Control of Linear Systems with Fractional Brownian Motions

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Abstract— **In this paper a control problem for a multidimensional linear stochastic system with a vector of fractional Brownian motions and a cost functional that is quadratic in the state and the control is solved. An optimal control is given explicitly using the (Riemann-Liouville) fractional calculus and the control is shown to be the sum of a prediction of the optimal system response to the future fractional Brownian motion and the well known linear feedback control for the deterministic linear-quadratic control problem. It is noted that the methods to obtain an optimal control extend to other noise processes with continuous sample paths and finite second moments.**

Key Words: linear quadratic Gaussian control, fractional Brownian motion, linear regulator, linear stochastic systems.

I. INTRODUCTION

The linear-quadratic Gaussian (LQG) control problem for the control of a linear stochastic system with a Brownian motion (white Gaussian noise) and a quadratic cost functional of the state and the control e.g. [5] is the most well known and basic solvable stochastic control problem for stochastic systems with continuous sample paths. Historically white Gaussian noise (the formal derivative of Brownian motion) was used to model ”noise” because it was justified as an approximation to the spectrum of the noise compared with the spectrum of a physical system. It was often acknowledged that this noise model was only an imprecise approximation. In fact empirical measurements of many physical phenomena suggest that Brownian motion is inappropriate to use in mathematical models of these phenomena. A family of processes that has empirical evidence of wide physical applicability is fractional Brownian motion. Fractional Brownian motion is a family of Gaussian processes that was defined by Kolmogorov [8] in his study of turbulence [9], [10]. While this family of processes includes Brownian motion, it also includes other processes that describe behavior that is bursty or has a long range dependence. The first empirical evidence of the usefulness of these latter processes was made by Hurst [6] in his statistical analysis of rainfall along the Nile River where he estimated a parameter that determines the specific fractional Brownian motion. Mandelbrot [11] used fractional Brownian motions to describe economic data and noted that Hurst's statistical analysis was identifying the appropriate fractional Brownian motion (FBM). Mandelbrot and van Ness [12] provided some of the initial theory for FBMs.

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Since FBMs have a wide variety of potential applications, it is natural to consider the control of a linear stochastic system with an arbitrary FBM and a quadratic cost functional. It is natural to call such problems, linear-quadratic fractional Gaussian (LQFG) control. Some initial work has been done on this problem. Kleptsyna, Le Breton and Viot [7] consider a scalar linear stochastic system with the index (or Hurst parameter) for the FBMs restricted to $(1/2, 1)$ instead of the full family with index set $(0, 1)$. Duncan and Pasik-Duncan [4] consider a multidimensional linear stochastic system with an FBM having the Hurst parameter also in $(1/2, 1)$.

In this paper a more general linear stochastic control system with a vector of independent scalar FBMs with (possibly) different Hurst parameters in $(0, 1)$ is solved by explicitly giving an optimal control. The approach used here employs (Riemann-Liouville) fractional calculus [13] to provide an explicit description of an optimal control for an arbitrary FBM. The optimal control is the sum of two terms where one term is the well known linear feedback control from the deterministic control problem (or equivalently the optimal control for the case of Brownian motion) and the other term

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is a suitable prediction of the response of the optimal system to the future noise.

The method to obtain an optimal control that is used here combines some elementary stochastic analysis for a fractional Brownian motion and a completion of squares. It is noted that this approach can also be applied to other processes, such as an arbitrary Gaussian process with continuous sample paths and more generally an arbitrary process with continuous sample paths and (uniformly) bounded second moments.

A brief outline of the paper is given now. In Section II some information on the family of fractional Brownian motions is given that includes some elementary stochastic calculus for these processes and the definitions of fractional integrals and fractional derivatives. Furthermore the controlled linear system, the quadratic cost functional, and the family of admissible controls are given. In Section III the main result for an optimal control is given for the case of an arbitrary fractional Brownian motion and an approach to compute the optimal cost. A generalization to other noise processes is noted. In Section IV some concluding remarks are made.

II. Preliminaries

Fractional Brownian motion (FBM) is a family of Gaussian processes that is indexed by the Hurst parameter $H \in (0,1)$. Initially the definition for an FBM is given.

Definition 1: Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and let $H \in (0,1)$ be fixed. On this probability space a real-valued standard fractional Brownian motion, $(B(t), t \geq 0)$, with Hurst parameter H is a Gaussian process with continuous sample paths such that

$$\mathbb{E}[B(t)] = 0$$
$$\mathbb{E}[B(s)B(t)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for all $s,t \in \mathbb{R}_+$.

An $\mathbb{R}^n$-valued standard fractional Brownian motion, $(B(t), t \geq 0)$, with Hurst parameter H is an n-vector of independent real-valued standard fractional Brownian motions each with the same Hurst parameter H. If B is an FBM with $H = 1/2$ then B is a standard Brownian motion.

A fractional calculus of Riemann and Liouville e.g. [13] plays an important role in the analysis of a FBM. Let $\alpha \in (0,1)$ be fixed. The left-sided and the right-sided fractional (Riemann-Liouville) integrals for $\varphi \in L^1([0,T])$ are defined for almost all $t \in [0,T)$ by

$$\left(I_{0+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)\,ds$$

and

$$\left(I_{T-}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}\varphi(s)\,ds$$

respectively, where $\Gamma(\cdot)$ is the Gamma function. The inverse operators of these fractional integrals are called fractional derivatives and can be given by their respective Weyl representations

$$\begin{aligned}\left(D_{0+}^{\alpha}\psi\right)(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\psi(t)}{(t-s)^{\alpha}}ds \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{t^{\alpha}} + \alpha\int_0^t \frac{\psi(t)-\psi(s)}{(t-s)^{\alpha+1}}\,ds\right)\end{aligned}$$

and

$$\begin{aligned}\left(D_{T-}^{\alpha}\psi\right)(t) &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{\psi(t)}{(s-t)^{\alpha}}ds \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{(T-t)^{\alpha}} + \alpha\int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\,ds\right)\end{aligned}$$

where $\psi \in I_{0+}^{\alpha}\left(L^1([0,T])\right)$ and $\psi \in I_{T-}^{\alpha}\left(L^1([0,T])\right)$ respectively. Often it is convenient to write D^{α} as $I^{-\alpha}$.

One indication of the relevance of fractional calculus to FBM is that the covariance of a real-valued FBM can be expressed in terms of fractional integrals or fractional derivatives. Let $(B(t), t \geq 0)$ be a real-valued standard fractional Brownian motion with Hurst parameter H. The covariance of B satisfies the following equality

$$\begin{aligned}\mathbb{E}[B(s)B(t)] = \rho(H)\int_0^T u^2_{\frac{1}{2}-H}(r) \\ (I_{T-}^{H-\frac{1}{2}}u_{H-\frac{1}{2}}1_{[0,s]})(r)(I_{T-}^{H-\frac{1}{2}}u_{H-\frac{1}{2}}1_{[0,t]})(r)dr\end{aligned}$$

where

$$\rho(H) = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}.$$

and $u_a(s) = s^a$ for $a > 0, s > 0$ and I^α is a fractional integral for $\alpha > 0$ and a fractional derivative for $\alpha \in (-1,0)$. The integral on the RHS shows that the covariance for a FBM can be factored in the Lebesgue space $L^2([0,T])$ from which it follows that an FBM can be expressed in terms of a Brownian motion and conversely using fractional calculus. This relation between processes is important for the solution of the prediction problem for a fractional Brownian motion. Associated with each Gaussian process is a "natural" Hilbert space. For a real-valued ractional Brownian motion this Hilbert space can be given explicitly from the above expression for the covariance. Let $H \in (0,1)$ be fixed and let $L^2_H([0,T])$ be the Hilbert space where $f,g \in L^2_H$ if $< f,f >_H < \infty$ and $< g,g >_H < \infty$ where the inner product is given by

$$\begin{aligned} < f,g >_H &= \rho(H) \int_0^T u^2_{\frac{1}{2}-H}(r) \\ & (I^{H-\frac{1}{2}}_{T-} u_{H-\frac{1}{2}} f)(r)(I^{H-\frac{1}{2}}_{T-} u_{H-\frac{1}{2}} g)(r) dr \end{aligned} \tag{II.1}$$

The linear operator I^α is understood to be the fractional derivative $D^{-\alpha}$ for $\alpha \in (-1,0)$.

If B is a real-valued FBM with the Hurst parameter H and $f \in L^2_H$ then $\int_0^T f dB$ is a zero mean Gaussian random variable with second moment $< f,f >_H$ and more generally if $f,g \in L^2_H$ then $\mathbb{E} \int_0^T f dB \int_0^T g dB = < f,g >_H$. An element $f \in L^2_H$ is necessarily a function if $H \in (0,1/2)$ but it may be a (Schwartz) distribution if $H \in (1/2,1)$.

Consider a control system given by the following controlled linear stochastic differential equation with a fractional Brownian motion

$$\begin{aligned} dX(t) &= (AX(t) + CU(t))dt + dB(t) \\ X(0) &= X_0 \end{aligned} \tag{II.2}$$

where X_0 is an $\mathbb{R}^n$-valued random variable with mean zero and covariance Σ, $X(t) \in \mathbb{R}^n, U(t) \in \mathbb{R}^m, A \in L(\mathbb{R}^n,\mathbb{R}^n), C \in L(\mathbb{R}^m,\mathbb{R}^n)$ and $(B(t), t \geq 0)$ is an $\mathbb{R}^n$-valued standard fractional Brownian motion whose components $(B_1,...,B_n)$ are independent real-valued standard fractional Brownian motions with Hurst parameters $(H_1,...,H_n)$ and $H_j \in (0,1)$ for $j \in \{1,...,n\}$. The random vector X_0 and the process B are independent and are defined on the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $(\mathscr{F}(t), t \in [0,T])$ is the natural filtration for B on $(\Omega, \mathscr{F}, \mathbb{P})$. The existence and the uniqueness of the solution of (II.1) (e.g. [1]) follows from the variation of parameters formula for linear ordinary differential equations. The quadratic cost functional J is

$$\begin{aligned} J(U) &= \frac{1}{2} E[\int_0^T < QX(s), X(s) > + < RU(s), U(s) > ds] \\ & + \frac{1}{2} E < MX(T), X(T) > \end{aligned} \tag{II.3}$$

where $Q > 0, R > 0$, and $M \geq 0$ are symmetric linear transformations, $T > 0$ is fixed, and $<,>$ is the standard inner product on $\mathbb{R}^n$. The family of admissible controls $\mathscr{U}$ is defined as

$\mathscr{U} = \{U : U$ is an $\mathbb{R}^n$-valued process adapted to $(\mathscr{F}(t), t \in [0,T])$ such that $U \in L^2([0,T])$ a.s$\}$

The prediction of a Wiener-type integral (linear functional) of a fractional Brownian motion plays an important role for an explicit description of an optimal control for the control problem (II.2) (II.3). The predictor minimizes the mean squared error so it is the conditional mean. It is necessary to predict the response of the optimal system to the future noise because the noise is correlated. An important property of the optimal predictor is that an explicit expression can be given for it using fractional calculus. The following result is a multidimensional generalization of [2].

Proposition. Let $(B(t), t \geq 0)$ be an $\mathbb{R}^n$-valued fractional Brownian motion whose components $(B_1,...,B_n)$ are independent real-valued standard fractional Brownian motions with Hurst parameters $(H_1,...,H_n)$ where $H_j \in (0,1)$ for $j \in \{1,...,n\}$. Let $(c(t), t \geq 0)$ be an $L(\mathbb{R}^n,\mathbb{R}^n)$-valued continuous function. Then for $0 < s < t$

$$\begin{aligned} \mathbb{E}\left[\int_s^t c dB \mid B(r),\ r \in [0,s]\right] = \Sigma_{j=1}^n \int_0^s u_{1/2-H_j} \\ \left(I^{1/2-H_j}_{s-}\left(I^{(H_j-1/2)}_{t-} u_{H_j-1/2}(c_{1j},....,c_{nj})^T\right)\right) dB \end{aligned} \tag{II.4}$$

where $u_a(s) = s^a$ for $a > 0$ and $s > 0$ and $c = \{c_{ij}\}$.

III. Main Result

The following theorem provides a solution to the control problem (II.1,II.3) by exhibiting an explicit optimal control. This result generalizes the results in [4], [7].

Theorem. For the control problem (II.2) and (II.3) and the family of admissible controls $\mathscr{U}$, there is an optimal control U^* that can be expressed as

$$U^*(t) = -R^{-1}C^T(P(t)X(t)+V(t)) \qquad \text{(III.1)}$$

where $(P(t), t \in [0,T])$ is the unique, symmetric, positive definite solution of the following Riccati equation and V is a prediction of the optimal system response to the future noise as given below

$$\frac{dP}{dt} = -PA - A'P + PCR^{=1}C^TP - Q \qquad \text{(III.2)}$$

$$P(T) = M$$

$$V(t) = \mathbb{E}[\int_t^T \Phi_P(s,t)P(s)dB(s)|\mathscr{F}(t)] \qquad \text{(III.3)}$$

$$= \Sigma_{j=1}^n \int_0^t u_{1/2-H_j} I_{t-}^{1/2-H_j}$$

$$(I_{T-}^{H_j-1/2} u_{H_j-1/2}(\Phi_P(.,t)P)_{.j})dB$$

and Φ_P is the fundamental solution for the matrix equation

$$\frac{dY}{dt} = (A - CR^{-1}C^TP(t))Y \qquad \text{(III.4)}$$

$$Y(0) = I$$

While the proof of this theorem is not given here, some of the ideas and methods for the proof are given. Since the cost functional $J(U)$ is a quadratic functional of the state and the control, the minimization problem has a natural Hilbert space setting. The stochastic system (II.2) is naturally related to an affine deterministic control system. It is natural to try to exploit this naive relationship. In fact by approximating the sample paths of the fractional Brownian motion with a sequence of piecewise smooth processes there is a precise association with a deterministic control problem. However the optimal control obtained by this approach is not adapted to the filtration for the stochastic system, but it can be shown that an optimal adapted control can be obtained from the optimal nonadapted control by an explicit projection method.

The optimal cost contains terms from the contribution of the linear feedback control and from the prediction of the response of the system to the future fractional Brownian motion.

Let $\mathscr{L}_2$ be the Hilbert space of Hilbert-Schmidt linear operators on $\mathbb{R}^n$. the inner product $< .,. >_H$ in (II.1) can be extended to $\mathscr{L}_2$-valued functions in a natural way recalling that if $K, L \in \mathscr{L}_2$ then the inner product on $\mathscr{L}_2$ is $< K, L >_{\mathscr{L}_2} = tr(K^TL) = tr(KL^T)$. The extension of the inner product on L_H^2 to $\mathscr{L}_2$-valued generalized functions K, L [3] is

$$< K, L >_{H,\mathscr{L}_2} = \rho(H) \int_0^T u^2_{\frac{1}{2}-H}(r) \qquad \text{(III.5)}$$

$$< (I_{T-}^{H-\frac{1}{2}} u_{H-\frac{1}{2}} K)(r)(I_{T-}^{H-\frac{1}{2}} u_{H-\frac{1}{2}} L)(r) >_{\mathscr{L}_2} dr$$

Corollary. Let $X_0 = x$ be a constant. The cost for the optimal control $\hat{U}^*$ is

$$J(U^*) = \frac{1}{2}\int_0^T [A_1(t)+A_2(t)+A_3(t)+A_4(t) \qquad \text{(III.6)}$$

$$+A_5(t)+A_6(t)+A_7(t)]dt + B_1 + B_2 + B_3 + B_4$$

where

$$A_1(t) = \quad < \tilde{Q}(t)\Phi_P(t,0)x, \Phi_P(t,0)x >_{\mathscr{L}_2}$$

$$A_2(t) = \int_0^t \int_0^t < \tilde{Q}(t)\Phi_P(t,s)CR^{-1}C^T\Phi_P(.,s)P,$$
$$\Phi_P(t,r)CR^{-1}C^T\Phi_P(.,r)P >_{H,\mathscr{L}_2} drds$$

$$A_3(t) = < \tilde{Q}(t)1_{[0,t]}\Phi_P(t,.), 1_{[0,t]}\Phi_P(t,.)) >_{H,\mathscr{L}_2}$$

$$A_4(t) = -2\int_0^t < \tilde{Q}(t)\Phi_P(t,s)CR^{-1}C^T\Phi_P(.,s)P,$$
$$1_{[0,t]}\Phi_P(t,.) >_{H,\mathscr{L}_2} ds$$

$$A_5(t) = 2\int_0^t < CR^{-1}C^TP(t)\Phi_P(t,s)\Phi_P(.,s),$$
$$\Phi_P(.,t)P >_{H,\mathscr{L}_2} ds$$

$$A_6(t) = 2 < 1_{[0,t]}\Phi_P(t,.), \Phi_P(.,t)P >_{H,\mathscr{L}_2}$$

$$A_7(t) =< CR^{-1}C^T 1_{[0,t]}\Phi_P(.,t)P, 1_{[0,t]}\Phi_P(.,t)P >_{H,\mathscr{L}_2}$$

$$B_2 = \int_0^T \int_0^T < M\Phi_P(T,s)CR^{-1}C^T\Phi_P(.,s)P, \Phi_P(T,r)CR^{-1}C^T\Phi_P(.,r)P >_{H,\mathscr{L}_2} drds$$

$$B_1 =< M\Phi_P(T,0)x, \Phi_P(T,0)x >_{\mathscr{L}_2}$$

$$B_3 =< M\Phi_P(T,.), \Phi_P(T,.)) >_{H,\mathscr{L}_2}$$

$$B_4 = -2\int_0^t < M\Phi_P(T,s)CR^{-1}C^T\Phi_P(.,s)P, \Phi_P(T,.) >_{H,\mathscr{L}_2} ds$$

and $\tilde{Q}(t) = Q + P(t)CR^{-1}C^T P(t)$.

An outline of the computation of the optimal cost, $J(U^*)$, is given now. For simplicity of notation it is assumed that the fractional Brownian motion B is a vector of independent, identically distributed real-valued standard fractional Brownian motions with the Hurst parameter H.

$$J(U^*) = \frac{1}{2}\mathbb{E}[\int_0^T < QX, X > + < RU^*, U^* > dt + < MX(T), X(T) >]$$
$$= \frac{1}{2}\mathbb{E}[\int_0^T (< \tilde{Q}X, X > + < CR^{-1}C^T V, V > +2 < CR^{-1}C^T PX, V >)dt + < MX(T), X(T) >]$$

where $\tilde{Q}(t) = Q + P(t)CR^{-1}C^T P(t)$. It is convenient to decompose the optimal solution X as $X = X_1 + X_2 + X_3$ where

$$X_1(t) = \Phi_P(t,0))X_0 \quad \text{(III.7)}$$
$$X_2(t) = -\int_0^t \Phi_P(t,s)CR^{-1}C^T V(s)ds \quad \text{(III.8)}$$
$$X_3(t) = \int_0^t \Phi_P(t,s)dB(s) \quad \text{(III.9)}$$

To compute the expectations in the cost functional, the above decomposition of X is quite useful. At most it is only necessary to compute expectations of products of Wiener-type stochastic integrals. While there are a number of terms to compute the result is explicit and only requires computing some explicit fractional integrals or derivatives so it does not reduce the optimal cost to the determination of the solution of a Riccati equation.

Since X_1 and (X_2, X_3) are independent and

$$\mathbb{E}X_1(t) = \mathbb{E}X_2(t) = \mathbb{E}X_3(t) = \mathbb{E}V(t) = 0$$

for all $t \in [0,T]$, it follows that

$$\mathbb{E} < \tilde{Q}(t)X(t), X(t) >= \mathbb{E}[< \tilde{Q}(t)X_1(t), X_1(t) > + < \tilde{Q}(t)X_2(t), X_2(t) > + < \tilde{Q}(t)X_3(t), X_3(t) > -2 < \tilde{Q}(t)X_2(t), X_3(t) >]$$

Now the following equalities are satisfied

$$\mathbb{E} < \tilde{Q}(t)X_2(t), X_2(t) >= \int_0^t \int_0^t < \tilde{Q}(t)\Phi_P(t,s)CR^{-1}C^T\Phi_P(.,s)P, \Phi_P(t,r)CR^{-1}C^T\Phi_P(.,r)P >_{H,\mathscr{L}_2} drds$$

$$\mathbb{E} < \tilde{Q}(t)X_3(t), X_3(t) >= < \tilde{Q}(t)1_{[0,t]}\Phi_P(t,.), 1_{[0,t]}\Phi_P(t,.)) >_{H,\mathscr{L}_2}$$

$$\mathbb{E} < \tilde{Q}(t)X_2(t), X_3(t) >= = -\int_0^t < \tilde{Q}(t)\Phi_P(t,s)CR^{-1}C^T \Phi_P(.,s)P, 1_{[0,t]}\Phi_P(t,.) >_{H,\mathscr{L}_2} ds$$

$$\mathbb{E} < CR^{-1}C^T P(t)X_2(t), V(t) >= \int_0^t < CR^{-1}C^T P(t)\Phi_P(t,s)\Phi_P(.,s), \Phi_P(.,t)P >_{H,\mathscr{L}_2} ds$$

$$\mathbb{E} < CR^{-1}C^T P(t)X_3(t), V(t) >= < CR^{-1}C^T 1_{[0,t]}\Phi_P(t,.), \Phi_P(.,t)P >_{H,\mathscr{L}_2}$$

$$\mathbb{E} < CR^{-1}C^T V(t), V(t) >= < CR^{-1}C^T 1_{[0,t]}\Phi_P(.,t)P, 1_{[0,t]}\Phi_P(.,t)P >_{H,\mathscr{L}_2}$$

While the result for the optimal cost (III.6) is explicit, it does not show how the solution of the Riccati equation (III.2) arises in the optimal cost. This is described now. The solution, X_1, is the process that contributes to a quadratic form depending on the solution of the Riccati equation(III.2).

Consider the equation for X_1.

$$dX_1(t) = (A - CR^{-1}C^T P(t))X_1(t)dt \quad \text{(III.10)}$$

$$X_1(0) = X_0 \quad \text{(III.11)}$$

Let $f(t) = \frac{1}{2} < P(t)X_1(t), X_1(t) >$ where P satisfies (III.2).

Integrating $df(t)$ in $[0,T]$ it follows that

$$\frac{1}{2} < P(T)X_1(T), X_1(T) > - \frac{1}{2} < P(0)X_0, X_0 >$$
$$= \int_0^T < P(t)X_1(t), (A - CR^{-1}C^T P(t))X_1(t) >$$
$$\frac{1}{2}\int_0^T < (-P(t)A - A^T P(t) + P(t)CR^{-1}C^T P(t) - Q)X_1(t), X_1(t) > dt$$

Thus

$$\frac{1}{2} < P(0)X_0, X_0 >= \frac{1}{2} < MX_1(T), X_1(T) > + \frac{1}{2}\int_0^T < \tilde{Q}(t)X_1(t), X_1(t) > dt \quad \text{(III.12)}$$

The method of proof of the theorem to obtain an optimal control does not depend on some special properties of a fractional Brownian motion. Thus it can be extended to an arbitrary Gaussian process with continuous sample paths. However, an explicit form for the conditional expectation in (III.1) may be more complicated though it is formally known. Furthermore the methods do not depend on the assumption that the processes are Gaussian so that an optimal control can be obtained for a (noise) process with continuous sample paths and a (uniformly) bounded second moment, though again the explicit form for the conditional expectation in (III.1) may be complicated.

IV. Concluding Remarks

The proof of the optimal control result in the theorem shows that processes other than Brownian motion can be used in linear control systems to model perturbations of a physical system. This control result demonstrates that the usual models of Markov processes or semimartingales for control systems can be extended to a significantly larger class of stochastic processes. The result also demonstrates a close relation between a stochastic control problem and a deterministic control problem and how an optimal control for the latter problem can be used to obtain an optimal control for the former problem.

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