}_2$  are self-adjoint (real) the solution is given by

$$\begin{aligned} \check{x}_3 &= -\check{a}_1 + \sqrt{\check{a}_1^2 + |\check{c}_1|^2}; \\ \check{x}_2 &= -\check{a}_2 + \sqrt{\check{a}_2^2 + |\check{c}_2|^2 + 2\check{x}_3}; \\ \check{x}_1 &= \check{x}_3 \check{x}_2 + \check{a}_1 \check{x}_2 + \check{x}_3 \check{a}_2. \end{aligned}$$

**Example II.4.** For the special case of no damping,  $\check{a}_2 = 0$ , the solution is

$$\begin{aligned} \check{x}_3 &= -\check{a}_1 + \sqrt{|\check{a}_1|^2 + |\check{c}_1|^2}; \\ \check{x}_2 &= \sqrt{|\check{c}_2|^2 + 2\text{Re } \check{x}_3}; \\ \check{x}_1 &= \sqrt{|\check{a}_1|^2 + |\check{c}_2|^2} \check{x}_2. \end{aligned}$$

■

In both Examples II.3, II.4 we see that, under the assumption that  $\mathfrak{W}_\beta$  satisfies the Gelfand-Raikov-Shilov condition, it has Properties I.1. Hence whenever (2) has an exponential stabilizing solution in  $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{2 \times 2})$ , it is also an exponentially stabilizing solution in  $\mathfrak{W}_\beta^{2 \times 2}$ . In the next section we generalize these last two results.

The general case is more complicated.

**Example II.5.** We introduce the simpler notation

$$\check{x}_3 = \alpha + i\beta, \quad \check{a}_1 = e_1 + ie_2, \quad \check{a}_2 = d_1 + id_2,$$

$$|\check{c}_1|^2 = h^2, \quad |\check{c}_2|^2 = s^2$$

for real  $\alpha, \beta, e_1, e_2, d_1, d_2, h, s$ . We obtain

$$\begin{aligned} \beta + e_2 &= \frac{d_1 e_2 + \alpha d_2}{\sqrt{d_1^2 + s^2 + 2\alpha}}; \\ \check{x}_2 &= -d_1 + \sqrt{d_1^2 + s^2 + 2\alpha}; \\ \check{x}_1 &= e_1 \check{x}_2 + \beta d_2 + \check{x}_2 \alpha (d_1 + \check{x}_2), \end{aligned}$$

where  $\alpha$  is a solution of the cubic equation

$$\begin{aligned} 2\alpha^3 + \alpha^2(d_1^2 + d_2^2 + s^2 + 4e_1) \\ + 2\alpha(e_1(d_1^2 + s^2) + d_1 d_2 e_2 - h^2 - e_2^2) \\ = h^2(d_1^2 + s^2) + e_2^2 s^2. \end{aligned}$$

So if the Riccati equation has a solution, it will be in terms of sums, products, quotients and square roots. We expect that it will also be in  $\mathfrak{W}_\beta^{2 \times 2}$ , but Property I.1.4 requires that the terms under the square roots be positive, and we cannot verify this.

This last example leads to the following interesting algebraic question:

Given that a third order polynomial equation with coefficients in  $\mathfrak{W}_\beta$  has a positive solution in  $\mathbf{L}_\infty(\mathbb{R}; \mathbb{C})$ , will this solution be in  $\mathfrak{W}_\beta$ ? ■

III. SOME SYSTEM THEORETIC PROPERTIES

In [2] the concept of algebraic reachability was introduced. We show that it is equivalent to the notion of exact controllability.

**Definition III.1.** *Let  $H$  be a Hilbert space. Then  $A, B \in \mathcal{L}(H)$  are exactly controllable if and only if there exist positive constants  $\tau, \beta$ , such that for all  $u \in H$  there holds*

$$\int_0^\tau \|B^* e^{A^* t} u\|_H^2 dt \geq \beta \|u\|_H^2.$$

Let  $\mathfrak{A}$  be a unital Banach algebra. For  $A, B \in \mathfrak{A}$  we call  $(A, B)$  algebraically reachable with respect to  $\mathfrak{A}$  if there exists an integer  $k > 0$  such that

$$\text{span}[B, AB, \dots, A^k B] = \mathfrak{A}. \tag{3}$$

■

We remark that when  $A, B \in \mathcal{L}(H)$  and  $H$  is a Hilbert space, then  $(A, B)$  is algebraically reachable with respect to the Banach algebra  $\mathcal{L}(H)$  if  $(A, B)$  is exactly controllable on  $[0, T]$  for  $T > 0$  (see [3, Theorem 3.18]).

**Proposition III.2.** *Suppose that  $\mathfrak{W}_\beta$  satisfies the Gelfand-Raikov-Shilov condition. Then  $(\check{A}, \check{B})$  is algebraically reachable wrt  $\mathfrak{W}_\beta^{n \times n}$  if and only if it is algebraically reachable wrt  $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$  if and only if  $(\check{A}, \check{B})$  is exactly controllable on  $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$ , if and only if  $(\check{A}(z), \check{B}(z))$  is controllable for all  $z \in \mathbb{T}$ .*

**Proof** a. The equivalence of the two algebraic reachability concepts is easy to see. First note that condition (3) is equivalent to the existence of  $n$  columns  $c_k, k = 1, \dots, n$  of  $[\check{B}, \check{A}\check{B}, \check{A}^2\check{B}, \dots, \check{A}^{n-1}\check{B}]$  with components in  $\mathfrak{W}_\beta$  such that the matrix  $S = [c_1 \ c_2 \ \dots \ c_n]$  is invertible over  $\mathfrak{W}_\beta^{n \times n}$ . But property 3 of Proposition I.2 implies that this is equivalent to it being invertible over  $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$  and this implies the algebraic reachability over  $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$ .

b. As in the finite-dimensional case, algebraic reachability is equivalent to the invertibility of the operator  $\int_0^\tau e^{\check{A}t} \check{B} \check{B}^* e^{-\check{A}^* t} dt$  for all  $\tau > 0$  and this is equivalent to the exact controllability of  $(\check{A}, \check{B})$  on  $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$  [4, Theorem 4.1.7].

c. The pointwise conditions were shown in [6, Theorem 3.3]. From [3, Theorem 3.18] exact controllability implies the existence of a positive integer  $k$  such that  $\text{span}(BH^n, ABH^n, A^2BH^n, \dots, A^k BH^n) = H^n$ .

The Cayley-Hamilton theorem shows that we can always take  $k = n - 1$ . So this implies (3) and algebraic reachability. ■

The study of Riccati equations for single input algebraically reachable systems can be reduced to the study of Riccati equations with  $A, B$  having the canonical controllable form.

$$A_c = \begin{bmatrix} 0 & 1 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 1 \\ -a_1 & -a_2 & \cdot & -a_{n-1} & -a_n \end{bmatrix} \tag{4}$$

$$B_c = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 1 \end{bmatrix}. \tag{5}$$

**Proposition III.3.** *Suppose that  $A, B, C \in \mathfrak{A}^{n \times n}$ , where  $\mathfrak{A}$  is a unital Banach algebra, and  $(A, B)$  is algebraically reachable and  $B$  has rank one. Then there exists an invertible matrix in  $T \in \mathfrak{A}^{n \times n}$  such that  $A_c = TAT^{-1}, B_c = -TB, C_c = CT^{-1}$  has the canonical form (4), (5). Moreover,  $X = T^* X_c T$  is a solution to the following Riccati equation*

$$A^* X + X A - X B B^* X + C^* C = 0, \tag{6}$$

if and only if  $X_c$  is a solution to the Riccati equation (6) for  $A_c, B_c, C_c$ .

**Proof** First we show the existence of the invertible matrix  $T \in \mathfrak{A}^{n \times n}$ . Since  $B$  has rank one, we can without loss of generality take  $B$  to have the structure with the first  $n - 1$  columns zero and the last column  $b$ . Since  $(A, B)$  is algebraically reachable, the reachability matrix  $S = [b A b \cdot \dots \cdot A^{n-1} b]$  is boundedly invertible in  $\mathfrak{A}^{n \times n}$ . As in the constant coefficients case, an invertible  $T \in \mathfrak{A}^{n \times n}$  can be constructed iteratively with

$$t_1^* = b_c^* S^{-1}, \quad t_{k+1}^* = t_1^* \check{A}^k,$$

such that  $A_c = TAT^{-1}, B_c = -TB = B_c$ , and  $A_c, B_c$  have the canonical form (4), (5). The rest is substitution in (6). ■

Combining the last two propositions gives the following corollary.

**Corollary III.4.** *Suppose that  $\check{A}, \check{B}, \check{C} \in \mathfrak{W}_\beta^{n \times n}$ , which satisfies the Gelfand-Raikov-Shilov condition and  $\check{B}$  has rank one. If  $(\check{A}(z), \check{B}(z))$  is controllable and  $(\check{A}(z), \check{C}(z))$  is detectable for all  $z \in \mathbb{T}$ , then there exists an invertible matrix in  $T \in \mathfrak{W}_\beta^{n \times n}$  such that  $A_c = T\check{A}T^{-1}, B_c = -T\check{B}, C_c = \check{C}T^{-1}$  has the canonical form (4),(5). Moreover,  $\check{X} = T^*X_cT$  is a solution to the Riccati equation (2) for  $\check{A}, \check{B}, \check{C}$  if and only if  $\check{X}$  is a solution to the Riccati equation (6) for  $A_c, B_c, C_c$ .*

The following is an easy application of the above corollary to Example II.3.

**Corollary III.5.** *Suppose that  $\check{A}, \check{B}, \check{C} \in \mathfrak{W}_\beta^{2 \times 2}$ , which satisfies the Gelfand-Raikov-Shilov condition,  $\check{A} = \check{A}, \check{C} = \check{C}$  and  $\check{B}$  has rank one. If  $(\check{A}(z), \check{B}(z))$  is controllable and  $(\check{A}(z), \check{C}(z))$  is detectable for all  $z \in \mathbb{T}$ , then there exists an exponentially stabilizing nonnegative solution to the Riccati equation (2) in  $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{2 \times 2}) \cap \mathfrak{W}_\beta^{2 \times 2}$ .*

#### IV. HIGHER ORDER REAL SYSTEMS

The following result can be verified by direct substitution.

**Lemma IV.1.** *Suppose that  $A, B, C \in \mathfrak{A}$ , a unital Banach algebra and that (6) has a solution  $X = X^* \in \mathfrak{A}$ . Then for all  $\lambda \in \rho(A) \cap i\mathbb{R}$  there holds*

$$\begin{aligned} I + B^*(\lambda I - A)^{-*}C^*C(\lambda I - A)^{-1}B \\ = Q(\lambda)^*Q(\lambda), \end{aligned} \quad (7)$$

where  $Q(\lambda) = I + B^*X(\lambda I - A)^{-1}B$ .

The following lemma gives conditions for (2) to have a nonnegative stabilizing solution with real components.

**Lemma IV.2.** *Suppose that  $\check{A}, \check{B}, \check{C} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$  with  $\check{A} = \check{A}, \check{B}\check{B}^* = \check{B}\check{B}^*, \check{C}^*\check{C} = \check{C}^*\check{C}$ . If  $(\check{A}(z), \check{B}(z))$  is stabilizable and  $(\check{A}(z), \check{C}(z))$  is detectable for all  $z \in \mathbb{T}$ , then there exists a unique stabilizing nonnegative solution  $\check{X} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$  to (2) such that  $\check{X} = \check{X}^* = \overline{\check{X}}$ .*

**Proof** The stabilizability and detectability assumptions guarantee the existence of a unique nonnegative, exponentially stabilizing, solution  $\check{X} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{n \times n})$  to (2) such that  $\check{X} = \check{X}^*$ . Note that whenever  $\check{X}$  is a solution to (2) so is  $\overline{\check{X}} = \check{X}^*$ . Moreover,  $\lambda \in \sigma(\check{A} - \check{B}\check{B}^*\check{X}) \iff \bar{\lambda} \in \sigma(\check{A} - \check{B}\check{B}^*\overline{\check{X}})$ . So the spectra of  $\check{A} - \check{B}\check{B}^*\check{X}$  and  $\check{A} - \check{B}\check{B}^*\overline{\check{X}}$  lie in the same left half-plane and  $\overline{\check{X}}$  is also self-adjoint

and exponentially stabilizing. The uniqueness implies that  $\overline{\check{X}} = \check{X}$ . ■

We now suppose that  $\check{A}, \check{B}, \check{C} \in \mathfrak{W}_\beta^{n \times n}$ , where  $\beta_k$  satisfy the Gelfand-Raikov-Shilov condition and  $\check{A}, \check{B}$  have the canonical controllable form (4), (5) and the components are real, i.e.,  $\check{a}_1^* = \check{a}_1$ . Then we have

$$\det(\lambda I - \check{A}) = \lambda^n + \check{a}_n\lambda^{n-1} + \dots + \check{a}_1.$$

$$(\lambda I - \check{A})^{-1}\check{B} = \frac{1}{\det(\lambda I - \check{A})} \begin{bmatrix} 1 \\ \lambda \\ \cdot \\ \cdot \\ \cdot \\ \lambda^{n-1} \end{bmatrix}$$

$$\begin{aligned} \check{B}^*\check{X}(\lambda I - \check{A})^{-1}\check{B} = \\ \frac{(\check{x}_{n1} + \check{x}_{n2}\lambda + \dots\check{x}_{nn}\lambda^{n-1})}{\det(\lambda I - \check{A})}. \end{aligned}$$

So substituting the above into the identity (7) with  $\lambda \in \rho(\check{A}) \cap i\mathbb{R}$  yields

$$\begin{aligned} 2\text{Re}\overline{\det(\lambda - \check{A})}(\check{x}_{n1} + \check{x}_{n2}\lambda + \dots\check{x}_{nn}\lambda^{n-1}) \\ + |(\check{x}_{n1} + \check{x}_{n2}\lambda + \dots\check{x}_{nn}\lambda^{n-1})|^2 \end{aligned} \quad (8)$$

$\in \mathfrak{W}_\beta$ . Since  $\check{A}$  is bounded, its spectrum on  $i\mathbb{R}$  is at most contained in a compact interval. Hence (8) is valid for all  $\lambda$  outside a compact interval. With  $\lambda = i\omega$ , both the left-hand and right-hand sides of (8) reduce to polynomials in  $\omega^2$ . All terms in the left-hand side of (8) are in  $\mathfrak{W}_\beta$  and the coefficients of the polynomial on the right-hand side (8) are in  $\mathfrak{W}_\beta$ . Under our assumptions we know that  $\check{x}_{n1}, \check{x}_{n2}, \dots, \check{x}_{nn}$  are real (see Lemma IV.2). Hence we obtain  $n$  nonlinear equations on  $\mathfrak{W}_\beta$  in the  $n$  unknowns  $\check{x}_{n1}, \check{x}_{n2}, \dots, \check{x}_{nn}$ . Suppose we can find solutions to these nonlinear equations in  $\mathfrak{W}_\beta$ . Then we can show that the remaining components of  $\check{X}$  are also in  $\mathfrak{W}_\beta$  by solving the matrix equation  $\check{A}^*\check{X} + \check{X}\check{A} = \check{X}\check{B}\check{B}^*\check{X} - \check{C}^*\check{C}$ , which is now linear, since the right hand side are known quantities in  $\check{C}^*\check{C}$  and  $\check{x}_{n1}, \check{x}_{n2}, \dots, \check{x}_{nn}$  (see (8)).

The remaining question is whether one can show that the solutions to the  $n$  nonlinear equations in the  $n$  unknowns  $\check{x}_{n1}, \check{x}_{n2}, \dots, \check{x}_{nn}$  are in  $\mathfrak{W}_\beta$ . For the cases  $n = 3, 4$  this reduces to the solution of a quartic equation over  $\mathfrak{W}_\beta$  given that it has a solution over  $\mathbf{L}_\infty(\mathbb{T}; \mathbb{C})$ .

**Example IV.3.**  $n=3$  case. The equation (8) with  $\lambda = i\omega$  yields

$$2\operatorname{Re}\{[(\check{a}_1 - \omega^2\check{a}_3) + i(\omega^3 - \check{a}_2\omega)][\check{x}_{31} - \omega^2\check{x}_{33} + i\omega\check{x}_{32}]\} + |\check{x}_{31} - \omega^2\check{x}_{33} + i\omega\check{x}_{32}|^2.$$

So equating coefficients we obtain

$$\begin{aligned} 2\check{a}_1\check{x}_{31} + \check{x}_{31}^2 &= (\check{x}_{31} + \check{a}_1)^2 - \check{a}_1^2 \in \mathfrak{W}_\beta; \\ -2\check{a}_1\check{x}_{33} - 2\check{a}_3\check{x}_{31} + 2\check{a}_2\check{x}_{32} - \\ &\quad 2\check{x}_{31}\check{x}_{33} + \check{x}_{32}^2 \in \mathfrak{W}_\beta; \\ \check{x}_{33}^2 + 2\check{a}_3\check{x}_{33} - 2\check{x}_{32} &\in \mathfrak{W}_\beta. \end{aligned}$$

Introducing the new variables  $y_k = \check{x}_{3k} + \check{a}_k$ ,  $k = 1, 2, 3$  we obtain

$$y_1^2, \quad y_3^2 - 2y_2, \quad y_2^2 - 2y_3y_1 \in \mathfrak{W}_\beta.$$

Hence  $y_1 \in \mathfrak{W}_\beta$  and for some  $\alpha \in \mathfrak{W}_\beta$  we have the quartic equation

$$(y_3^2 + \alpha)^2 - 8y_1y_3 \in \mathfrak{W}_\beta.$$

So whenever  $y_3 \in \mathfrak{W}_\beta$ , we can conclude that  $\check{X} \in \mathfrak{W}_\beta^{3 \times 3}$ . ■

**Example IV.4.**  $n=4$  case. The equation (8) with  $\lambda = i\omega$  yields

$$\begin{aligned} 2\operatorname{Re}\{[(\check{a}_1 - \check{a}_3\omega^2 + \omega^4) + i(\check{a}_4\omega^3 - \check{a}_2\omega)] \\ [(\check{x}_{41} - \check{x}_{43}\omega^2) + i(\check{x}_{42}\omega - \check{x}_{44}\omega^3)]\} \\ + |(\check{x}_{41} - \check{x}_{43}\omega^2) + i(\check{x}_{42}\omega - \check{x}_{44}\omega^3)|^2 \in \mathfrak{W}_\beta. \end{aligned}$$

So equating coefficients we obtain

$$\begin{aligned} 2\check{a}_1\check{x}_{41} + \check{x}_{41}^2 &\in \mathfrak{W}_\beta; \\ 2(-\check{a}_3\check{x}_{41} - \check{a}_1\check{x}_{43} + \check{a}_2\check{x}_{42} - \check{x}_{41}\check{x}_{43}) + \check{x}_{42}^2 &\in \mathfrak{W}_\beta; \\ 2(\check{x}_{41} + \check{a}_3\check{x}_{43}) - 2(\check{a}_2\check{x}_{44} + \check{a}_4\check{x}_{42}) \\ &\quad + \check{x}_{43}^2 - 2\check{x}_{44}\check{x}_{42} \in \mathfrak{W}_\beta; \\ \check{x}_{44}^2 + 2\check{a}_4\check{x}_{44} - 2\check{x}_{43} &\in \mathfrak{W}_\beta. \end{aligned}$$

Introducing the new variables  $y_k = \check{x}_{3k} + \check{a}_k$ ,  $k = 1 - 4$ , we obtain

$$y_1^2, \quad y_2^2 - 2y_1y_3, \quad y_3^2 - 2y_2y_4, \quad y_4^2 - y_3 \in \mathfrak{W}_\beta.$$

Hence  $y_1 \in \mathfrak{W}_\beta$  and we can eliminate  $y_2$  and  $y_3$  to obtain the following quartic equation in  $y_3$ :

$$(y_3^2 + \alpha_1)^2 = 4(y_3 + \alpha_2)(2y_3y_1 + \alpha_3),$$

where  $\alpha_k, k = 1, 2, 3 \in \mathfrak{W}_\beta$ . So whenever  $y_3 \in \mathfrak{W}_\beta$ , we can conclude that  $\check{X} \in \mathfrak{W}_\beta^{4 \times 4}$ . ■

For both examples we obtain a quartic polynomial equation with coefficients in  $\mathfrak{W}_\beta$ . We know that this

has a real solution  $y_3 \in \mathcal{L}_2(\mathbb{T}; \mathbb{C})$  and it is in terms of sums, quotients and square and cubic roots of the coefficients. So this leads to the following interesting algebraic question:

Given that a fourth order polynomial equation with coefficients in  $\mathfrak{W}_\beta$  has a positive solution in  $\mathcal{L}_\infty(\mathbb{R}; \mathbb{C})$ , will this solution be in  $\mathfrak{W}_\beta$ ?

## V. NON-COMMUTATIVE ALGEBRAS

The aim in [12] was to find conditions under which the LQR Riccati equation (6) for spatially distributed systems on the state space  $H = \ell_2(\mathbb{Z}^d; \mathbb{C}^n)$  will have solutions with a spatial decaying property. This property was formulated as lying in a certain non-commutative subalgebra  $\mathfrak{A}$  of  $\mathcal{L}(H^n)$ . They claimed that whenever the operators  $A, B, C \in \mathfrak{A}$  of  $\mathcal{L}(H^n)$ , and (6) has a nonnegative exponentially stabilizing solution  $X$ , then  $X \in \mathfrak{A}$ . While it is easy to show that this is false in general (see [5]), it is more challenging to give conditions under which it is true. Here we examine the case where the non-commutative subalgebra  $\mathfrak{A}$  satisfies Properties I.1. The problem in [12] then translates into a non-commutative version of the problem already treated in the previous sections. Consider the subalgebras of  $\mathcal{L}(\ell_2(\mathbb{Z}^d; \mathbb{C}^n))$  given by

$$\mathfrak{A}_v := \{A \in \mathcal{L}(\ell_2(\mathbb{Z}^d; \mathbb{C})) : \|A\|_v < \infty\},$$

where

$$\begin{aligned} \|A\|_v &:= \max\left\{\sup_{k \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} |a_{kl}|v(k-l)\right\}, \\ &\sup_{l \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |a_{kl}|v(k-l)\}, \end{aligned}$$

and  $v : \mathbb{R}^d \rightarrow \mathbb{R}^+$  is an *admissible weight function* with the properties:

- $v(x) = e^{\rho(\|x\|)}$  where  $\rho : [0, \infty) \rightarrow [0, \infty)$  is a continuous concave function with  $\rho(0) = 0$ .
- $\lim_{k \rightarrow \infty} v(kx)^{1/k} = 1$  for all  $x \in \mathbb{R}^d$ .
- there exist positive constants  $C, \delta, 0 < \delta \leq 1$  such that  $v(x) \geq C(1 + |x|)^\delta$ .

The first condition implies that  $v$  is submultiplicative, ( $v(x+y) \leq v(x)v(y)$ ), while the second condition is a generalized Gelfand-Raikov-Shilov condition. Typical admissible weights are polynomial weights  $v(x) = (1 + |x|)^s$ ,  $s > 0$ , and subexponential weights  $v(x) = e^{\alpha|x|^\beta}$ ,  $\alpha > 0, 0 < \beta < 1$ . In [11, Theorem 3.1, Corollary 3.2] it is shown that  $\mathfrak{A}_v$  defines a unital Banach algebra which satisfies Properties I.1, 1–4, with  $H = \ell_2(\mathbb{Z}^d; \mathbb{C})$ . Let  $\mathfrak{A}_v^{n \times n}$  denote the matrix subalgebra of  $\mathcal{L}(H^n)$  with components in  $\mathfrak{A}_v$ .

**Example V.1.** Consider the Riccati equation (2) for the scalar system with  $a, b, c \in \mathfrak{A}_v$  when  $b = 1$ ,  $a^* = a$ . Then the nonnegative solution in  $\mathcal{L}(\ell_2(\mathbb{Z}^d; \mathbb{C}^n))$  is given by

$$x = a + \sqrt{a^2 + c^2}$$

Properties I.1 show that it is also in  $\mathfrak{A}_v$ . Applying Proposition III.3 we conclude that if  $a, b, c \in \mathfrak{A}_v$ ,  $a$  is self adjoint,  $(a, b)$  is algebraically controllable and  $(a, c)$  is exponentially detectable, then (2) has a unique exponentially stabilizing solution in  $\mathfrak{A}_v$ .

**Example V.2.** Suppose that the matrices  $A, B, C \in \mathfrak{A}_v^{2 \times 2}$ , where

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and  $C = \text{diag}(c_1, c_2)$ . If  $a_1^* = a_1, a_2^* = a_2, c_1^* = c_1, c_2^* = c_2$ , then, as in Example II.3 we can solve for explicit solutions to (2) to obtain

$$\begin{aligned} x_3 &= -a_1 + \sqrt{a_1^2 + c_1^2}, \\ x_2 &= -a_1 + \sqrt{a_2^2 + c_2^2 + 2x_3}, \\ x_1 &= x_3x_2 + a_1x_2 + x_3a_2. \end{aligned}$$

Applying Proposition III.3 we conclude that if  $A, B, C \in \mathfrak{A}_v^{2 \times 2}$ ,  $A, C$  have self-adjoint components,  $B$  has rank one and  $(A, B)$  is algebraically controllable and  $(A, C)$  is exponentially detectable, then (2) has a unique exponentially stabilizing solution in  $\mathfrak{A}_v^{2 \times 2}$ .

While these results are rather special, they do cover typical applications discussed in [12].

## VI. CONCLUSIONS

In this paper we have examined the question whether the LQR Riccati equation for matrices with components in a subalgebra  $\mathfrak{A}$  of  $\mathcal{L}(H)$ , where  $H$  is a Hilbert space will have a unique nonnegative exponentially stabilizing solution with components in  $\mathfrak{A}$ . We give counterexamples to results claimed in the literature. For  $2 \times 2$  matrices where the control input has rank one we obtain some positive results. In addition, we pose a conjecture and an algebraic problem.

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