Approximate Solution to Nonlinear Optimal Regulator Problem using Quantum and Stochastic Theories

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Abstract— **This paper proposes a new constraction scheme of nonlinear optimal regulator by using two methods. Nishimura and Yamashita has introduced the method of obtaining approximate Lyapunov functions for deterministic and stochastic systems, which is based on difference apporximation with directions and quantization of Markov processes. On the other hand, Itami has proposed the method replacing the problem of obtaining nonlinear optimal regulators by the issue of solving Schrödinger equations. We combine above two schemes for constructing our new method.**

I. INTRODUCTION

It is difficult to obtain optimal regulators for nonlinear dynamical systems. The controllers depend on value functions, which is calculated as solutions to Hamilton-Jacobi-Bellman equations. The equations are very hard to solve because they are nonlinear partial differential equations.

From half a century ago, the nonlinear regulator problems have been studied in many researchers. However, the problems are not fully resolved yet. Recently, Itami [3] has been proposed an apporixmate method for the problems. He approximates partial derivatives of the value functions in state variables by using the canonical quantization, which is the way that transforms classical dynamical systems to quantum dynamical systems. Itami's approximation implies that optimal regulators can be obtained from solutions of the Schrödinger equations, which are linear partial differential equations. The problem becomes easier to solve. However, we have to consider how we can solve the Schrödinger equations. In this paper, we apply Nishimura and Yamashita's method [6] to the Schrödinger equations in order to approximate nonlinear optimal regulators. The method of [6] is a construction scheme of approximate Lyapunov functions for nonlinear Ito-type stochastic systems without inputs. In this method, we deal with Zubov's equation [2] whose form is analogous to the Schrdöinger equations. Therefore, we try to obtain a new approximation scheme for nonlinear optimal regulators by combining Itami's method and Nishimura and Yamashita's method.

In the next section, we describe the optimal regulator problems for nonlinear input-affine systems and show simple overviews of Itami's method and Nishimura and Yamashita's scheme. In Section 3, we propose a new method for obtaining nonliner optimal regulator with numerical calculation. In Section 4, we illustate effectiveness of our method via a simple example. In Section 5, we conclude the paper.

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II. PRELIMINARY

A. Optimal Regulator Problems

In this paper, we consider an input-affine system of the form

$$\dot{x} = G(x)u + F(x),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{1}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ are smooth enough. We aim to obtain an optimal regulator that minimizes a function J of the form

$$J(x,u;t) \equiv \int_t^T L(x,u;t)dt = \int_t^T \left\{u^T Ru + L_0(x)\right\} dt, \tag{2}$$

where R is an $m \times m$ positive definite symmetric matrix and $L_0 : \mathbb{R}^n \to \mathbb{R}$ is a positive definite function.

To solve this problem, we often consider the following Hamilton-Jacobi-Bellman equation (HJB-eq):

$$\frac{\partial V}{\partial t} = H\left(x, \frac{\partial V}{\partial x}\right), \tag{3}$$

where Hamiltonian H is defined by

$$H\left(x, \frac{\partial V}{\partial x}\right) := \frac{\partial V}{\partial x} G(x)^T R^{-1} G(x) \left[\frac{\partial V}{\partial x}\right]^T + \frac{\partial V}{\partial x} F(x) - L_0(x). \tag{4}$$

It is well known that the solution of HJB-eq, $V(x,t)$, satisfies $V(x,t) = \inf_u J(x,u;t)$ if and only if

$$u^* = -\frac{1}{2} R^{-1} G(x)^T \left[\frac{\partial V}{\partial x}\right]^T \tag{5}$$

holds. We obtain the optimal regulator u^* if the HJB-eq is solved. However, (3) is hard to solve because it is a nonlinear partial differential equation.

B. Approximation of Optimal Regulator Problems using Canonical Quantization

Itami [3] has constructed the problem based on constrained analytic mechanics. Itami considers system (1) as constraint conditions of a mass system moving on space $M = \{w | w = (x,u,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n\}$, where λ is a vector of Lagrange multipliers. He derives a Dirac bracket of the system, which is a Poisson bracket for a constraint system. Then, he applies *the canonical quantization* to the Dirac bracket and obtains a Schrödinger equation (Sch-eq). We now explain this way

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briefly. At first, we consider a mass system Σ, which moves on M and has Lagrangian

$$L'(w,\dot{w}) := L(x,u) + \lambda^T(f(x,u) - \dot{x}). \tag{6}$$

Any trajectory of Σ has to be constrained to a subspace $M' := \{w | w \in M,\ d^k\phi_w(w,p_w)/dt^k = 0,\ k = 0,1,2,\ldots\}$, where p_w denotes a vector of the generalized momenta and

$$\phi_w := (\phi_x^T, \phi_u^T, \phi_\lambda^T)^T = (p_x + \lambda^T, p_u, p_\lambda)^T. \tag{7}$$

A candidate of the Hamiltonian of Σ is defined by

$$H'(w,p_w,\dot{w}) := -L'(w,\dot{w}) + p_w\dot{w} + \mu_w\phi_w(w,p_w), \tag{8}$$

where $\mu_w = (\mu_x, \mu_u, \mu_\lambda)$ is a vector of Lagrange multipliers. We obtain μ_x, μ_λ, and μ_u by calculating $d\phi_\lambda/dt = 0$, $d\phi_x/dt = 0$, $d^2\phi_u/dt^2 = 0$, respectively. Moreover, $\phi_H := d\phi_u/dt = 0$ yields the relation between u and λ:

$$u = -\frac{1}{2}R^{-1}G(x)^T\lambda. \tag{9}$$

Then, we obtain the Hamiltonian

$$H^*(w) := -L(x,u) - \lambda^T(G(x)u + F(x)) \tag{10}$$

and Dirac brackets

$$\{v_1,v_2\}_{\mathrm{D}} := \{v_1,v_2\}_{\mathrm{P}} - \{v_1,\phi\}_{\mathrm{P}}^T\{\phi,\phi\}_{\mathrm{P}}^{-1}\{\phi,v_2\}_{\mathrm{P}}, \tag{11}$$

where $v_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{r_1}$, $v_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{r_2}$, $r_1, r_2 = 1,2,\ldots$, $\phi = (\phi_w^T, \phi_H^T)^T$, and

$$\{v_1,v_2\}_{\mathrm{P}} := \frac{\partial v_1}{\partial w}\left(\frac{\partial v_2}{\partial p_w}\right)^T - \frac{\partial v_1}{\partial p_w}\left(\frac{\partial v_2}{\partial w}\right)^T. \tag{12}$$

Next, we apply the canonical quantization to Σ. In the case of constraint systems, the canonical quantization means that we change variables to linear operators and replace Dirac brackets by communicators for the linear operators, i.e.,

$$\mathbf{i}\hbar\{v_1,v_2\}_{\mathrm{P}} \Rightarrow [\hat{v}_1,\hat{v}_2] := \hat{v}_1\hat{v}_2 - \hat{v}_2\hat{v}_1, \tag{13}$$

where $\mathbf{i}$ denotes the imaginary unit, $\hbar \in \mathbb{R}$ is a design parameter, and $\hat{v}_1$ and $\hat{v}_2$ are linear operators corresponding to v_1 and v_2, respectively. The linear operators are obtained by calculating Dirac brackets $\{\phi_{z_1},\phi_{z_2}\}_{\mathrm{D}}$, $z_1, z_2 = x, u, \lambda, H$.

Then, we obtain the Schrödinger eqaution

$$\mathbf{i}\hbar\frac{\partial\psi}{\partial t} = -\hat{H}\psi, \tag{14}$$

where ψ is called a wave function and A linear operator $\hat{H}$ is called *a Hamiltonian operator*, which is obtained from (10) and the canonical quantization.

Moreover, Itami uses a transformation $\tilde{\hbar} = \mathbf{i}\hbar$, where $\tilde{\hbar}$ is considered as a real value. The Schrödinger equation is represented by the following equation:

$$\tilde{\hbar}\frac{\partial\tilde{\psi}}{\partial t} = -\tilde{\hat{H}}\tilde{\psi}. \tag{15}$$

According to Itami, we obtain an apprixmation of the optimal regulator u^* by

$$u^* \approx -\frac{\tilde{\hbar}}{2}R^{-1}G(x)^T\frac{\psi_R\frac{\partial\psi_R}{\partial x} + \psi_I\frac{\partial\psi_I}{\partial x}}{\psi_R^2 + \psi_I^2}, \tag{16}$$

where $\psi_R(x)$ is a real part and $\psi_I(x)$ is an imaginary part of $\Psi(x)$.

In the case of (1) and (2), $\tilde{\hat{H}}$ is calculated as

$$\begin{aligned}\tilde{\hat{H}} =& L_0(x) - \frac{\hbar^2}{4}\left\langle\left\langle G(x)R^{-1}G(x)^T, \frac{\partial}{\partial x}\right\rangle, \frac{\partial}{\partial x}\right\rangle \\ &- \left\langle\frac{\tilde{\hbar}^2}{4}\left\langle\frac{\partial}{\partial x}, G(x)R^{-1}G(x)^T\right\rangle + \tilde{\hbar}F(x), \frac{\partial}{\partial x}\right\rangle \\ &+ \frac{\tilde{\hbar}^2}{16}\left\langle\frac{\partial}{\partial x}, G(x)\right\rangle R^{-1}\left\langle\frac{\partial}{\partial x}, G(x)\right\rangle^T - \frac{\tilde{\hbar}}{2}\left\langle\frac{\partial}{\partial x}, F(x)\right\rangle.\end{aligned} \tag{17}$$

C. Approximate Method of Constructing Lyapunov Functions using Quantization of Markov Processes

Nishimura and Yamashita propose an approximate method of constructing Lyapunov functions [6]. In the scheme, we use the windup difference approximation [4] and the quantization of Markov processes [1]. This subsection shows the method briefly. We consider a stochastic system

$$dx = \xi(x)dt + \sigma(x)dw,\ x \in \mathbb{R}^n,\ w \in \mathbb{R}^d, \tag{18}$$

where w is the standard Wiener process and σ is a diffusion coefficient matrix satisfying $a(x) = \sigma(x)\sigma^T(x)$. If (18) is stable in probability, there exists a stochastic Lyapunov function W that satisfies

$$\frac{\partial W}{\partial t} + \mathscr{L}W(x) = -q(x)W(x), \quad q : \mathbb{R}^n \to \mathbb{R}_{>0}, \tag{19}$$

where $\mathscr{L}$ is an infinitesimal operator of (18), i.e.,

$$\mathscr{L} := \left\langle\xi(x), \frac{\partial}{\partial x}\right\rangle + \frac{1}{2}\left\langle\left\langle a(x), \frac{\partial}{\partial x}\right\rangle, \frac{\partial}{\partial x}\right\rangle. \tag{20}$$

Let

$$\xi_i^+(x) = \max(\xi_i(x), 0), \quad \xi_i^-(x) = \max(-\xi_i(x), 0), \tag{21}$$

$$a_{ij}^+(x) = \max(a_{ij}(x), 0), \quad a_{ij}^-(x) = \max(-a_{ij}(x), 0). \tag{22}$$

We define a discrete state space as

$$M_d := \left\{\delta\sum_{i=1}^n \gamma_i e_i,\ \forall\gamma_i \in \mathbb{Z}\right\} = \{x_d, x_d', x_d'', \ldots\}, \tag{23}$$

where δ is a spatial step and $e_1, e_2, \ldots, e_n$ are the orthogonal bases of the state vector. Let us consider the first-order difference quotients

$$\mathscr{D}_i^+v(x_d,t) := \frac{v(x_d + \delta_i e_i, t) - v(x_d,t)}{\delta_i}, \tag{24}$$

$$\mathscr{D}_i^-v(x_d,t) := \frac{v(x_d,t) - v(x_d - \delta_i e_i, t)}{\delta_i}, \tag{25}$$

and the second-order difference quotients

$$\mathscr{D}_{ii}v(x_d,t) := \frac{1}{\delta_i}(\mathscr{D}_i^+v(x_d,t) - \mathscr{D}_i^-v(x_d,t)), \tag{26}$$

$$\begin{aligned}\mathscr{D}_{ij}^+v(x_d,t) := \frac{1}{2\delta_j}\{&-\mathscr{D}_i^+v(x_d,t) + \mathscr{D}_i^-v(x_d,t) \\ &+ \mathscr{D}_i^+v(x_d + \delta_j e_j, t) - \mathscr{D}_i^-v(x_d - \delta_j e_j, t)\},\end{aligned} \tag{27}$$

$$\begin{aligned}\mathscr{D}_{ij}^-v(x_d,t) := \frac{1}{2\delta_j}\{&\mathscr{D}_i^+v(x_d,t) - \mathscr{D}_i^-v(x_d,t) \\ &- \mathscr{D}_i^+v(x_d - \delta_j e_j, t) + \mathscr{D}_i^-v(x_d + \delta_j e_j, t)\},\end{aligned} \tag{28}$$

where $v : M_d \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $i \neq j$. Let

$$\begin{aligned}\mathscr{L}_d(\cdot) := \sum_{i=1}^{n} \left\{ \xi_i^+(x_d)\mathscr{D}_i^+(\cdot) - \xi_i^-(x_d)\mathscr{D}_i^-(\cdot) \right\} \\ + \frac{1}{2}\sum_{i=1}^{n}\Big[a_{ii}(x_d)\mathscr{D}_{ii}(\cdot) \\ + \sum_{i\neq j}\left\{ a_{ij}^+(x_d)\mathscr{D}_{ij}^+(\cdot) - a_{ij}^-(x_d)\mathscr{D}_{ij}^-(\cdot)\right\}\Big]. \quad (29)\end{aligned}$$

Then, $\mathscr{L}_d$ is said to be a difference operator of (18). In this way, we approximate $\mathscr{L}$ by $\mathscr{L}_D$, i.e.,

$$\frac{\partial V}{\partial x_i}(x,t) \approx \begin{cases} \mathscr{D}_i^+ V_d(x_d,t) & \text{if} \quad f_i(x) \geq 0 \\ \mathscr{D}_i^- V_d(x_d,t) & \text{if} \quad f_i(x) < 0 \end{cases}, \quad (30)$$

$$\frac{\partial^2 V}{\partial x_i^2}(x,t) \approx \mathscr{D}_{ii} V_d(x,t), \quad (31)$$

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(x,t) \approx \begin{cases} \mathscr{D}_{ij}^+ V_d(x_d,t) & \text{if} \quad a_{ij}(x) \geq 0 \\ \mathscr{D}_{ij}^- V_d(x_d,t) & \text{if} \quad a_{ij}(x) < 0 \end{cases}, \quad (32)$$

where $i \neq j$.

Kushner [4] approximates $\partial V/\partial t$ by using the backward difference and defines the one-step transition probabilities. We keep the time continuous and define $p^\flat(x_d'|x_d)$ as the transition probability rates from x_d to x_d'. Thus,

$$p^\flat(x_d + \delta e_i|x_d) = \frac{1}{\delta} f_i^+(x_d) + \frac{1}{2\delta^2}\Big\{ a_{ii}(x_d) - \sum_{j\neq i}|a_{ij}(x_d)|\Big\}, \quad (33)$$

$$p^\flat(x_d - \delta e_i|x_d) = \frac{1}{\delta} f_i^-(x_d) + \frac{1}{2\delta^2}\Big\{ a_{ii}(x_d) - \sum_{j\neq i}|a_{ij}(x_d)|\Big\}, \quad (34)$$

$$p^\flat(x_d + \delta e_i + \delta e_j|x_d) = p^\flat(x_d - \delta e_i - \delta e_j|x_d) = \frac{a_{ij}^+(x_d)}{2\delta^2}, \quad i \neq j, \quad (35)$$

$$p^\flat(x_d - \delta e_i + \delta e_j|x_d) = p^\flat(x_d + \delta e_i - \delta e_j|x_d) = \frac{a_{ij}^-(x_d)}{2\delta^2}, \quad i \neq j, \quad (36)$$

$$p^\flat(x_d|x_d) = -\sum_{i=1}^{n}\Big\{ \frac{|f_i(x_d)|}{\delta} + \frac{a_{ii}(x_d)}{\delta^2} - \sum_{j\neq i}\frac{|a_{ij}(x_d)|}{2\delta^2}\Big\}, \quad (37)$$

$$p^\flat(x_d'|x_d) = 0, \quad (38)$$

where $x_d' \neq x_d, x_d + \delta e_i, x_d - \delta e_i, x_d + \delta e_i + \delta e_j, x_d + \delta e_i - \delta e_j, x_d - \delta e_i + \delta e_j, x_d - \delta e_i - \delta e_j$.

Then, (19) is approximated by *a master equation*:

$$\frac{\partial V_d}{\partial t}(x_d,t) = -L_d(x_d,t) - \sum_{x_d' \in M_d} p^\flat(x_d|x_d')V_d(x_d',t). \quad (39)$$

In order to combine windup difference approximations and Alcaraz et al.'s quantization of Markov processes [1], we use the bra-ket notation. Consider a complex separable Hilbert space $\mathbb{H}$ whose orthogonal bases are elements of M_d. The space $\mathbb{H}^*$ denotes a dual space of $\mathbb{H}$. A vector in $\mathbb{H}$ is written as $|\cdots\rangle$ and a vector in $\mathbb{H}^*$ is written as $\langle\cdots|$. The vectors $|\cdots\rangle$ and $\langle\cdots|$ are said to be a ket vector and a bra vector, respectively. The inner product of a bra vector and a ket vector is expressed as $\langle\cdots|\cdots\rangle$. We present specific examples of bra-ket notations as follows:

$$\langle\psi_1|\mathscr{O}|\psi_2\rangle = \sum_{x_d' \in M_d}\psi_1^*(x_d')(\mathscr{O}\psi_2)(x_d'), \quad (40)$$

$$\langle\psi_1|\psi_2\rangle = \sum_{x_d' \in M_d}\psi_1^*(x_d')\psi_2(x_d'), \quad (41)$$

$$\langle x_d|\psi_1\rangle = \psi_1(x_d), \quad (42)$$

$$\langle x_d|x_d'\rangle = \delta(x_d|x_d'), \quad (43)$$

where ψ_1^* is the complex conjugate of ψ_1.

For a function $V_d : M_d \times [0,\infty) \to \mathbb{R}$, we define

$$|\bar{V}(t)\rangle := \sum_{x_d \in M_d} V_d(x_d,t)|x_d\rangle. \quad (44)$$

Then,

$$V_d(x_d,t) = \langle x_d|\bar{V}(t)\rangle \quad (45)$$

is satisfied. An operator $\mathscr{H}$ is called a Hamiltonian operator if

$$\frac{\partial}{\partial t}|\bar{V}(t)\rangle = -\mathscr{H}|\bar{V}(t)\rangle. \quad (46)$$

Equation (46) is called a Schrödinger-like equation.

When we choose a Hamiltonian operator

$$\langle x_d|\mathscr{H} = \sum_{x_d' \in M_d}\{p^\flat(x_d|x_d')\langle x_d'| + p^\sharp(x_d'|x_d)\langle x_d|\}, \quad (47)$$

where

$$p^\sharp(x_d'|x_d) := p^\flat(x_d|x_d'), \quad x_d' \neq x_d \quad (48)$$

$$p^\sharp(x_d|x_d) := q(x_d) - \sum_{x_d' \neq x_d} p^\sharp(x_d'|x_d), \quad (49)$$

the master equation (39) is equivalent to the Schrödinger equaiton (46).

III. New Construction Scheme of Nonlinear Optimal Regulator Problem

In this study, we extend Nishimura and Yamashita's method to Itami's result. We assume that

$$a(x) = -\frac{\hbar}{2}G(x)R^{-1}G(x)^T \quad (50)$$

$$\xi(x) = -\left(\frac{\tilde{\hbar}}{4}\left\langle\frac{\partial}{\partial x}, G(x)R^{-1}G(x)^T\right\rangle + F(x)\right) \quad (51)$$

$$q(x) = \frac{\tilde{\hbar}}{16}\left\langle\frac{\partial}{\partial x}, G(x)\right\rangle R^{-1}\left\langle\frac{\partial}{\partial x}, G(x)\right\rangle^T - \frac{1}{2}\left\langle\frac{\partial}{\partial x}, F(x)\right\rangle + \frac{1}{\tilde{\hbar}}L_0(x) \quad (52)$$

hold. Then, we can apply Nishimura and Yamashita's method to Itami's result. In the original method, we choose positive definite functions as solutions to the problems because the purpose is constructing approximate Lyapunov functions. In contrast, in our way, we choose functions that have a different features with positive definite functions. Recall that

$$\frac{\partial V(x)}{\partial x} \approx \hbar\frac{\psi_R(x)\dfrac{\partial\psi_R(x)}{\partial x} + \psi_I(x)\dfrac{\partial\psi_I(x)}{\partial x}}{\psi_R(x)^2 + \psi_I(x)^2}. \quad (53)$$

This equation implies that the approximate value functions are represented as

$$V(x) \approx \frac{\hbar}{2} \log \left(\psi_R(x)^2 + \psi_I(x)^2 \right). \tag{54}$$

If $V(x)$ is the optimal regulator for (1), $V(x)$ is the Lyapunov function. Thereby we choose functions such that $\psi_R(x)^2 + \psi_I(x)^2 - 1$ becomes a positive definite function at least around the origin. We choose the wave function of the form

$$\psi(x) = \sum_{i=1}^{\infty} C_i \exp\left(-\frac{E_i}{\tilde{\hbar}} t \right) \phi_i(x), \tag{55}$$

where E_i, $\phi_i(x)$, $i = 1, 2, \ldots$ are the eigenvalues and the eigenfunctions of the Hamiltonian operator $\hat{\tilde{H}}$, respectively. If the Schrödinger equation is *time-independent*, i.e.,

$$\hat{\tilde{H}} |\psi(x)\rangle = -\frac{E_i}{\tilde{\hbar}} |\psi(x)\rangle, \tag{56}$$

the wave function is represented by

$$\psi(x) = \sum_{i=1}^{\infty} C_i \phi_i(x). \tag{57}$$

We choose this wave function because our aim is obtaining a time-invariant value function. That is, if there exists j such that $E_j = 0$ holds, we choose $C_i = \delta_{ij}$. Then, we obtain the time-independent Schrödinger equation (56). However, there may not exist j such that satisfying $E_j = 0$ when we design $\tilde{\hbar} > 0$. We should determine $\tilde{\hbar}$ near zero because the parameter causes numerical errors.

IV. Numerical Example

In this section, we calculate a simple example in order to show validity of our method. Let us consider

$$\dot{x}_1 = x_2, \tag{58}$$

$$\dot{x}_2 = -x_1 - x_2 + x_3 + u_1, \tag{59}$$

$$\dot{x}_3 = -x_3 + u_2, \tag{60}$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, $u = (u_1, u_2) \in \mathbb{R}^2$. For linear systems, (50), (51), and (52) are

$$\xi(x) = -Ax, \quad a(x) = -\frac{\tilde{\hbar}}{2} bR^{-1}b^T, \tag{61}$$

$$q(x) = -\frac{1}{2}\mathrm{tr}[A] + \frac{1}{\tilde{\hbar}} L_0(x), \tag{62}$$

where matrix A and matrix b are defined by $\dot{x} = Ax + bu$. In the case of (58)-(60),

$$\xi = \begin{bmatrix} -x_2 \\ x_1 + x_2 - x_3 \\ x_3 \end{bmatrix}, \; a = -\frac{\tilde{\hbar}}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & r_{11}^{-1} & r_{12}^{-1} \\ 0 & r_{21}^{-1} & r_{22}^{-1} \end{bmatrix}$$

$$q = 1 + \frac{L_0(x)}{\tilde{\hbar}}, \; R^{-1} := \begin{bmatrix} r_{11}^{-1} & r_{12}^{-1} \\ r_{21}^{-1} & r_{22}^{-1} \end{bmatrix}. \tag{63}$$

We compute a numerical approximation of the value function to the problem. The parameters are as follows: $\tilde{\hbar} = 0.3$, $R = ((1,0),(0,1))$, $L_0(x) = x_1^2 + x_2^2 + x_3^2$, and the computational region is $x_1 = [-1,1]$, $x_2 = [-1,1]$, $x_3 = [-1,1]$. Fig. 1 shows the approximate value function $\tilde{V}$ and Fig. 2 shows the time derivative of approximate value function $\dot{\tilde{V}}$, which is calculated by using multi-linear approximation. In the region Ω, $\tilde{V}$ is positive definite and $\dot{\tilde{V}}$ is negative definite. Therefore, system (58)-(60) is at least stabilized by the approximate optimal regulator. Fig. 3 describes a numerical solution of Riccati equation, i.e.,

$$V_{riccati} = x^T P x, \; P = \begin{bmatrix} 1.3673 & 0.4129 & 0.0605 \\ 0.4129 & 0.6658 & 0.2259 \\ 0.0605 & 0.2259 & 0.5494 \end{bmatrix}. \tag{64}$$

These figures show that our approximate solution has error, in particular, near the origin. However, we can obtain a regulator for the system because the approximate solution is a local Lyapunov function. The fact implies that our scheme can be used for regulator problems for various nonlinear controlled systems.

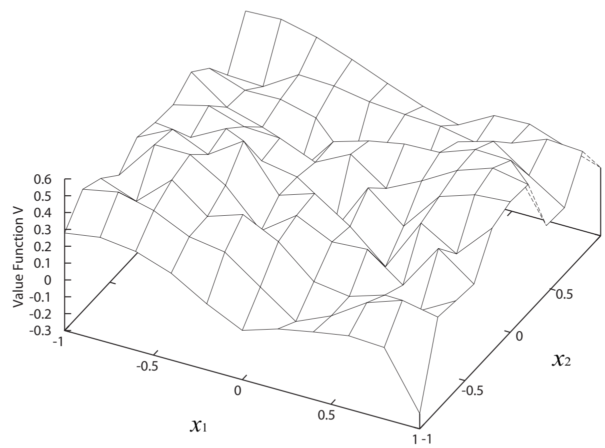

Fig. 1. Approximate Value Fuction $V(x)$

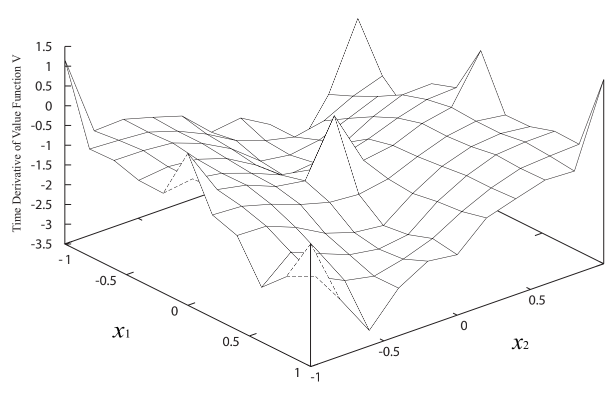

Fig. 2. Time Derivative of Approximate Value Fuction $V(x)$

V. Conclusions

In this paper, we have introduced a new method of obtaining approximate optimal regulator for nonlinear input-affine systems. We combine two schemes: Itami's approximation via canonical quantizations and Nishimura and Yamashita's

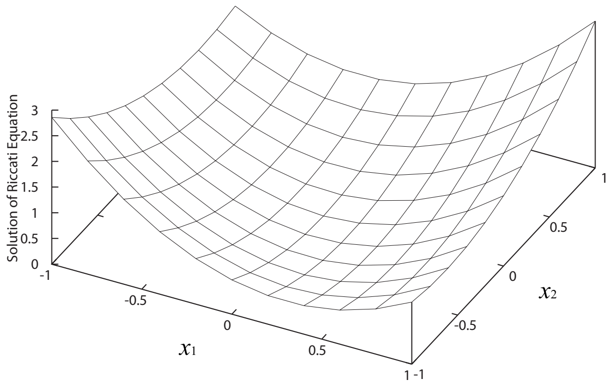

Fig. 3. Solution to Riccati Equation

construction scheme of approximate Lyapunov functions for stochastic systems. We show the effectiveness of our result by using a simple model.

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