On periodic solutions of linear difference equations

Eva Zerz

***Abstract*— We study systems of linear difference equations with constant coefficients in a commutative quasi-Frobenius ring F, that is, F is Noetherian and self-injective. For instance, F could be a field or a residue class ring of the integers. Given a positive integer p, we first answer the following basic questions: Does there exist a p-periodic solution? When are all solutions p-periodic? Then we address the more interesting question of how to determine candidates for the period p. We characterize strong autonomy (i.e., finitely many initial data) and weak autonomy (i.e., no free variables), which are non-equivalent concepts, in general. If F is finite, all trajectories of a strongly autonomous system eventually become periodic, and we characterize the case where they are purely periodic (i.e., no pre-period), as well as the minimal period in this case. These methods can be applied to periodically time-varying systems as studied by Kuijper/Willems and Aleixo/Polderman/Rocha, and the question whether a p-periodic system admits (only) p-periodic solutions can be tackled using the known lifting technique to rewrite a periodic system as an equivalent shift-invariant system.**

I. MOTIVATING EXAMPLE

The Fibonacci equation

$$w(t+2) = w(t+1) + w(t) \quad \text{for all } t \in \mathbb{N} \qquad (1)$$

is usually considered over the field $\mathbb{R}$, where one has, with $w(0) = 0$ and $w(1) = 1$, the famous solution

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Considering (1) over a finite field, for instance over $\mathbb{F}_2$, we get (with $w(0) = a$ and $w(1) = b$) the solutions

$$a, b, a+b, a, b, a+b, \ldots$$

which are periodic with period 3. Analogously, over $\mathbb{F}_3$, all solutions are periodic with period 8. Over $\mathbb{F}_5$, the period equals 20 etc.

How to determine these periods systematically? Let $\chi := s^2 - s - 1 \in F[s]$, where F is a finite field. Since the constant term of χ is non-zero, we have $\langle \chi, s \rangle = F[s]$ and we may conclude that $\bar{s}$ is a unit in the ring $F[s]/\langle \chi \rangle$. By elementary group theory, $\bar{s}$ has finite order, that is, $\bar{s}^p = \bar{1}$ holds in the unit group of $F[s]/\langle \chi \rangle$ for some positive $p \in \mathbb{N}$ that is minimal with this property. This means that $s^p - 1 \in \langle \chi \rangle$, that is, the Fibonacci equation implies p-periodicity. By Lagrange's theorem, one also knows that p must be a divisor of the order of the unit group of $F[s]/\langle \chi \rangle$. If χ is irreducible in $F[s]$, then $F[s]/\langle \chi \rangle$ is a field with $|F|^2$ elements and we conclude that $p \mid (|F|^2 - 1)$. This holds for $F = \mathbb{F}_2$ and $F = \mathbb{F}_3$ from above. For $F = \mathbb{F}_5$ however, χ factors as $\chi = (s+2)^2$ and hence $\mathbb{F}_5[s]/\langle \chi \rangle$ has only 20 units (all elements except $\overline{c(s+2)}$, $c \in \mathbb{F}_5$).

To decide whether χ is irreducible over $F = \mathbb{F}_q$, where $q \geq 7$ is a prime number, we observe that

$$\begin{array}{ll} \lambda^2 - \lambda - 1 = 0 & \Leftrightarrow \\ 4\lambda^2 - 4\lambda - 4 = 0 & \Leftrightarrow \\ (2\lambda - 1)^2 = 5. & \end{array}$$

Thus χ has two distinct zeros $\lambda_i \in F$ if and only if 5 is a square modulo q. By the law of quadratic reciprocity, this is true if and only if $q \equiv \pm 1 \pmod 5$. In that case, the solutions to (1) take the form $w(t) = c_1 \lambda_1^t + c_2 \lambda_2^t$ for some $c_i \in F$. Since $\lambda_i \neq 0$, we have $\lambda_i^{q-1} = 1$ by Fermat's little theorem. We conclude that all solutions to (1) are periodic with a period p which divides $q - 1$. If $q \equiv \pm 2 \pmod 5$, then χ is irreducible and we only have $p \mid (q^2 - 1) = (q-1)(q+1)$.

The following table shows the so-called *Pisano period* p for the first ten values of prime numbers $q = |F|$. The cases where χ splits are typed in bold.

q	2	3	**5**	7	**11**	13	17	**19**	23	**29**
p	3	8	**20**	16	**10**	28	36	**18**	48	**14**

In this paper, we study several problems as outlined above, in a more general setting. Firstly, we will treat systems of equations rather than scalar equations, and secondly, we will admit certain coefficient rings which are not necessarily fields. Therefore, we start with some preliminaries about these rings.

II. QUASI-FROBENIUS RINGS

The material of this section can be found in algebra textbooks such as [4]. We just restate it, without proofs, for the sake of completeness.

Let F be a commutative ring (with 1). One says that F is *self-injective* if it is injective as a module over itself, that is, if $\operatorname{Hom}_F(\cdot, F)$ is an exact functor. By Baer's criterion, this is equivalent to saying that any homomorphism $\phi : I \to F$, where I is an ideal of F, can be extended to all of F, that is, there exists a homomorphism $\psi : F \to F$ such that $\psi|_I = \phi$.

For example, if F is a field, then F has only the trivial ideals 0 and F, and thus, F is self-injective. The exact functor $\operatorname{Hom}_F(\cdot, F)$ is well-known in linear algebra: It takes an F-vector space V to its dual vector space V^*, and an F-linear map $f : V \to W$ to its dual map $f^* : W^* \to V^*$. By contrast, the ring $\mathbb{Z}$ is not self-injective: For instance, the homomorphism $\phi : 2\mathbb{Z} \to \mathbb{Z}$ which maps an even integer $2k$ to k cannot be extended to all of $\mathbb{Z}$.

E. Zerz is with Lehrstuhl D für Mathematik, RWTH Aachen University, 52062 Aachen, Germany eva.zerz@math.rwth-aachen.de

ISBN 978-963-311-370-7

To understand the structure of self-injective rings better, we consider the set

$$S := \{c \in F \mid \operatorname{ann}(c) \text{ is essential in } F\},$$

where an ideal of F is called *essential* in F if it has a non-zero intersection with any non-zero ideal of F. One observes that S is an ideal of F, called the *singular ideal* of F.

Recall that the *nilradical* N of F is the set of all nilpotent elements of F. It equals the intersection of all prime ideals of F. A ring is called *reduced* if its nilradical is zero. The *Jacobson radical* J of F is defined to be the intersection of all maximal ideals of F. Clearly, $N \subseteq J$.

Theorem 1: We always have $N \subseteq S$. Moreover, $N = 0$ is equivalent to $S = 0$. If F is Noetherian, then $N = S$. If F is self-injective, then $J = S$.

A commutative ring C is called *von Neumann regular* if for all $c \in C$, we have $\langle c \rangle = \langle c^2 \rangle$, that is, there exists $d \in C$ such that $c = c^2 d$. Then d is a kind of generalized inverse of c. Clearly, any field is von Neumann regular. Also $\mathbb{Z}/6\mathbb{Z}$ is von Neumann regular, since $2 = 2^2 \cdot 2$, $3 = 3^2$, $4 = 4^2$. However, $\mathbb{Z}/4\mathbb{Z}$ is not von Neumann regular, since $2 = 2^2 d$ has no solution. The following theorem implies that $\mathbb{Z}/n\mathbb{Z}$ is von Neumann regular if and only if n is square-free.

Theorem 2: The following are equivalent:

1) C is von Neumann regular.
2) C is reduced and has Krull dimension zero.
3) For every maximal ideal $\mathfrak{m}$ of C, the localization $C_\mathfrak{m}$ is a field.

Theorem 3: Let F be self-injective. Then F/S is von Neumann regular.

Thus, every reduced ring of Krull dimension at least one is not self-injective. For instance, this holds for $\mathbb{Z}$.

Theorem 4: The following are equivalent:

1) F is Noetherian and self-injective.
2) F is Noetherian and satisfies $\operatorname{ann}(\operatorname{ann}(I)) = I$ for all ideals I of F.
3) F is Artinian and satisfies $\operatorname{ann}(\operatorname{ann}(I)) = I$ for all ideals I of F.

If the equivalent conditions are satisfied, then F is called a *quasi-Frobenius* ring.

Any field is a quasi-Frobenius ring. If D is a principal ideal domain and $0 \neq d \in D$, then $F := D/\langle d \rangle$ is a quasi-Frobenius ring. For instance, this holds for $\mathbb{Z}/n\mathbb{Z}$ and $k[s]/\langle s^n - 1 \rangle$, where k is a field and $n \neq 0$.

Theorem 5 (Structure theorem for Artinian rings): Any Artinian ring is isomorphic to a finite direct sum of local Artinian rings.

For example, we have $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{e_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p_l^{e_l}\mathbb{Z}$, where $n = p_1^{e_1} \cdots p_l^{e_l}$ is the prime factorization of $n \geq 2$.

Local Artinian rings have the following nice properties: They are Noetherian and have Krull dimension zero, hence there exists only one prime ideal, namely the maximal ideal $\mathfrak{m}$. Thus $N = S = J = \mathfrak{m}$, and every element is either a unit or nilpotent. Moreover, $\mathfrak{m}$ is nilpotent. Every proper ideal is $\mathfrak{m}$-primary (that is, it is primary and its radical equals $\mathfrak{m}$). However, unlike the example $\mathbb{Z}/p^e\mathbb{Z}$, a proper ideal is not necessarily a power of the maximal ideal.

A commutative ring F is called a *cogenerator ring* if it is a cogenerator as a module over itself, that is, if $\operatorname{Hom}_F(\cdot, F)$ is a faithful functor. This means that for any non-zero homomorphism $f : N \to M$, where M, N are F-modules, the map

$$\operatorname{Hom}_F(M, F) \to \operatorname{Hom}_F(N, F), \quad \varphi \mapsto \varphi \circ f$$

is also non-zero, that is, there exists a homomorphism $\varphi : M \to F$ such that $\varphi \circ f \neq 0$. Equivalently,

$$\bigcap_{\varphi \in \operatorname{Hom}_F(M,F)} \ker(\varphi) = 0$$

for all F-modules M. In other words, for any F-module M and any $0 \neq m \in M$, there exists $\varphi \in M^* := \operatorname{Hom}_F(M, F)$, that is, a homomorphism $\varphi : M \to F$, such that $\varphi(m) \neq 0$. In that case, i.e., if the canonical homomorphism $M \to M^{**}$, $m \mapsto (\varphi \mapsto \varphi(m))$ is injective, one calls M *torsionless.* Thus F is a cogenerator ring if and only if any F-module is torsionless.

Theorem 6: Let F be Noetherian. Then F is self-injective if and only if it is a cogenerator ring.

Summing up: If F is quasi-Frobenius, then $\operatorname{Hom}_F(\cdot, F)$ is exact and faithful, that is, it preserves and reflects exactness. According to [5], [9], [12], this implies that $\operatorname{Hom}_\mathcal{D}(\cdot, \mathcal{A})$ is also exact and faithful, where $\mathcal{D} := F[s]$, $\mathcal{A} := F^\mathbb{N}$ and the module action of $s \in \mathcal{D}$ on $a \in \mathcal{A}$ is given by $(s \cdot a)(t) := a(t+1)$. In other words, $\mathcal{A}$ as a $\mathcal{D}$-module is an injective cogenerator [8]. This is due to the fact that

$$\operatorname{Hom}_F(\mathcal{D}, F) \cong \mathcal{A}, \quad \varphi \mapsto (\varphi(s^t))_{t \in \mathbb{N}}$$

and hence

$$\operatorname{Hom}_F(M, F) \cong \operatorname{Hom}_\mathcal{D}(M, \mathcal{A})$$

for every $\mathcal{D}$-module M. (The $\mathcal{D}$-module structure on the left hand side is given by $d\varphi := \varphi(d\cdot)$ for $d \in \mathcal{D}$.)

III. PERIODIC SOLUTIONS

Let F be a non-zero commutative ring, $\mathcal{D} = F[s]$ and $\mathcal{A} = F^\mathbb{N}$. Let $R \in \mathcal{D}^{g \times q}$ be a polynomial matrix. Then $R(\sigma)w = 0$ is a linear system of difference equations with coefficients in F, where σ denotes the shift operator defined by $(\sigma w)(t) := w(t+1)$ for $t \in \mathbb{N}$. The solution set

$$\mathcal{B} := \{w \in \mathcal{A}^q \mid R(\sigma)w = 0\}$$

is called a *behavior*. Let $M := \mathcal{D}^{1 \times q}/\mathcal{D}^{1 \times g}R$. We have [8]

$$\mathcal{B} \cong \operatorname{Hom}_F(M, F) \cong \operatorname{Hom}_\mathcal{D}(M, \mathcal{A}).$$

From now on, let F be a quasi-Frobenius ring. Then $\mathcal{A}$ is an injective cogenerator over $\mathcal{D}$. Therefore we have

$$^\perp\mathcal{B} := \{r \in \mathcal{D}^{1 \times q} \mid r(\sigma)w = 0 \, \forall w \in \mathcal{B}\} = \mathcal{D}^{1 \times g}R.$$

Given two matrices $R_i \in \mathcal{D}^{g_i \times q}$ and the corresponding behaviors $\mathcal{B}_i = \{w \in \mathcal{A}^q \mid R_i(\sigma)w = 0\}$, this implies that $\mathcal{B}_1 \subseteq \mathcal{B}_2$ holds if and only if $\mathcal{D}^{1 \times g_1}R_1 \supseteq \mathcal{D}^{1 \times g_2}R_2$, that is, if $R_2 = ZR_1$ holds for some $Z \in \mathcal{D}^{g_2 \times g_1}$.

In particular, we have $\mathcal{B} = \{w \in \mathcal{A}^q \mid R(\sigma)w = 0\} = \{0\}$ if and only if $\mathcal{D}^{1 \times g}R = \mathcal{D}^{1 \times q}$, that is, if R is left invertible.

Theorem 7: Let p be a positive integer.

1) The system $R(\sigma)w = 0$ does not admit any p-periodic solutions besides zero if and only if the matrix
$$\hat{R} := \begin{bmatrix} R \\ (s^p - 1)I_q \end{bmatrix} \in \mathcal{D}^{(g+q)\times q}$$
is left invertible, that is, if $X\hat{R} = I_q$ holds for some $X \in \mathcal{D}^{q\times(g+q)}$.
2) All solutions of the system $R(\sigma)w = 0$ are p-periodic if and only if
$$(s^p - 1)I_q = YR$$
holds for some $Y \in \mathcal{D}^{q\times g}$. In algebraic terms, this condition is also equivalent to
$$s^p - 1 \in \operatorname{ann}(M), \text{ where } M = \mathcal{D}^{1\times q}/\mathcal{D}^{1\times g}R.$$

Proof: These are direct consequences of the facts explained above. □

If all elements of a system $\mathcal{B}$ are p-periodic, then in particular, every trajectory w is uniquely determined by $w(0), \ldots, w(p-1)$, that is, by a finite number of initial values. This is a strong form of autonomy which will be called "past-determinedness" below. The last statement of the theorem above shows that
$$\operatorname{ann}(M) = \{d \in \mathcal{D} \mid \exists Z \in \mathcal{D}^{q\times g} : dI_q = ZR\}$$
plays an important role for autonomy properties [10], [11].

We have $\operatorname{ann}(M)^q \subseteq J_q(R) \subseteq \operatorname{ann}(M)$, where $J_q(R)$ is the ideal generated by the $q \times q$ subdeterminants of R. We say that R has *full column rank* if $J_q(R) \neq 0$, and that R has *reduced full column rank* if $\operatorname{ann}(J_q(R)) = 0$. Since F and hence $\mathcal{D} = F[s]$ are Noetherian, we have $\operatorname{ann}(I) = 0$ if and only if I contains a non-zero-divisor, for any ideal I in $\mathcal{D}$.

Lemma 1 ([10]): We have the following chain of implications: $\operatorname{ann}(M)$ contains a non-zero-divisor $\Leftrightarrow$
$J_q(R)$ contains a non-zero-divisor $\Leftrightarrow$
R has reduced full column rank, that is, $\operatorname{ann}(J_q(R)) = 0 \Rightarrow$
R has full column rank, that is, $J_q(R) \neq 0 \Rightarrow$
$\operatorname{ann}(M) \neq 0$.

Next, we investigate the non-zero-divisors of $\mathcal{D} = F[s]$. Recall that for a commutative ring F, a polynomial $d = \sum_{i=0}^n d_i s^i \in F[s]$ with coefficients $d_i \in F$ is

1) a unit if and only if d_0 is a unit and all d_i for $i \neq 0$ are nilpotent;
2) nilpotent if and only if all d_i are nilpotent;
3) a zero-divisor if and only if there exists a constant $0 \neq c \in F$ such that $cd = 0$.

Since F is a quasi-Frobenius ring, it is Artinian, and hence it is isomorphic to a finite direct sum of local Artinian rings F_i. Over a local Artinian ring F_i, any non-zero-divisor in $F_i[s]$ is associated to a monic polynomial. For the sake of completeness, we restate and prove this fact.

Theorem 8 ([6, Theorem XIII.6]): Let F be a local Artinian ring and $\mathcal{D} = F[s]$. Let $d \in \mathcal{D}$ be a non-zero-divisor. Then d is associated to a monic polynomial.

Proof: Let $\mathfrak{m}$ be the maximal ideal of F. As remarked above, we have $\mathfrak{m}^m = 0$ for some positive integer m. Let $d = \sum_{i=0}^n d_i s^i$ with $d_i \in F$ be given. By assumption, d is a non-zero-divisor, which means that not all coefficients d_i can be contained in $\mathfrak{m}$. Thus there exists $0 \leq t \leq n$ such that $d_t \in F$ is a unit. Without loss of generality, let t be maximal with this property and let $d_t = 1$. We show that there exist, for $1 \leq j \leq m$, monic polynomials f_j such that
$$f_j \equiv f_{j+1} \pmod{\mathfrak{m}^j}$$
and polynomials $g_j \in \mathfrak{m}[s]$ such that
$$d \equiv (1 + g_j)f_j \pmod{\mathfrak{m}^j}.$$
Then the claim follows from $\mathfrak{m}^m = 0$, which implies that $d = (1 + g_m)f_m$, where f_m is monic and $1 + g_m$ is a unit in $\mathcal{D}$, since g_m is nilpotent because all its coefficients are.

For $j = 1$, let $g_1 := 0$ and $f_1 := \sum_{i=0}^t d_i s^i$. Since $d_t = 1$ and $d_i \in \mathfrak{m}$ for all $i > t$, we have $d \equiv f_1$ modulo $\mathfrak{m}$, and f_1 is monic. Assume that $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ have already been constructed as required. Then $d = (1 + g_k)f_k + h$ for some $h \in \mathfrak{m}^k[s]$. Since f_k is monic, we can use division with remainder to obtain a representation $h = qf_k + r$, where $r = 0$ or $\deg(r) < \deg(f_k)$. Set $f_{k+1} := f_k + r$ and $g_{k+1} := g_k + q$.

Then f_{k+1} is monic, and analyzing the division with remainder process in detail, we check that $q \in \mathfrak{m}^k[s]$ and hence $r = h - qf_k \in \mathfrak{m}^k[s]$. Thus, $f_{k+1} \equiv f_k$ modulo $\mathfrak{m}^k$. Moreover, $g_{k+1} = g_k + q \in \mathfrak{m}[s]$. Finally, $d = (1 + g_k)f_k + h = f_k + g_k f_k + qf_k + r = f_{k+1} + g_{k+1}f_k = f_{k+1} + g_{k+1}(f_{k+1} - r) \equiv f_{k+1} + g_{k+1}f_{k+1}$ modulo $\mathfrak{m}^{k+1}$, since $g_{k+1}r \in \mathfrak{m}^{k+1}[s]$. □

As a consequence, any non-zero-divisor in $F[s]$, where F is an arbitrary Artinian ring, has a monic multiple.

Theorem 9: Let F be an Artinian ring and $\mathcal{D} = F[s]$. Let $d \in \mathcal{D}$ be a non-zero-divisor. Then there exists a monic multiple of d.

Proof: Let $F \cong \oplus_{i=1}^n F_i$ be the decomposition of F into a finite direct sum of local Artinian rings. It induces an isomorphism $\Phi : F[s] \cong \oplus_{i=1}^n F_i[s]$. Then $\Phi_i(d) \in F_i[s]$ is a non-zero-divisor as well. By the previous theorem, $\Phi_i(d)$ is associated to a monic polynomial, say, $\Phi_i(d)u_i = p_i \in F_i[s]$ with p_i monic. Let N denote the maximal degree of the p_i. Then the n polynomials $q_i := p_i s^{N-\deg(p_i)}$ are all monic of degree N. Set $q := (q_1, \ldots, q_n) \in \oplus_{i=1}^n F_i[s]$. Then $\Phi^{-1}(q)$ is monic of degree N, and since q is a multiple of $\Phi(d)$, it is a multiple of d. □

We conclude that $R \in \mathcal{D}^{g\times q}$ has reduced full column rank if and only if $\operatorname{ann}(M)$, where $M = \mathcal{D}^{1\times q}/\mathcal{D}^{1\times g}R$, contains a monic polynomial. This is the key to the following characterization of "past-determinedness" of a system [11].

Theorem 10: The following are equivalent:

1) R has reduced full column rank, that is, $\operatorname{ann}(M)$ contains a monic polynomial.
2) There exists $t_0 \in \mathbb{N}$ such that each $w \in \mathcal{B}$ is uniquely determined by its initial values $w(0), \ldots, w(t_0 - 1)$.

Then we call $\mathcal{B}$ *past-determined* (or *strongly autonomous*).

Proof: Let $d \in \mathrm{ann}(M)$ be monic. Then $R(\sigma)w = 0$ implies $d(\sigma)w = 0$ and one may take $t_0 := \deg(d)$. Conversely, if R does not have reduced full column rank, then there exists $0 \neq M \in \mathcal{D}^q$ such that $RM = 0$ by McCoy's theorem [7, Chapter 3, Theorem 6]. Then we have $M(\sigma)\ell \in \mathcal{B}$ for all $\ell \in \mathcal{A}$. From this, one can construct nonzero trajectories of $\mathcal{B}$ with support in any interval $[t_0, \infty)\cap\mathbb{N}$. This contradicts past-determinedness. □

Theorem 11: Let F be finite and $\mathcal{D} = F[s]$.

1) $\mathcal{B}$ is past-determined if and only if all solutions of $R(\sigma)w = 0$ become periodic at some point, that is, there exists $t_0 \in \mathbb{N}$ and a positive integer p such that any solution w satisfies $w(t+p) = w(t)$ for all $t \geq t_0$. Here, t_0 and p can be taken to be minimal with respect to the property $\bar{s}^{t_0+p} = \bar{s}^{t_0}$ in $\mathcal{D}/\mathrm{ann}(M)$.
2) There exists a positive p such that all solutions of $R(\sigma)w = 0$ are p-periodic if and only if $\mathrm{ann}(M)$ contains a polynomial whose constant term is a unit. Then the minimal such p is the order of $\bar{s}$ in the unit group of $\mathcal{D}/\mathrm{ann}(M)$.

Proof: Part 1 follows from the characterization of past-determinedness and the finiteness of F. If $\mathrm{ann}(M)$ contains a monic polynomial, then $\mathcal{D}/\mathrm{ann}(M)$ is a finite ring, which guarantees the existence of t_0, p as desired. For part 2, $\mathrm{ann}(M)$ contains a polynomial with a unit constant term if and only if $\bar{s}$ is a unit in $\mathcal{D}/\mathrm{ann}(M)$. □

Finally, we mention that the absence of free variables, which is equivalent to past-determinedness for systems over fields, is a weaker property over rings [3]. In fact, it is even weaker than the requirement $\mathrm{ann}(M) \neq 0$ [10].

Theorem 12 ([9]): The following are equivalent:

1) $\mathcal{B}$ has no free variables, i.e., there exists no $1 \leq i \leq q$ such that the projection of $\mathcal{B}$ on the i-th component, $\pi_i : \mathcal{B} \to \mathcal{A}$, $w \mapsto w_i$, is surjective.
2) There exist $Z \in \mathcal{D}^{q\times g}$ and $0 \neq d_i \in \mathcal{D}$ such that $ZR = \mathrm{diag}(d_1, \ldots, d_q)$.

In that case, we call $\mathcal{B}$ *weakly autonomous*.

Proof: This follows from

$$\exists w \in \mathcal{B} : \pi_i(w) = a \Leftrightarrow d(\sigma)a = 0 \,\forall d \text{ with } de_i \in \mathcal{D}^{1\times g}R$$

and

$$d = 0 \quad \Leftrightarrow \quad \{a \in \mathcal{A} \mid d(\sigma)a = 0\} = \mathcal{A}.$$

Both facts are consequences of the injective cogenerator property of the $\mathcal{D}$-module $\mathcal{A}$. □

IV. PERIODICALLY TIME-VARYING SYSTEMS

A natural next question concerns the existence of p-periodic solutions to p-periodic systems as studied in [1], [2]. Consider the p-periodically time-varying system described by

$$(R_\tau(\sigma)w)(pt+\tau) = 0 \text{ for all } t \in \mathbb{N},\ 0 \leq \tau < p, \qquad (2)$$

where $R_\tau \in F[s]^{g_\tau\times q}$. Using the lifting techniques described in the above-mentioned papers, one can rewrite these equations as $R^L(\sigma)w^L = 0$, where

$$w^L(t) := \begin{bmatrix} w(pt) \\ w(pt+1) \\ \vdots \\ w(pt+p-1) \end{bmatrix}$$

and R^L is defined as follows: Let

$$R := \begin{bmatrix} R_0 \\ sR_1 \\ \vdots \\ s^{p-1}R_{p-1} \end{bmatrix} \in F[s]^{g\times q},$$

where $g = \sum_{\tau=0}^{p-1} g_\tau$. There exists a unique representation

$$R = R_0^L(s^p) + sR_1^L(s^p) + \ldots + s^{p-1}R_{p-1}^L(s^p)$$

where $R_i^L \in F[s]^{g\times q}$. Set

$$R^L := [R_0^L, \ldots, R_{p-1}^L] \in F[s]^{g\times pq}.$$

Then (2) is equivalent to [1], [2]

$$R^L(\sigma)w^L = 0. \qquad (3)$$

Since a p-periodic solution of (2) corresponds to a constant solution of the lifted system (3), we have the following result.

Corollary 1: The p-periodically time-varying system (2) has no p-periodic solution besides zero if and only if

$$\begin{bmatrix} R^L \\ (s-1)I_{pq} \end{bmatrix}$$

is left invertible. All solutions of (2) are p-periodic if and only if

$$(s-1)I_{pq} = YR$$

for some $Y \in \mathcal{D}^{pq\times g}$.

ACKNOWLEDGMENT

This paper is gratefully dedicated to my Ph.D. advisor Professor Ulrich Oberst, on the occasion of his 70th birthday.

REFERENCES

[1] J. C. Aleixo, J. W. Polderman, P. Rocha. Representations and structural properties of periodic systems. *Automatica* 43, 1921–1931 (2007).

[2] M. Kuijper, J. C. Willems. A behavioral framework for periodically time-varying systems. *Proc. 36th IEEE Conference on Decision and Control (CDC 97)*, San Diego, USA, 1997.

[3] M. Kuijper, R. Pinto, J. W. Polderman, P. Rocha. Autonomicity and the absence of free variables for behaviors over finite rings. *Proc. 7th Portuguese Conference on Automatic Control*, Lisbon, Portugal, 2006.

[4] T. Y. Lam. *Lectures on Modules and Rings.* Springer, 1999.

[5] P. Lu, M. Liu, U. Oberst. Linear recurring arrays, linear systems and multidimensional cyclic codes over quasi-Frobenius rings. *Acta Applicandae Mathematicae* 80, 175–198 (2004).

[6] B. R. McDonald. *Finite Rings with Identity.* Marcel Dekker, 1974.

[7] D. G. Northcott. *Finite Free Resolutions.* Cambridge University Press, 1976.

[8] U. Oberst. Multidimensional constant linear systems. *Acta Applicandae Mathematicae* 20, 1–175 (1990).

[9] E. Zerz. Discrete multidimensional systems over $\mathbb{Z}_n$. *Systems and Control Letters* 56, 702–708 (2007).

[10] E. Zerz. Autonomy properties of multidimensional linear systems over rings. *Proc. 5th Int. Workshop on Multidimensional Systems (NDS 07)*, Aveiro, Portugal, 2007.

[11] E. Zerz. Strongly autonomous behaviors over finite rings. *Proc. 6th Int. Workshop on Multidimensional Systems (NDS 09)*, Thessaloniki, Greece, 2009.

[12] E. Zerz, L. Wagner. Finite multidimensional behaviors. To appear in *Multidimensional Systems and Signal Processing*.