Linear quadratic optimal control, dissipativity, and para-Hermitian matrix polynomials

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Abstract— **In this paper we will look at two results in which a special para-Hermitian matrix polynomial appears in linear quadratic systems theory. The first result constitutes the first step in a dissipativity check. The second result shows that dissipativity is equivalent to the solvability of the infinite-horizon linear quadratic optimal control problem and that its solutions are given by the behavior specified by the special para-Hermitian matrix polynomial. The results can be used to derive efficient eigenvalue methods for linear first-order state-space systems.**

I. INTRODUCTION

In this paper the results from [1] and [2] are explained as briefly as possible.

The following notation will be used. i denotes the imaginary unit. $\mathbb{C}[\lambda]$ denotes the set of polynomials and $\mathbb{C}(\lambda)$ denotes the set of rational functions. $\mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q)$ denotes the set of infinitely often differentiable functions mapping from $\mathbb{R}$ to $\mathbb{C}^q$ and $\mathcal{C}_c^\infty(\mathbb{R},\mathbb{C}^q)$ denotes the set of infinitely often differentiable functions with compact support. For a matrix $A \in \mathbb{C}^{n,m}$, by $\operatorname{rank}(A)$ we denote the rank of A in the usual sense and for a rational or polynomial matrix $R \in \mathbb{C}(\lambda)^{p,q}$, by $\operatorname{rank}_{\mathbb{C}(\lambda)}(R)$ we denote the rank of R over the field of rational functions. $\operatorname{rank}_{\mathbb{C}(\lambda)}(R)$ is also called generic rank of R. For a rational matrix $R \in \mathbb{C}[\lambda]^{p,q}$ we denote the set of all complex $\lambda \in \mathbb{C}$ such that $R(\lambda)$ is well defined (i.e., no element of R has a pole at λ) by $\mathfrak{D}(R)$ and call it the domain of definition of R. Further, we denote the set of all complex $\lambda_0 \in \mathfrak{D}(R)$ such that that

$$\operatorname{rank}(R(\lambda_0)) < \operatorname{rank}_{\mathbb{C}(\lambda)}(R),$$

by $\zeta(R)$ and call it the set of zeros of R.

The following definition will be used later.

Definition 1: Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial and let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ be its generic rank. Then the rational matrix $U \in \mathbb{C}(\lambda)^{q,q-r}$ and the polynomial matrix $V \in \mathbb{C}[\lambda]^{q,r}$ are called *kernel matrix and co-kernel matrix of* P, resp., if they fulfill the following properties

1) $PU = 0$,
2) $\operatorname{rank}_{\mathbb{C}(\lambda)}(PV) = r$,
3) $\operatorname{rank}_{\mathbb{C}(\lambda)}(U) = \operatorname{rank}(U(\lambda)) = q - r$ for all $\lambda \in \mathfrak{D}(U)$,
4) $\operatorname{rank}_{\mathbb{C}(\lambda)}(V) = \operatorname{rank}(V(\lambda)) = r$ for all $\lambda \in \mathbb{C}$,
5) $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular, i.e., its determinant is a non-zero constant.

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One can show that for every matrix polynomial P there exists a polynomial kernel matrix and a co-kernel matrix, see [1, Theorem 10]. However, the kernel matrix in Definition 1 is allowed to be a rational function, because for regular first-order state-space systems we will later present a kernel matrix in explicit form, see (11), which happens to be a rational matrix.

A. Linear systems

Let $P \in \mathbb{C}[\lambda]^{p,q}$ be an p-by-q matrix polynomial of degree $\pi \in \mathbb{N}$, i.e., let P take the form

$$P(\lambda) = P_\pi \lambda^\pi + \ldots + P_1\lambda + P_0$$

with $P_i \in \mathbb{C}^{p,q}$ for $i = 0, \ldots, \pi$. Consider the linear dynamical system given by the equations

$$P_\pi z^{(\pi)}(t) + \ldots + P_1 z^{(1)}(t) + P_0 z(t) = 0, \qquad (1)$$

for all $t \in \mathbb{R}$, where $z \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q)$ is an infinitely often differentiable function mapping from $\mathbb{R}$ to $\mathbb{C}^q$ and $z^{(i)} = \left(\frac{d}{dt}\right)^i z$ denotes the i-th derivative of z with respect to t. We will frequently use the notation $P\left(\frac{d}{dt}\right) z = 0$ as a shortcut for (1). The set of all solutions of (1) is called the *behavior of* P and denoted by

$$\mathfrak{B}(P) := \left\{ z \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q) \middle| P\left(\tfrac{d}{dt}\right) z = 0 \right\}.$$

Similar, we call the set of all solutions of (1) with compact support $z \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{C}^q)$ the *compact behavior of* P and denote it by

$$\mathfrak{B}_c(P) := \left\{ z \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{C}^q) \middle| P\left(\tfrac{d}{dt}\right) z = 0 \right\}.$$

For obvious reasons one often also writes

$$\begin{aligned} \mathfrak{B}(P) &= \operatorname{kernel}_{\mathcal{C}^\infty}\left(P\left(\tfrac{d}{dt}\right)\right), \\ \mathfrak{B}_c(P) &= \operatorname{kernel}_{\mathcal{C}_c^\infty}\left(P\left(\tfrac{d}{dt}\right)\right), \end{aligned}$$

and says that P is a kernel representation of the dynamical system (1). In contrast to this, a system can also be given in image representation. This is emphasized by the following Lemma.

Lemma 1: Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial with generic rank $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P (which we know always exists). Then

$$\begin{aligned} \mathfrak{B}_c(P) &= \operatorname{range}_{\mathcal{C}_c^\infty}\left(U\left(\tfrac{d}{dt}\right)\right) \\ &= \left\{ U\left(\tfrac{d}{dt}\right)\alpha \middle| \alpha \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{C}^{q-r}) \right\} \end{aligned}$$

Proof: See, e.g., [1, Lemma 18]. ∎

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Clearly, we have that $\mathfrak{B}_c(P) \subset \mathfrak{B}(P)$. Also we see that for every trajectory $z \in \mathfrak{B}(P)$ which has the form $z = U\left(\frac{d}{dt}\right)\alpha$ with $\alpha \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ and for every $T_1, T_2 \in \mathbb{R}$ with $T_1 < T_2$ there exists an $\tilde{\alpha} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ such that $\tilde{z} := U\left(\frac{d}{dt}\right)\tilde{\alpha}$ fulfills $z(t) = \tilde{z}(t)$ for all $t \leq T_1$ and $\tilde{z}(t) = 0$ for $t \geq T_2$. This observation in connection with Lemma 1 suggests that $\mathfrak{B}_c(P)$ can be thought of as the controllable subsystem of $\mathfrak{B}(P)$.

Definition 2: We say that $P \in \mathbb{C}[\lambda]^{p,q}$ is controllable if it has no zeros, i.e., if

$$\zeta(P) = \emptyset.$$

Definition 2 is motivated by [5, Theorem 5.2.5]

With these definitions we have encapsulated the notion of a (linear) system into a matrix polynomial P. Therefore, in the following we may also speak of P as a system.

B. Quadratic cost (or energy supply)

For linear systems a measure of cost (or energy supply) is often introduced by means of a quadratic cost functional, i.e., for a trajectory of the system $z \in \mathfrak{B}(P)$ there is a given $\ell \in \mathbb{N}$ and matrices $H_{i,j} = H_{j,i}^* \in \mathbb{C}^{q,q}$ for $i, j = 0, \ldots, \ell$ such that the cost caused by z (or the energy supplied to the system along the trajectory z) at the time point t is measured by

$$\begin{bmatrix} z(t) \\ z^{(1)}(t) \\ \vdots \\ z^{(\ell)}(t) \end{bmatrix}^* \underbrace{\begin{bmatrix} H_{0,0} & H_{0,1} & \cdots & H_{0,\ell} \\ H_{1,0} & H_{1,1} & \cdots & H_{1,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\ell,0} & H_{\ell,1} & \cdots & H_{\ell,\ell} \end{bmatrix}}_{=:\tilde{H}} \begin{bmatrix} z(t) \\ z^{(1)}(t) \\ \vdots \\ z^{(\ell)}(t) \end{bmatrix}. \quad (2)$$

With $\tilde{H} \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ defined through (2) we see that $\tilde{H} = \tilde{H}^*$ is Hermitian. By introducing the symbol $\Delta_\ell^q \in \mathbb{C}[\lambda]^{q(\ell+1),q}$ through

$$\Delta_\ell^q(\lambda) := \begin{bmatrix} I_q \\ \lambda I_q \\ \vdots \\ \lambda^\ell I_q \end{bmatrix},$$

and the symbol $\Delta_\ell z := \Delta_\ell^q\left(\frac{d}{dt}\right) z$ for $z \in \mathfrak{B}(P)$ we can rewrite (2) through

$$\left(\Delta_\ell z(t)\right)^* \tilde{H} \left(\Delta_\ell z(t)\right).$$

With these definitions we have encapsulated the notion of a (quadratic) cost (or energy supply) into the Hermitian matrix $\tilde{H}$. Therefore, in the following we may also speak of $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ as a cost (or energy supply).

C. Dissipativity

Definition 3: Let $R \in \mathbb{C}(\lambda)^{p,q}$ be a rational matrix. Then we call $R^\sim$ defined by

$$R^\sim(\lambda) := R^*(-\overline{\lambda}),$$

the *para-Hermitian* of R. Furthermore, if we have that $R^\sim = R$ we say that R itself is *para-Hermitian*.

With this definition, for a given $\tilde{H}$, we introduce the matrix polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through

$$H(\lambda) := \left(\Delta_\ell^q(\lambda)\right)^\sim \tilde{H} \left(\Delta_\ell^q(\lambda)\right). \quad (3)$$

One can easily check that H is a para-Hermitian.

Note, that often the matrix H is introduced as a two-variable polynomial, see [8]. We avoid this here to make the presentation simpler, although it is sometimes more convenient to work with two variable polynomials.

Definition 4: Let the system $P \in \mathbb{C}[\lambda]^{p,q}$ and the cost $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ be given. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$, let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P, and let $H \in \mathbb{C}[\lambda]^{q,q}$ be defined through (3). Then we call $\Pi \in \mathbb{C}(\lambda)^{q-r,q-r}$ defined through

$$\Pi(\lambda) := U^\sim(\lambda) H(\lambda) U(\lambda)$$

a *Popov function* of P with respect to $\tilde{H}$.

Popov functions are also para-Hermitian. The Popov function is not unique, although any two Popov functions Π_1 and Π_2 are related by $\Pi_1 = S^\sim \Pi_2 S$, where $S \in \mathbb{C}(\lambda)^{q-r,q-r}$ is a unimodular matrix, see [1, Lemma 12].

Definition 5: System $P \in \mathbb{C}[\lambda]^{p,q}$ is called *dissipative with respect to* $\tilde{H}$ if we have that

$$0 \leq \int_{-\infty}^{\infty} \left(\Delta_\ell z(t)\right)^* \tilde{H} \left(\Delta_\ell z(t)\right) dt, \quad (4)$$

for all trajectories with compact support $z \in \mathfrak{B}_c(P)$.
If $\tilde{H}$ is thought of as a cost term, dissipativity means the system can not generate a negative costs along trajectories with compact support. If $\tilde{H}$ is measuring the energy supplied to the system, dissipativity means that the system can not generate energy internally. Note, that if there was a trajectory $\tilde{z} \in \mathfrak{B}_c(P)$ with compact support such that the integral term in (4) would become negative one could concatenate this trajectory over and over again, say k times, to obtain new trajectories $\tilde{z}_k \in \mathfrak{B}_c(P)$ with compact support which generate an arbitrary amount of energy. This means that one would have a model of a perpetual motion machine, which might possibly not be a good thing.

There are several equivalent characterizations of dissipativity, which are summed up in the following Theorem.

Theorem 1: Consider the system $P \in \mathbb{C}[\lambda]^{p,q}$ together with the cost $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$, let $U \in \mathbb{C}(\lambda)q, q-r$ be a kernel matrix of P, and let $H \in \mathbb{C}[\lambda]^{q,q}$ be defined through (3). Then the following are equivalent:

1) P is dissipative with respect to $\tilde{H}$.
2) The Popov function is positive semi-definite along the imaginary axis, i.e., for all $\omega \in \mathbb{R}$ such that $i\omega \in \mathfrak{D}(U)$ we have

$$\begin{aligned} \Pi(i\omega) &= U^\sim(i\omega) H(\omega) U(i\omega) \\ &= U^*(i\omega) H(i\omega) U(i\omega) \geq 0 \end{aligned}$$

3) P admits a storage function, i.e., there exists $\ell_1 \in \mathbb{N}$ and a Hermitian matrix $\tilde{S} = \tilde{S}^* \in \mathbb{C}^{q(\ell_1+1),q(\ell_1+1)}$

such that

$$\frac{d}{dt}\left((\Delta_{\ell_1} z(t))^* \tilde{N} (\Delta_{\ell_1} z(t))\right) \leq (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t)),$$

for all $z \in \mathfrak{B}_c(P)$ and all $t \in \mathbb{R}$.

4) P admits a dissipation function, i.e., there exists $\ell_2 \in \mathbb{N}$ and a Hermitian matrix $\tilde{D} = \tilde{D}^* \in \mathbb{C}^{q(\ell_2+1),q(\ell_2+1)}$ such that

$$\int_{-\infty}^{\infty} (\Delta_{\ell_2} z(t))^* \tilde{D} (\Delta_{\ell_2} z(t)) = \int_{-\infty}^{\infty} (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))$$

for all $z \in \mathfrak{B}_c(P)$.

Proof: See [1, Theorem 42] and [1, Theorem 36] or [8]. ■

Also, it should be possible to show that in Theorem 1 we have $l = l_1 = l_2$. This will be discussed in a forthcoming paper.

II. CHECKING DISSIPATIVITY

In Theorem 1 we saw that dissipativity is equivalent to the positive semi-definiteness of a Popov function along the imaginary axis. This property gives a clue at how to check dissipativity.

Theorem 2: Let the system $P \in \mathbb{C}[\lambda]^{p,q}$ and the cost $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ be given. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$, let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P, and define the para-Hermitian matrix polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through (3). Let

$$\lambda_0 \in \left(\mathfrak{D}(U) \setminus -\overline{\zeta(P)}\right) \cap \left(\mathfrak{D}(U^\sim) \setminus \zeta(P)\right). \qquad (5)$$

Then λ_0 is a zero of the Popov function $\Pi := U^\sim H U$ if and only if λ_0 is a zero of

$$N := \begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix} \in \mathbb{C}[\lambda]^{p+q,p+q}.$$

Note, that N is again a para-Hermitian polynomial.

Proof: Let $V \in \mathbb{C}[\lambda]^{q,r}$ be a co-kernel matrix, see Definition 1. Then $\begin{bmatrix} U & V \end{bmatrix}$ is a unimodular matrix and thus we can perform the structure preserving similarity transformation

$$\begin{aligned} N &\sim \begin{bmatrix} I & \\ & \begin{bmatrix} U & V \end{bmatrix} \end{bmatrix}^\sim \begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix} \begin{bmatrix} I & \\ & \begin{bmatrix} U & V \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & PU & PV \\ U^\sim P^\sim & U^\sim H U & U^\sim H V \\ V^\sim P^\sim & V^\sim H U & V^\sim H V \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & U^\sim H V \\ V^\sim P^\sim & V^\sim H U & V^\sim H V \end{bmatrix}. \end{aligned} \qquad (6)$$

Since PV has full column rank by Definition, the blocks in (6) below the (1,3)-block and right to the (3,1)-block can also be eliminated and we see that the zeros of N are essentially given by the union of the zeros of PV, Π, and $V^\sim P^\sim$. For a complete proof, see [1, Theorem 44]. ■

If the kernel matrix U in Theorem 2 is polynomial we have $\mathfrak{D}(U) = \mathbb{C}$ and if P is controllable we have $\zeta(P) = \emptyset$. Thus every $\lambda_0 \in \mathbb{C}$ fulfills condition (5) and we obtain the following Corollary.

Corollary 1: Let the system $P \in \mathbb{C}[\lambda]^{p,q}$ be controllable and let $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$, let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a *polynomial* kernel matrix (which we know always exits), and define the para-Hermitian matrix polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through (3). Then λ_0 is a zero of the Popov function $\Pi := U^\sim H U$ (which in this case is polynomial) if and only if λ_0 is a zero of the para-Hermitian matrix polynomial

$$\begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix}.$$

Proof: A complete proof can be found in [1, Corollary 45]. ■

How can Theorem 2 (or Corollary 1) help to check dissipativity of systems given in behavior form? For this consider Fig. 1, which depicts three possible Popov functions Π_1, Π_2, and Π_3 along the imaginary axis.

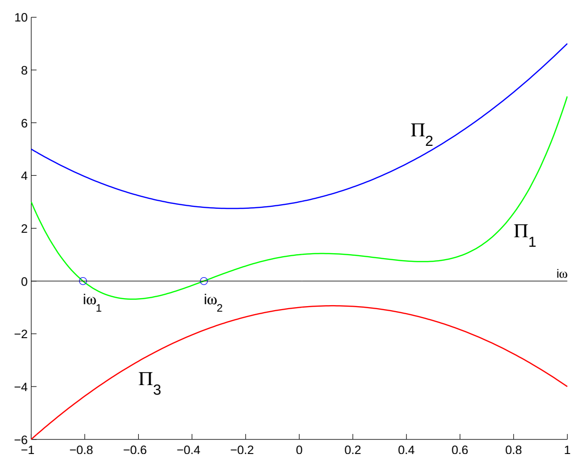

Fig. 1. Three possible Popov functions along the imaginary axis

Each Popov function is represented by a single line which means that we are considering systems which are represented by polynomials $P_j \in \mathbb{C}[\lambda]^{p,q}$ such that with $r_j := \operatorname{rank}_{\mathbb{C}(\lambda)}(P_j)$ we have $q - r_j = 1$, for $j = 1, 2, 3$ since in this case we have that the associated kernel matrices $U_j \in \mathbb{C}[\lambda]^{q,q-r_j} = \mathbb{C}[\lambda]^{q_j,1}$ are column vectors and thus the Popov functions

$$\Pi_j = U_j^\sim H U_j \in \mathbb{C}[\lambda],$$

for $j = 1, 2, 3$ are scalar polynomials, where $H = H^* = H^\sim \in \mathbb{C}^{q,q}$ is some fixed (para-)Hermitian matrix (polynomial). This corresponds to the single-input setting in state-space systems. Using Theorem 1 we see that P_j is dissipative with respect to H if and only if its Popov function Π_j is positive semi-definite along the imaginary axis. Examining Fig. 1 clearly shows that P_2 is dissipative, while P_1 and P_3 are not. However, P_1 seems to be somehow close to dissipative, while P_3 is not dissipative at all. Clearly, none of the polynomials Π_1, Π_2, and Π_3 is the zero polynomial from which we conclude that $\mathrm{rank}_{\mathbb{C}(\lambda)}(\Pi_i) = 1$ for $i = 1, 2, 3$. When, however, we look at the rank of the Popov functions at specific points on the imaginary axis we find that the rank of Π_2 and Π_3 does not drop anywhere (from 1 to 0) on the imaginary axis, while the rank of Π_1 drops at $i\omega_1$ and $i\omega_2$, i.e., Π_1 has the purely imaginary zeros $i\omega_1$ and $i\omega_2$, while Π_2 and Π_3 have no purely imaginary zeros. Using Theorem 2 (or Corollary 1) we see that to compute the values $i\omega_1$ and $i\omega_2$ we have to determine the purely imaginary zeros of

$$\begin{bmatrix} 0 & P_1 \\ P_1 & H \end{bmatrix},$$

while the connected para-Hermitian matrix polynomials for P_2 and P_3 have no purely imaginary zeros. Thus Theorem 2 (or Corollary 1) can help to distinguish Π_1 from Π_2 (but not Π_2 from Π_3) without explicitly computing the kernel matrices U_j and the associated Popov functions Π_j. This is especially handy since the imaginary zeros of para-Hermitian matrix polynomials are computable in a numerically stable way [3, 7]. This idea can also be generalized to systems where the kernel matrix has more than one column but then one cannot depict the Popov functions in such a convenient way.

III. LINEAR QUADRATIC OPTIMAL CONTROL

In this section we are going to consider the following optimal control problem. Let a system $P \in \mathbb{C}[\lambda]^{p,q}$ be given and let $\hat{z} \in \mathfrak{B}(P)$ be a given and fixed trajectory on the interval $(-\infty, 0]$. We now ask the question, how this trajectory can be continued after 0 in such a way that the cost given by a Hermitian matrix $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ is minimized. In other words we are looking for a solution of

$$\inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(t)=\hat{z}(t), t \le 0}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt, \tag{7}$$

where $\mathfrak{B}_+(P)$ denotes the *exponentially decaying behavior* of P. To be more specific, we define the set of *exponentially decaying functions* by

$$\mathcal{E}_q^+ := \left\{ z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q) \middle| \exists a_i, b_i > 0 \text{ such that } \|z^{(i)}(t)\|_2 \le a_i e^{-b_i t}, \text{ for all } i \in \mathbb{N}_0 \right\},$$

and set $\mathfrak{B}_+(P) := \mathfrak{B}(P) \cap \mathcal{E}_+$. Of course, we are not only interested in the value of (7) but also in a trajectory for which this infimum is assumed. The following two Theorems give the answer to this infinite-horizon optimal control problem.

Theorem 3: Let the system $P \in \mathbb{C}[\lambda]^{p,q}$ and the cost $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ be given. Define the para-Hermitian matrix polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through

$$H(\lambda) := (\Delta_\ell^q(\lambda))^\sim \tilde{H} (\Delta_\ell^q(\lambda)).$$

Let P be dissipative with respect to $\tilde{H}$. Let $\hat{z} \in \mathcal{E}_q^+$ and $\hat{\mu} \in \mathcal{E}_p^+$ be such that

$$\begin{bmatrix} 0 & P\left(\frac{d}{dt}\right) \\ P^\sim\left(\frac{d}{dt}\right) & H\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{z} \end{bmatrix} = 0.$$

Then, we have

$$\int_{t_0}^\infty (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt = \inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(t)=\hat{z}(t),\, t \le t_0}} \int_{t_0}^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt,$$

for all $t_0 \in \mathbb{R}$, i.e., $\hat{z}$ solves the optimal control problem.

Proof: Calculus of variation, see [2, Theorem 26]. ■

The converse direction is stated in the following theorem.

Theorem 4: Let the system $P \in \mathbb{C}[\lambda]^{p,q}$ and the cost $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ be given. Define the para-Hermitian polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through

$$H(\lambda) := (\Delta_\ell^q(\lambda))^\sim \tilde{H} (\Delta_\ell^q(\lambda)).$$

Let $\hat{z} \in \mathfrak{B}_+(P)$ be such that

$$\int_{t_0}^\infty (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt = \inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(t)=\hat{z}(t),\, t \le t_0}} \int_{t_0}^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt,$$

for all $t_0 \in \mathbb{R}$. Then, P is dissipative with respect to $\tilde{H}$ and there exists a co-state function $\hat{\mu} \in \mathcal{E}_p^+$ such that we have

$$\begin{bmatrix} 0 & P\left(\frac{d}{dt}\right) \\ P^\sim\left(\frac{d}{dt}\right) & H\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{z} \end{bmatrix} = 0. \tag{8}$$

Proof: Calculus of variation, see [2, Theorem 28]. ■

Theorem 3 and Theorem 4 together show that dissipativity is equivalent to the solvability of the optimal control problem (7) and also that the exponentially decaying behavior of N, i.e., $\mathfrak{B}_+(N)$ (with N as in Theorem 2) gives the solution of (7).

IV. SPECIALIZATION TO PASSIVITY OF STATE-SPACE SYSTEMS

In this section we will see how the above results apply to state-space descriptor systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{9}$$

where $E, A \in \mathbb{C}^{\rho,n}$, $B \in \mathbb{C}^{\rho,m}$, $C \in \mathbb{C}^{m,n}$, $D \in \mathbb{C}^{m,m}$ are matrices, $x \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^n)$ is called the state, $u \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^m)$ is called the input, and $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^m)$ is called the output. Note, that we are considering systems for which the input and output dimension is equal. Defining the polynomial $P \in \mathbb{C}[\lambda]^{\rho,n+m}$ through

$$P(\lambda) := \begin{bmatrix} \lambda E - A & -B \end{bmatrix} = \lambda \begin{bmatrix} E & 0 \end{bmatrix} + \begin{bmatrix} -A & -B \end{bmatrix}, \tag{10}$$

we immediately see that $z := \begin{bmatrix} x^T & u^T \end{bmatrix}^T$ is in the behavior of P if and only if x together with u solves the upper equation in (9).

A system of the form (9) is called *passive* if for all $(u,x,y) \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{C}^{m+n+m})$ that solve (9) we have

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} y^*(t)u(t) + u^*(t)y(t)dt \\ &= \int_{-\infty}^{\infty} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} dt \\ &= \int_{-\infty}^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} 0 & C^* \\ C & D+D^* \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \end{aligned}$$

i.e., if P is dissipative with respect to

$$H := \begin{bmatrix} 0 & C^* \\ C & D^*+D \end{bmatrix}.$$

If we further assume that $\rho = n$ and the matrix polynomial $\lambda E - A$ is invertible over $\mathbb{C}(\lambda)$ one can easily show that $U \in \mathbb{C}(\lambda)^{n+m,m}$ defined by

$$U(\lambda) := \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}, \tag{11}$$

is a kernel matrix of P as defined in (10), see [1, Lemma 46]. Thus, if $\lambda E - A$ is regular we see from Theorem 1 that the system (9) is passive if and only if the Popov function $\Pi = U^{\sim}HU$ is positive semi-definite along the imaginary axis, i.e., if and only if for all $\omega \in \mathbb{R}$ such that $i\omega E - A$ is invertible over $\mathbb{C}$ we have

$$\begin{aligned} &\Pi(i\omega) \\ = \; & U^{\sim}(i\omega)HU(i\omega) \\ = \; & U^*(-\overline{i\omega})HU(i\omega) \\ = \; & U(i\omega)^* \begin{bmatrix} 0 & C^* \\ C & D^*+D \end{bmatrix} \begin{bmatrix} (i\omega E - A)^{-1}B \\ I_m \end{bmatrix} \\ = \; & \begin{bmatrix} (i\omega E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} C^* \\ C(i\omega E - A)^{-1}B + D^* + D \end{bmatrix} \\ = \; & G^*(i\omega) + G(i\omega) \geq 0, \end{aligned} \tag{12}$$

where $G(\lambda) := C(\lambda E - A)^{-1}B + D$ is the so called *transfer function* of (9). The inequality (12) is one important property of positive realness. To check this property we can compute the zeros of the associated para-Hermitian matrix polynomial from Theorem 2. (which in this case may also be called eigenvalues, since we are considering a pencil). The above construction proves the following Corollary.

Corollary 2: Consider system (9) with $\rho = n$ and let $\lambda E - A$ be regular. Let $G(\lambda) := C(\lambda E - A)^{-1}B + D$ be the transfer function. Assume that $i\omega$ is not an zero/eigenvalue of $\lambda E - A$. Then the Popov function $\Pi := G^{\sim} + G$ has the zero $i\omega$ if and only if the para-Hermitian matrix pencil

$$\lambda \begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^* & 0 & C^* \\ B^* & C & D^*+D \end{bmatrix}$$

has the zero/eigenvalue $i\omega$.

Proof: For a complete proof, see [1, Corollary 47]. ■

The same construction can be applied if $\lambda E - A$ is not regular and even if $\rho \neq n$, i.e., if E and A are rectangular matrices, in which case, however, one can not give such a nice and explicit representation of the Popov function.

The results about the infinite horizon optimal control problem can be applied to system (9) in the same way.

Corollary 3: Let system (9) be passive, let $(\hat{u}, \hat{x}, \hat{\mu}) \in \mathcal{E}_\rho^+ \times \mathcal{E}_n^+ \times \mathcal{E}_m^+$ be such that

$$\begin{aligned} &\begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{\mu}}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{u}}(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & A & B \\ A^* & 0 & C^* \\ B^* & C & D+D^* \end{bmatrix} \begin{bmatrix} \hat{\mu}(t) \\ \hat{x}(t) \\ \hat{u}(t) \end{bmatrix}, \end{aligned}$$

and let $\hat{y} = C\hat{x} + D\hat{u}$. Then, we have

$$\begin{aligned} &\int_0^\infty \hat{y}^*(t)\hat{u}(t) + \hat{u}^*(t)\hat{y}(t)dt \\ &= \inf_{\substack{(u,x,y) \text{ solve } (9) \\ x(0)=\hat{x}(0)}} \int_0^\infty y^*(t)u(t) + u^*(t)y(t)dt, \end{aligned}$$

where the infimum is taken over all exponentially decaying trajectories.

Proof: In particular, one has to show that the infimum in Theorem 3 only depends on the initial condition $x(0)$ and not on the complete history $x(t)$ with $t \leq 0$. This is not a trivial task. For a complete proof, see [2, Corollary 35]. ■

Corollary 4: Let $(\hat{u}, \hat{x}, \hat{\mu}) \in \mathcal{E}_\rho^+ \times \mathcal{E}_n^+ \times \mathcal{E}_m^+$ be a solution of (9) that solves the optimal control problem, i.e., let

$$\begin{aligned} &\int_{t_0}^\infty \hat{y}^*(t)u(t) + u^*(t)y(t)dt \\ &= \inf_{\substack{(u,x,y) \text{ solves } (9) \\ x(t_0)=\hat{x}(t_0)}} \int_{t_0}^\infty y^*(t)u(t) + u^*(t)y(t)dt, \end{aligned}$$

for all $t_0 \in \mathbb{R}$, where the infimum is taken over all exponentially decaying trajectories.

Then, (9) is passive and there exists a co-state function $\hat{\mu} \in \mathcal{E}_\rho^+$ such that we have

$$\begin{aligned} &\begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{\mu}}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{u}}(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & A & B \\ A^* & 0 & C^* \\ B^* & C & D+D^* \end{bmatrix} \begin{bmatrix} \hat{\mu}(t) \\ \hat{x}(t) \\ \hat{u}(t) \end{bmatrix}, \end{aligned}$$

for all $t \in \mathbb{R}$.

Proof: As Corollary 3, see [2, Corollary 36]. ■

V. CONCLUSIONS AND FUTURE WORKS

We established a tight connection between problems of linear systems P with quadratic cost term $\tilde{H}$ and the para-Hermitian matrix polynomial

$$\begin{bmatrix} 0 & P \\ P^{\sim} & H \end{bmatrix},$$

where $\tilde{H}$ and H are related by

$$H(\lambda) := \left(\Delta_\ell^q(\lambda)\right)^{\sim} \tilde{H} \left(\Delta_\ell^q(\lambda)\right).$$

For first-order systems of the type (9) this approach leads to para-Hermitian matrix pencils. For para-Hermitian matrix pencils (sometimes also called even matrix pencils) there exists structure preserving eigenvalue methods [7]. The importance of para-Hermitian matrix pencils for linear quadratic systems theory has long been known [4]. In [6], problem (9) is considered with $E = I$, i.e., the standard control system, and then linked to the associated para-Hermitian matrix pencil. It should be possible to adapt this approach to descriptor systems with singular E. In this way one can generalize the algebraic Riccati equation to systems of the form (9) and also devise a method to compute its solutions, the so called available storage and the required supply, in a numerically efficient way.

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