Elimination, fundamental principle and duality for analytic linear systems of partial differential-difference equations with constant coefficients

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Abstract— **Partial differential-difference equations are the multidimensional generalization of ordinary delay-differential equations. We investigate behaviors of analytic signals governed by equations of this type, i.e., solution modules of linear systems with constant coefficients of such equations, and especially the problems of elimination and duality. The first concerns the question whether the images of behaviors are again behaviors and in particular the existence of solutions of inhomogeneous linear systems which satisfy the obvious necessary compatibility conditions. Duality refers to the determination of the module of equations by the behavior. Our theory is presently restricted to analytic signals because the proofs make substantial use of the Stein algebra of multivariate entire functions and of Stein modules over it, but the extension to smooth or distributional signals is of course an important task for the future. We especially prove the validity of elimination for delay-differential equations with incommensurate delays and thus solve, for analytic signals, an open problem stated by Gluesing-Luerssen, Vettori and Zampieri. Duality is expressed and derived by means of the polar theorem for locally convex spaces in duality. Gluesing-Luerssen's rather complete and seminal behavioral theory of delay-differential equations with commensurate delays relies on the fact that the appropriate ring of operators is a Bezout domain and especially coherent. Coherence of the relevant rings of operators in the more general situations is important, but has not yet been proven. Further contributors to the module theoretic or behavioral approach to delay-differential equations are Fliess, Habets, Mounier, Rocha, Willems et al..**

AMS-classification: 93B25, 93C05, 93C23, 93C35

Keywords: partial differential-difference equation, fundamental principle, elimination, duality, multidimensional system, Stein algebra

I. Introduction

Partial differential-difference equations are the multidimensional generalization of ordinary delay-differential equations. The fundamental principle (terminology of Ehrenpreis) states that in suitable function spaces W over rings $\mathbf{E}$ of operators an inhomogeneous system of $\mathbf{E}$-linear equations has a solution with components in W if and only if it satisfies the (obvious) necessary compatibility conditions. Duality or cogenerator properties refer to the determination of the modules of equations of homogeneous linear systems by their modules of solutions, called ${}_{\mathbf{E}}W$-behaviors in systems theory. Elimination signifies that the images of behaviors are again behaviors. In famous difficult work Ehrenpreis, Malgrange and Palamodov proved the validity of the fundamental principle for linear systems of partial differential equations with constant coefficients and distributional or C^{∞}-solutions around 1960. The cogenerator properties were derived in [17] where also the systems theoretic significance of these results was explained.

The present work first establishes the fundamental principle and duality properties for the non-noetherian Stein algebra $\mathbf{D} := \mathcal{O}(\mathbb{C}^n_s)$ of entire functions on $\mathbb{C}^n$, $n \geq 1$, in variables $s = (s_1, \cdots, s_n)$ which acts canonically on its dual space of analytic functionals. The latter can be identified with the space $W := \mathcal{O}(\mathbb{C}^n_x; \exp)$ of entire functions of at most exponential growth in n variables $x = (x_1, \cdots, x_n)$, the canonical action $\circ$ being given by partial differentiation $(s_i \circ w)(x) = \partial w / \partial x_i$ and, in particular, by translation $\left(e^{\lambda \bullet s} \circ w\right)(x) = w(x + \lambda)$ where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ and $\lambda \bullet s := \lambda_1 s_1 + \cdots + \lambda_n s_n$. Spectral controllability is introduced and characterized by the freeness of the behavior module.

Consider the subalgebras $\mathbf{A}$ of polynomials resp. $\mathbf{C}$ of polynomial-exponential functions of $\mathbf{D}$:

$$\mathbf{A} := \mathbb{C}[s] \subset \mathbf{C} := \mathcal{PE}(s) := \oplus_{\lambda \in \mathbb{C}^n} \mathbb{C}[s] e^{\lambda \bullet s} \subset \mathbf{D}. \quad (1)$$

A finitely generated subgroup $\mathbf{G}$ of $\mathbb{C}^n$ with a $\mathbb{Z}$-basis $\lambda^1, \ldots, \lambda^m$ gives rise to its group ring or the Laurent polynomial subalgebra

$$\begin{aligned} \mathbf{B} := \mathbf{A}[\mathbf{G}] = \mathbb{C}[s, \sigma, \sigma^{-1}] := \\ \mathbb{C}[s_1, \cdots, s_n, \sigma_1, \sigma_1^{-1}, \cdots, \sigma_m, \sigma_m^{-1}] \subset \mathbf{C} \subset \mathbf{D} \end{aligned} \quad (2)$$

where $\sigma_j := e^{\lambda^j \bullet s}$, $j = 1, \cdots, m$ are the translation operators. The ring $\mathbf{B}$ acts on W by partial differentiation and by the translations $(\sigma_j \circ w)(x) = w(x + \lambda^j)$ and is thus a ring of partial differential-difference operators proper. We consider intermediate rings $\mathbf{E}$ with $\mathbf{B} \subseteq \mathbf{E} \subseteq \mathbf{D}$ and *generalized* $\mathbf{E}$*-behaviors*

$$\begin{aligned} \mathcal{B} := V^{\perp} := \left\{ w \in W^{\ell};\ V \circ w = 0 \right\} \text{ where} \\ V \subseteq \mathbf{E}^{1 \times \ell} \text{ is any } \mathbf{E}\text{-submodule.} \end{aligned} \quad (3)$$

ISBN 978-963-311-370-7

If V is $\mathbf{E}$-finitely generated (f.g.) $V^\perp$ is just called an $\mathbf{E}$-*behavior*. We investigate the *elimination and duality properties* for (generalized) $\mathbf{E}$-behaviors. The exact conditions will be described below. In this context the following rings appear naturally:

$$\mathbf{H}(\mathbf{B}) := \mathrm{quot}(\mathbf{B}) \bigcap \mathbf{D} = \mathbb{C}(s)[\sigma, \sigma^{-1}] \bigcap \mathbf{D} \subseteq \\ \mathbf{H}(\mathbf{E}) := \mathrm{quot}(\mathbf{E}) \bigcap \mathbf{D}. \tag{4}$$

The intersection is taken in the quotient field $\mathrm{quot}(\mathbf{D})$ of meromorphic functions. The description of $\mathbf{H}(\mathbf{B})$ as $\mathbb{C}(s)[\sigma, \sigma^{-1}] \bigcap \mathbf{D}$ was derived in [2]. Gluesing-Luerssen has shown in her seminal work (see [8], [9]) that strong elimination and duality properties hold for $\mathbf{H}(\mathbf{B})$-behaviors in the case $n = m = 1$, i.e., for delay-differential equations with commensurate delays, but not for $\mathbf{B}$-behaviors. In this case the ring $\mathbf{H}(\mathbf{B})$ is a *Bezout domain* [8, Th. 3.1.6], [3, Ex. VII.1.20], i.e., each finitely generated ideal is principal, and this implies the properties according to [loc.cit] or the results of the present paper. Under additional assumptions on the ring $\mathbf{E}$ we prove elimination for generalized $\mathbf{E}$-behaviors and, if in addition $\mathbf{E}$ is *coherent* [3, Ex. I.2.12], for $\mathbf{E}$-behaviors. The additional assumptions are satisfied for $n = 1,\ m \geq 1$ and the ring $\mathbf{H}(\mathbf{B})$, and we thus solve the open problem 1 from [9, foll. Th. 3.14] for ordinary delay-differential equations with non-commensurate delays and analytic signals. For this case we also prove the equivalence of spectral controllability with the existence of an image representation in contrast to the case of smooth signals where this equivalence does not hold [9, Sect. 4]. We also *characterize* the rings $\mathbf{E}$ with the strongest elimination and duality properties as those coherent rings $\mathbf{E}$ for which $\mathbf{E} \subseteq \mathbf{D}$ is a faithfully flat ring extension, especially $\mathbf{A} \subset \mathbf{D}$ is faithfully flat [1]. For $n = 1$ this signifies that $\mathbf{E}$ is a *Prüfer domain* [3, Ex. VII.2.12] (compare [18]), i.e., each finitely generated ideal is projective, and that for each finitely generated *proper* ideal $\mathfrak{a}$ of $\mathbf{E}$ also $\mathbf{D}\mathfrak{a} \neq \mathbf{D}$. For the case $n = 1$, but $m > 1$, the inclusion $\mathbf{H}(\mathbf{B}) \subset \mathbf{D}$ is not faithfully flat which follows from counter-examples in [9]. Module theoretic aspects of delay-differential equations (n=m=1) were also investigated by the French school (M. Fliess, H. Mounier [16] et al.).

II. Generalized behaviors

The theory of several complex variables and especially of Stein algebras and modules used below is taken from the excellent sources [7], [14] and [10]. The algebra $\mathbf{D} = \mathcal{O}(\mathbb{C}^n_s)$ of entire or holomorphic functions on $\mathbb{C}^n$ is a Fréchet algebra with the topology of compact convergence and is a *Stein algebra*. The free modules $\mathbf{D}^{1\times\ell},\ \ell \in \mathbb{N}$, are topological *Stein modules* over $\mathbf{D}$, i.e., the modules of global sections of coherent analytic sheaves on the Stein manifold $\mathbb{C}^n$. Each finitely generated submodule of $\mathbf{D}^{1\times\ell}$ is closed. An entire function $w(x) \in \mathcal{O}(\mathbb{C}^n_x)$ is called of *at most exponential growth* if there is a vector

$$R \in \mathbb{R}^n_{>0} \text{ and C>0 such that} \\ \mid w(x) \mid \leq Ce^{R_1|x_1|+\cdots+R_n|x_n|} \text{ for } x \in \mathbb{C}^n. \text{ Then} \\ W := \mathcal{O}(\mathbb{C}^n_x; \exp) \subset \mathcal{O}(\mathbb{C}^n_x) \tag{5}$$

is the subspace of entire functions of at most exponential growth. The bilinear form

$$< -, - >: \mathbf{D} \times W \to \mathbb{C}, \\ < \sum_{\mu\in\mathbb{N}^n} f_\mu s^\mu, \sum_{\mu\in\mathbb{N}^n} w_\mu x^\mu >:= \sum_{\mu\in\mathbb{N}^n} f_\mu w_\mu \mu!, \tag{6}$$

is well-defined and non-degenerate. The continuous linear functions on $\mathbf{D}$ are called *analytic functionals* and form the *topological* dual space $\mathbf{D}' := \mathrm{Hom}_{\mathbb{C},\mathrm{cont}}(\mathbf{D}, \mathbb{C})$. The bilinear form (6) induces the isomorphism (compare [14, Sect. 4.5], [17, pp. 64-70])

$$\mathbf{D}' \cong W = \mathcal{O}(\mathbb{C}^n_x; \exp), \\ \phi =< -, w >\leftrightarrow w = \sum_{\mu\in\mathbb{N}^n} w_\mu x^\mu = \phi\left(e^{s\bullet x}\right). \tag{7}$$

Since $\mathbf{D}$ is a topological algebra its dual space $\mathbf{D}'$ is canonically a $\mathbf{D}$-module and so is W with the action $f \circ w$ given by

$$\langle f \circ w, g\rangle := \langle fg, w\rangle,\ f, g \in \mathbf{D},\ w \in W, \\ \text{especially } s_i \circ w = \partial w/\partial x_i \text{ and} \\ \left(e^{\lambda\bullet s} \circ w\right)(x) = w(x + \lambda) \text{ for } \lambda \in \mathbb{C}^n. \tag{8}$$

For $f = (f_1, \cdots, f_\ell) \in \mathbf{D}^{1\times\ell}$ and $w = (w_1, \cdots, w_\ell)^\top \in W^\ell := W^{\ell\times 1}$ we write $f \circ w := \sum_{i=1}^\ell f_i \circ w_i$. Any *subset* $U \subseteq \mathbf{D}^{1\times\ell}$ induces the *generalized* $\mathbf{D}$-*behavior*

$$\mathcal{B} := U^\perp := \left\{w \in W^\ell;\ U \circ w = 0\right\} \subseteq W^\ell \text{ and obviously} \\ U^\perp =_{\mathbf{D}} \langle U\rangle^\perp = \overline{_{\mathbf{D}}\langle U\rangle}^\perp \tag{9}$$

where $_{\mathbf{D}}\langle U\rangle$ is the $\mathbf{D}$-module generated by U and $\overline{_{\mathbf{D}}\langle U\rangle}$ is its closure. In particular, a generalized behavior can always be defined by a closed $\mathbf{D}$-submodule of $\mathbf{D}^{1\times\ell}$. The generalized $\mathbf{D}$-behavior is itself a $\mathbf{D}$-submodule of W^ℓ. If $U = \mathbf{D}^{1\times k}R \subseteq \mathbf{D}^{1\times\ell},\ R \in \mathbf{D}^{k\times\ell}$, is a f.g. $\mathbf{D}$-module then $\mathcal{B} := U^\perp = \left\{w \in W^\ell;\ R \circ w = 0\right\}$ is just called a $\mathbf{D}$-*behavior*. Likewise a subset $\mathcal{B}$ of W^ℓ induces the $\mathbf{D}$-submodule $\mathcal{B}^\perp \subseteq \mathbf{D}^{1\times\ell}$. The bipolar theorem (compare [4, Th. 2.23]) furnishes

$$\overline{_{\mathbf{D}}\langle U\rangle} = U^{\perp\perp} = \mathcal{B}^\perp,\ \mathcal{B} := U^\perp. \tag{10}$$

Consider rings $\mathbf{B} \subseteq \mathbf{E} \subseteq \mathbf{D}$ as in the Introduction. Any $\mathbf{E}$-submodule $V \subseteq \mathbf{E}^{1\times\ell}$ gives rise to the *generalized* $\mathbf{E}$-*behavior*

$$\mathcal{B} := V^\perp = (\mathbf{D}V)^\perp = \overline{\mathbf{D}V}^\perp = \\ \left\{w \in W^\ell;\ V \circ w = 0\right\} \cong \mathrm{Hom}_{\mathbf{E}}(M, W) \tag{11}$$

where $\mathbf{D}V =_{\mathbf{D}} \langle V\rangle$ is the $\mathbf{D}$-submodule of $\mathbf{D}^{1\times\ell}$ generated by V and $M := \mathbf{E}^{1\times\ell}/V$. If V is $\mathbf{E}$-f.g. then $\mathcal{B} = V^\perp$ is simply called an $\mathbf{E}$-*behavior*. The topology of $\mathbf{D}^{1\times\ell}$ induces a topology on $\mathbf{E}^{1\times\ell}$. The closure of an $\mathbf{E}$-submodule V of $\mathbf{E}^{1\times\ell}$ in $\mathbf{E}^{1\times\ell}$ is again denoted by $\overline{V}$.

III. Duality, elimination and fundamental principle

The assumptions of the preceding section are in force.

Theorem III.1. (Elimination and fundamental principle)

1) *Assume that for each $\lambda \in \mathbb{C}^n$ with its associated closed maximal ideal*

$$\mathfrak{m}(\lambda) := \{f \in \mathbf{D};\ f(\lambda) = 0\} = \sum_{i=1}^{n} \mathbf{D}(s_i - \lambda_i) \subset \mathbf{D}$$

of $\mathbf{D}$ the maximal ideal $\mathfrak{n}(\lambda) := \mathfrak{m}(\lambda) \bigcap \mathbf{E}$ of $\mathbf{E}$ satisfies

$$\mathfrak{n}(\lambda) = \mathfrak{m}(\lambda) \bigcap \mathbf{E} = \sum_{i=1}^{n} \mathbf{E}(s_i - \lambda_i)$$

and the local ring $\mathbf{E}_{\mathfrak{n}(\lambda)}$ is noetherian. Then:

a) *For each $\lambda \in \mathbb{C}^n$ the inclusion*

$$\mathbf{E}_{\mathfrak{n}(\lambda)} \subset \mathcal{O}_\lambda = \mathbb{C}\langle s - \lambda \rangle$$

of local rings is faithfully flat where $\mathbb{C}\langle s - \lambda \rangle$ denotes the ring of locally convergent power series near $\lambda \in \mathbb{C}^n$.

b) *The image of a* generalized *$\mathbf{E}$-behavior is again such, more precisely*

$$\begin{gathered} P \circ V^\perp = (\circ P)_{\mathbf{E}}^{-1}(V)^\perp = \left\{\xi \in \mathbf{E}^{1\times\ell'};\ \xi P \in V\right\}^\perp \\ \text{for } P \in \mathbf{E}^{\ell'\times\ell},\ V \subseteq \mathbf{E}^{1\times\ell}, \\ \text{and especially } P \circ W^\ell = \ker_{\mathbf{E}}(\circ P)^\perp, \\ V_1^\perp + V_2^\perp = (V_1 \bigcap V_2)^\perp,\ V_1^\perp \bigcap V_2^\perp = (V_1 + V_2)^\perp \\ \text{for } V_1, V_2 \subseteq \mathbf{E}^{1\times\ell}. \end{gathered} \tag{12}$$

c) *If, in particular,*

$$\ker_{\mathbf{E}}(\circ P) = \mathbf{E}^{1\times k'} R',\ R' \in \mathbf{E}^{k'\times\ell'}, \tag{13}$$

is f.g. or, sufficiently, if

$$\ker_{\mathbf{D}}(\circ P) = \mathbf{D}^{1\times k'} R',\ R' \in \mathbf{E}^{k'\times\ell'}, \tag{14}$$

then

$$P \circ W^\ell = \left\{u \in W^{\ell'};\ R' \circ u = 0\right\}.$$

This signifies that the equation $P \circ y = u$ is solvable for given right side u if and only if $R' \circ u = 0$, and is Ehrenpreis' original form of the fundamental principle.

2) *The ring $\mathbf{E}$ is coherent if and only if each f.g. ideal of $\mathbf{E}$ is finitely presented or, equivalently, if for all $P \in \mathbf{E}^{\ell'\times 1}$, $\ell' \in \mathbb{N}$, the kernel $\ker_{\mathbf{E}}(\circ P)$ is $\mathbf{E}$-f.g.. If in the preceding item $\mathbf{E}$ is indeed coherent and V is $\mathbf{E}$-f.g then so is $(\circ P)_{\mathbf{E}}^{-1}(V)$, and therefore each image of an $\mathbf{E}$-behavior is an $\mathbf{E}$-behavior.*
3) *Under the assumption of 1) the extension $\mathbf{E} \subseteq \mathbf{D}$ is flat if and only if for any $P \in \mathbf{E}^{\ell'\times 1}$ the module $\mathbf{D}\ker_{\mathbf{E}}(\circ P)$ is closed in $\mathbf{D}^{1\times\ell'}$. This holds if $\mathbf{E}$ is coherent and thus $\ker_{\mathbf{E}}(\circ P)$ is f.g.*
4) *The assumptions and conclusion of item 1) hold for $\mathbf{E} = \mathbf{D}$.*
5) *The assumptions and conclusion of item 1) hold for $n = 1$, $m \geq 1$ and any ring $\mathbf{E}$ with $\mathbf{E} = \mathrm{quot}(\mathbf{E}) \bigcap \mathbf{D}$, in particular for $\mathbf{E} = \mathbf{H}(\mathbf{B})$. Then the local rings $\mathbf{E}_{\mathfrak{n}(\lambda)}$ are discrete valuations rings with the unique prime element $s - \lambda$, up to association.*
This solves the open problem 1 in [9, foll. Th. 3.14] for ordinary delay-differential equations with non-commensurate delays and analytic signals.
6) ($\mathbb{C}$-finite-dimensional behaviors) *If $U \subseteq \mathbf{D}^{1\times\ell}$ is any $\mathbf{D}$-submodule such that $M := \mathbf{D}^{1\times\ell}/U$ is $\mathbb{C}$-finite-dimensional then U is closed and*

$$U = \mathbf{D}(\mathbf{A}^{1\times\ell} \bigcap U) \text{ and } \mathcal{B} := U^\perp = (\mathbf{A}^{1\times\ell} \bigcap U)^\perp$$

is a $\mathbb{C}$-finite dimensional $\mathbf{A}$-behavior of dimension $\dim_{\mathbb{C}}(\mathcal{B}) = \dim_{\mathbb{C}}(M)$ and thus defined by a finite system of partial differential equations without translation operators. If, in addition, $P \in \mathbf{D}^{\ell'\times\ell}$ then

$$P \circ \mathcal{B} = (\circ P)_{\mathbf{D}}^{-1}(U)^\perp$$

is also a $\mathbb{C}$-finite dimensional $\mathbf{A}$-behavior.

Remark III.2.
1) For $n = 1$ the ring $\mathbf{D} = \mathcal{O}(\mathbb{C})$ is a Bezout domain and thus coherent.
2) For $n > 1$ the coherence of $\mathbf{D}$ is unknown, but unlikely (oral communication of O. Forster). This suggested to consider generalized behaviors in this paper.
3) For $n = 1$ and $m > 1$ the coherence of $\mathbf{H}(\mathbf{B})$ is unknown too.
4) For $n > 1$ and $m \geq 1$ it is presently not clear whether $\mathbf{H}(\mathbf{B})$ or another suitable ring $\mathbf{E}$ satisfies the assumption of item 1) of Theorem III.1

Theorem III.3. (Duality)

1) *The closure of a submodule $V \subseteq \mathbf{E}^{1\times\ell}$ in $\mathbf{E}^{1\times\ell}$ is*

$$\begin{gathered} \overline{V} = \mathbf{E}^{1\times\ell} \bigcap \overline{\mathbf{D}V} = \bigcap_{\lambda\in\mathbb{C}^n,\ q\in\mathbb{N}} (V + V(\lambda, q+1)) \\ \text{where } V(\lambda, q+1) := \\ \left\{f \in \mathbf{E}^{1\times\ell};\ \partial^{|\mu|} f/\partial s^\mu(\lambda) = 0 \text{ for } \mu \in \mathbb{N}^n, |\ \mu\ | \leq q\right\} \\ \text{and } |\ \mu\ | := \mu_1 + \cdots + \mu_n. \end{gathered}$$

A f.g. $\mathbf{D}$-submodule of $\mathbf{D}^{1\times\ell}$ is closed, but a f.g. $\mathbf{E}$-submodule V of $\mathbf{E}^{1\times\ell}$ is not necessarily closed.

2) *For submodules $V_i \subseteq \mathbf{E}^{1\times\ell}$, $i = 1, 2$:*

$$\begin{gathered} \overline{V_2} \subseteq \overline{V_1} \iff \overline{\mathbf{D}V_2} \subseteq \overline{\mathbf{D}V_1} \iff V_1^\perp \subseteq V_2^\perp \\ \iff V_1^\perp \bigcap \mathcal{PE}(x)^\ell \subseteq V_2^\perp \bigcap \mathcal{PE}(x)^\ell. \end{gathered} \tag{15}$$

For f.g. and closed *(see item 1.) $V_i = \mathbf{E}^{1\times k_i} R_i$, $R_i \in \mathbf{E}^{k_i\times\ell}$, this signifies the existence of $X \in \mathbf{E}^{k_2\times k_1}$ with $R_2 = XR_1$.*

3) *If in item 2) the assumptions of Theorem III.1,(1) or (4) or (5), are satisfied then its equivalent conditions*

hold if and only for all $\lambda \in \mathbb{C}^n$

$$V_{2,\mathfrak{n}(\lambda)} = \mathbf{E}_{\mathfrak{n}(\lambda)} V_2 \subseteq V_{1,\mathfrak{n}(\lambda)}.$$

If, in addition, $V_i = \mathbf{E}^{1\times k_i} R_i,\ R_i \in \mathbf{E}^{k_i\times \ell},\ i = 1, 2,$ *is f.g this signifies that for all* $\lambda \in \mathbb{C}^n$ *there is a matrix*

$$X_\lambda \in \mathbf{E}_{\mathfrak{n}(\lambda)}^{k_2\times k_1} \text{ with } R_2 = X_\lambda R_1.$$

4) *If in item 2)* $M_1 := \mathbf{E}^{1\times\ell}/V_1$ *is torsionfree and especially, by definition,* $V_1^\perp$ *is controllable then* V_1 *is closed and*

$$V_1^\perp \subseteq V_2^\perp \iff V_2 \subseteq V_1.$$

5) *If in item 2)* $V_i = \mathbf{E}^{1\times k_i} R_i$ *and* $\operatorname{rank}(R_1) = k_1$ *then*

$$V_1^\perp \subseteq V_2^\perp \iff$$
$$\exists X \in \mathbf{H}(\mathbf{E})^{k_2\times k_1} \text{ with } R_2 = XR_1.$$

Item 5) suggested the introduction of $\mathbf{H}(\mathbf{B})$ *in [8] and [12].*

The next theorem was essentially proven in [7, §6].

Theorem and Definition III.4. (Controllability of $\mathbf{D}$-behaviors) *The following properties are equivalent for*

$$U = \mathbf{D}^{1\times k} R,\ M := \mathbf{D}^{1\times\ell}/U \text{ and}$$
$$\mathcal{B} := U^\perp \cong \operatorname{Hom}_{\mathbf{D}}(M, W)$$

(compare [9, Th. 3.12]):

1) $\mathcal{B}$ *is* spectrally controllable, *i.e.,* $\operatorname{rank}(R(\lambda)) = \operatorname{rank}(R)$ *for all* $\lambda \in \mathbb{C}^n$.
2) *For all* $\lambda \in \mathbb{C}^n$ *the module*

$$\mathcal{O}_\lambda \otimes_{\mathbf{D}} M = \mathcal{O}_\lambda^{1\times\ell}/\mathcal{O}_\lambda U = \mathcal{O}_\lambda^{1\times\ell}/\mathcal{O}_\lambda^{1\times k} R$$

is free.

3) M *is projective, i.e., there is a matrix* $G \in \mathbf{D}^{\ell\times k}$ *such that* $R = RGR$.
4) M *is free, hence torsionfree and thus* $\mathcal{B}$ *is* controllable.

Theorem III.5. (Controllability of $\mathbf{E}$-behaviors) *Assume that the assumptions of Theorem III.1,(1), are satisfied, for instance for* $n = 1$, $m \geq 1$ *and* $\mathbf{E} = \mathbf{H}(\mathbf{B})$, *and let the matrix, modules and* $\mathbf{E}$*-behavior*

$$R \in \mathbf{E}^{k\times\ell},\ V := \mathbf{E}^{1\times k} R \subseteq \mathbf{E}^{1\times\ell},$$
$$M := \mathbf{E}^{1\times\ell}/V,\ \mathcal{B} := V^\perp \subseteq W^\ell$$

be given. Consider the following four properties:

1) M *is projective.*
2) *For all* $\lambda \in \mathbb{C}^n$ *the localized module* $M_{\mathfrak{n}(\lambda)}$ *is free.*
3) $\mathcal{B}$ *is spectrally controllable, i.e.,* $\operatorname{rank}(R(\lambda)) = \operatorname{rank}(R)$ *for all* $\lambda \in \mathbb{C}$.
4) *The behavior* $\mathcal{B}$ *admits an image representation, i.e., there is a matrix* $P \in \mathbf{E}^{\ell\times q}$ *such that* $\mathcal{B} = P \circ W^q$.

Then the following implications hold:

1) $1) \Longrightarrow 2) \iff 3) \Longrightarrow 4)$.
2) *If* $n = 1$: $1) \Longrightarrow 2) \iff 3) \iff 4)$.

The last assertion proves the equivalence of the conditions (a) and (g) in [9, Th. 3.12] for analytic signals. *The implication* $2), 3) \Longrightarrow 1)$ *is false for* $n = 1$ *and* $m > 1$ *according to the counter-example in [9, Sect. 4].*

Remark III.6. For $n = 1$, $m > 1$ and *smooth signals* the implication $3) \Longrightarrow 4)$ of the preceding theorem is false according to the counter-example in [9, Sect. 4]. This is no contradiction to the preceding theorem since the trajectory constructed in [loc.cit.] is smooth, but not analytic.

The following theorem characterizes the rings $\mathbf{E}$ with the strongest elimination and duality properties.

Theorem III.7. *For* $\mathbf{B} \subseteq \mathbf{E} \subseteq \mathbf{D}$ *the following properties 1) and 2) are equivalent:*

1) *The set of* $\mathbf{E}$*-behaviors satisfies the following elimination and duality properties:*
   a) *The images of* $\mathbf{E}$*-behaviors are* $\mathbf{E}$*-behaviors, i.e., for matrices* $R \in \mathbf{E}^{k\times\ell}$ *and* $P \in \mathbf{E}^{\ell'\times\ell}$ *there is a matrix* $R' \in \mathbf{E}^{k'\times\ell'}$ *such that*

   $$P \circ \left\{w \in W^\ell;\ R \circ w = 0\right\} =$$
   $$\left\{w \in W^{\ell'};\ R' \circ w = 0\right\}.$$

   b) *If* $V_i = \mathbf{E}^{1\times k_i} R_i,\ R_i \in \mathbf{E}^{k_i\times\ell}$, *then*

   $$V_1^\perp \subseteq V_2^\perp \iff$$
   $$\exists X \in \mathbf{E}^{k_2\times k_1} \text{ with } R_2 = XR_1.$$

2) *The ring* $\mathbf{E}$ *is coherent and the extension* $\mathbf{E} \subseteq \mathbf{D}$ *is faithfully flat [3, Prop. I.3.9].*
   Under the equivalent conditions 1) and 2) the following assertions hold:
3) *The assumption of item 1) in Theorem III.1 holds.*
4) *If* $P \in \mathbf{E}^{\ell'\times\ell}$ *and* $V \subseteq \mathbf{E}^{1\times\ell}$ *is* $\mathbf{E}$*-finitely generated then so is* $(\circ P)_{\mathbf{E}}^{-1}(V)$ *and*

$$P \circ V^\perp = (\circ P)_{\mathbf{E}}^{-1}(V)^\perp,$$
$$\text{especially } P \circ W^\ell = \ker_{\mathbf{E}}(\circ P)^\perp.$$

5) $\mathbf{H}(\mathbf{B}) := \operatorname{quot}(\mathbf{B}) \bigcap \mathbf{D} \subseteq \mathbf{E} = \mathbf{H}(\mathbf{E}) := \operatorname{quot}(\mathbf{E}) \bigcap \mathbf{D}$.
6) *For a finitely presented module* $M = \mathbf{E}^{1\times\ell}/V$, $V = \mathbf{E}^{1\times k} R$, *and* $\mathcal{B} := V^\perp$ *the following properties are equivalent:*
   a) M *is projective or, equivalently, there is a matrix* $G \in \mathbf{E}^{\ell\times k}$ *with* $R = RGR$.
   b) $\mathbf{D}^{1\times\ell}/\mathbf{D}^{1\times k} R = \mathbf{D} \otimes_{\mathbf{E}} M$ *is projective and thus free.*
   c) *The behavior* $\mathcal{B}$ *is spectrally controllable, i.e,* $\operatorname{rank}(R(\lambda)) = \operatorname{rank}(R)$ *for all* $\lambda \in \mathbb{C}^n$.
7) *For* $n = 1$ *and* $m \geq 1$ *the equivalent properties 1) and 2) signify that* $\mathbf{E}$ *is a Prüfer domain and that for a finitely generated proper ideal* $\mathfrak{a}$ *of* $\mathbf{E}$, *i.e.,* $\mathfrak{a} \neq \mathbf{E}$, *also* $\mathbf{D}\mathfrak{a} \neq \mathbf{D}$.

Corollary III.8. *For* $n = 1$ *and* $m > 1$ *the ring* $\mathbf{E} := \mathbf{H}(\mathbf{B})$ *does not satisfy the equivalent properties 1) and 2) of the* *preceding theorem. Indeed, for $n = 1,\ m = 2$, the following inclusions hold [9, Ex. 3.5]:*

$$\begin{gathered}\mathfrak{a}_1 := \mathbf{E}(\sigma_1 - 1) + \mathbf{E}(\sigma_2 - 1) \subsetneq \mathfrak{a}_2 := \mathbf{E}s,\ \textit{but}\\ \mathbf{D}\mathfrak{a}_1 = \mathbf{D}(\sigma_1 - 1) + \mathbf{D}(\sigma_2 - 1) = \mathbf{D}\mathfrak{a}_2 = \mathbf{D}s,\ \textit{hence}\\ \mathfrak{a}_1 \subsetneq \mathfrak{a}_2 = \mathbf{E}s = \mathbf{E}\bigcap \mathbf{D}s = \mathbf{E}\bigcap \mathbf{D}\mathfrak{a}_1.\end{gathered}$$

Thus $\mathbf{H}(\mathbf{B}) \subset \mathbf{D}$ is surely not faithfully *flat. Also the example in [9, Sect. 4] is a behavior which satisfies 6),(c), but not 6),(a), of the preceding theorem.*

Remark III.9. (Compare [4, Sect. 4.3]) The dual of the space $\mathcal{E} := \mathrm{C}^\infty(\mathbb{R}^n_x, \mathbb{C})$ is the convolution algebra $\mathcal{E}'$ of distributions of compact support. It acts on $\mathcal{E}$ via convolution and can be embedded into $\mathbf{D}$ via the Laplace transform, the image being characterized in the Paley-Wiener-Schwartz theorem. We assume $\lambda^j \in \mathbb{R}^n$ and identify

$$\begin{aligned}&\mathbf{C} = \mathcal{P}\mathcal{E}(s) \subseteq \mathcal{E}' \subseteq \mathbf{D},\ \partial\delta_0/\partial x_i = s_i,\\ &\delta_\lambda = e^{\lambda\bullet s},\ \lambda \in \mathbb{R}^n,\ \delta_{\lambda^j} = e^{\lambda^j\bullet s} = \sigma_j.\end{aligned} \tag{16}$$

Then the canonical injection $W \to \mathcal{E},\ w \mapsto w \mid \mathbb{R}^n$, is $\mathcal{E}'$-linear. While the $\mathcal{E}'$-module $\mathcal{E}$ is not divisible Ehrenpreis [6],[13, Def. 16.3.12, Th. 16.5.7] proved that $\mathcal{E}$ is a divisible $\mathbf{C}$-module, and this was essentially used in [12], [19], [9] and [4]. But no fundamental principle is known for $\mathbf{C}$-behaviors in the signal space $\mathcal{E}$. Gurevič [11] gave an example of a nonzero (autonomous) $\mathcal{E}'$-behavior in $\mathcal{E}$ without nonzero polynomial-exponential solutions, i.e., where the conditions in (15) are not equivalent.

IV. Discussion

1) Elimination, the fundamental principle and duality are important for and ,at least implicitly, used by almost all approaches to linear systems theory and not only by the behavioral approach since they furnish a strong connection between analytic and algebraic properties of systems. If these properties do not hold, perhaps only in a weakened form, one cannot draw conclusions from the algebraic properties of the equations to the properties of actual trajectories which appear in engineering setups.
2) Although the signal space $\mathcal{E} = \mathrm{C}^\infty(\mathbb{R}^n, \mathbb{C})$ of smooth functions is more important and flexible than the signal space W of analytic signals it is reasonable and done in this paper to first solve the problems for analytic signals. It is, of course, an important task to extend the results to smooth signals as used, for instance, in [9] or even to distributional ones.
3) Since $\mathbf{E}$-behaviors are described by *finitely* many operator equations they are more significant than *generalized* behaviors, especially for practical applications. Unless $\mathbf{E}$ is coherent the use of generalized behaviors is, however, necessary for the proofs of Theorem III.1,(1), and of Theorem III.3,(2). That images of behaviors may be generalized behaviors only is not an artificial mathematical difficulty, but may be inherent to (partial) differential-difference systems. This complicates their study significantly.
4) Theorem III.1,(1) and (4), and Theorem III.3,(2), furnish a complete elimination and duality theory for generalized $\mathbf{D}$-behaviors. Disadvantages of this theory are the probable non-coherence of the ring $\mathbf{D}$ and the ensuing need for generalized $\mathbf{D}$-behaviors instead of $\mathbf{D}$-behaviors only, and the size of the operator ring $\mathbf{D}$ which deviates too much from the ring $\mathbf{B}$ of pure differential-difference operators.
5) Theorem III.1,(5), seems to be the first elimination result for delay-differential systems with incommensurate delays and represents a non-negligible advance in this theory. The same holds for Theorem III.5. The coherence of the ring $\mathbf{E} := \mathbf{H}(\mathbf{B})$ would substantially add to the usefulness of these results.
6) Theorem III.7 characterizes systems theoretic properties of $\mathbf{E}$-behaviors by algebraic properties of the extension $\mathbf{E} \subseteq \mathbf{D}$ and thus suggests to establish such properties for suitable $\mathbf{E}$. The present problem with this result is that except for the standard cases $n = m = 1$ and $n \geq 1,\ m = 0$ no examples are known to us.
7) Equation (14) shows that coherence may not be necessary for the validity of elimination for $\mathbf{E}$-behaviors.
8) **Open problems**:
    a) Prove the coherence of $\mathbf{D}$ or give a counterexample.
    b) Prove or disprove the coherence of $\mathbf{E} = \mathbf{H}(\mathbf{B})$ for $n = 1,\ m > 1$, i.e., for delay-differential equations with incommensurate delays. Or prove elimination for $\mathbf{E}$-behaviors by means of equation (14) without using the coherence of $\mathbf{E}$.
    c) Prove or disprove the assumptions of Theorem III.1,(1), and perhaps the coherence of $\mathbf{E}$ for the case $n > 1,\ m \geq 1$ and suitable $\mathbf{E}$, for instance for $\mathbf{E} := \mathbf{H}(\mathbf{B})$.

References

[1] J. Apel, J. Stückrad, P. Tworzewski, T. Winiarski, *Properties of Entire Functions over Polynomial Rings via Gröbner Bases*, AAECC 14(2003), 1-10

[2] C.A. Berenstein, M.A. Dostal, *The Ritt theorem in several variables*, Arkiv för matematik 12(1974), 267-280

[3] N. Bourbaki, *Commutative Algebra*, Hermann, Paris, 1972

[4] H. Bourlès, U. Oberst, *Duality for Differential-Difference Systems over Lie Groups*, SIAM J. Contr. Opt. 48(2009), pp. 2051-2084

[5] H. Bourlès, U. Oberst, *Fundamental principle and duality for analytic linear systems of partial differential-difference equations with constant coefficients*, submitted to MCSS, December 2008

[6] L. Ehrenpreis, *Solutions of some problems of division III*, Amer. J. Math. 78(1956), 685-715

[7] O. Forster, *Zur Theorie der Steinschen Moduln und Algebren*, Math.Z. 97(1967), 376-405

[8] H. Gluesing-Luerssen, *Linear Delay-Differential Systems with Commensurate Delays: An Algebraic Approach*, Lecture Notes in Mathematics 1770, Springer, Berlin, 2002

[9] H. Gluesing-Luerssen, P. Vettori, S. Zampieri, *The algebraic structure of DD systems: a behavioral perspective*, Kybernetica 37(2001), 397-426

[10] H. Grauert, R. Remmert, *Theorie der Steinschen Räume*, Springer, Heidelberg, 1977 (English translation: *Theory of Stein Spaces*, Springer, New York, 2004)

[11] D.I. Gurevič, *Counterexamples to a problem of L. Schwartz*, Funct. Anal. Appl. 9(1975), pp.116-120

[12] L.C. Habets, *System equivalence for AR-systems over rings-With an application to delay-differential systems*, Math. Contr. Sign. Syst. 12(1999), 219-244

[13] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer, Berlin, 1983

[14] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Third Edition, North Holland, Amsterdam, 1991

[15] B. Malgrange, *Existence et approximation des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier 6(1956), 271-355

[16] H. Mounier, *Algebraic interpretations of the spectral controllability of a linear delay system*, Forum Math. 10(1998), 39-58

[17] U. Oberst, *Multidimensional Constant Linear Systems*, Acta Appl. Math. 20(1990), 1-175

[18] A. Quadrat, *The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. Part II: Internal stabilization*, SIAM J. Control Optim. 42(2003), 300-320

[19] P. Vettori, S. Zampieri, *Some Results on Systems Described by Convolution Equations*, IEEE Trans. Autom. Control 46(2001), 793-797