Modeling for control of an inflatable space reflector, the linear 2-D case

T. Voß[1] and J.M.A Scherpen[2]
[1] University of Technology Eindhoven, Faculty of Electrical Engineering, P.O. Box 513, 5600MB Eindhoven, NL
T.Voss@tue.nl
[2] University of Groningen, Faculty of Math. & Natural Sciences, Nijenborgh 4 9747 AG Groningen, NL
J.M.A.Scherpen@rug.nl

Abstract— **In this paper we develop a mathematical model for the dynamics of a linear plate with piezoelectric actuation. This model can then be used to design controllers with the goal of achieving a desired shape of the plate. This control scheme can be used for several applications, e.g., vibration control in structures or shape control for high precision structures like inflatable space reflectors. The starting point of the control design is modeling for control. We will do this in the framework of port-Hamiltonian (pH) modeling, since the pH modeling framework has very nice properties which can be exploited if one wants to design a controller for a specific task. One property for example is that it facilitates modeling multi physics systems or systems which consist of several systems by first modeling all parts separate and then interconnecting them. This is possible because any interconnection of pH systems yields again a pH system. Hence, the pH framework is useful for our multi-domain modeling purpose.**

I. Introduction

Inflatable structures are a very promising technology for space applications [3]. With this emerging technology one is able to build bigger space crafts, which are cheaper in terms of costs but still use the same space in the orbiting device. As a consequence, the developments may enable us to build bigger solar panels and reflectors.

Due to the fact that any inflatable structure is built out of a polymer casing, an inflatable structure cannot have the same surface accuracy as a rigid body. As a possibility for changing the shape of a reflector one could use smart materials which have the possibility to change their properties on demand, e.g., piezoelectric polymers [14]. This means that with smart materials it is actually possible to change the shape of an element by means of an applied voltage.

Fig. 1. An inflatable space reflector test setup of the company L'garde (www.lgarde.com)

In this paper we consider a state-of-the-art inflatable space reflector such as the one illustrated in Figure 1. The structure consists of an inflatable torus to which an inflatable lens is fixed. The torus itself is then attached to the space craft by three or more inflatable booms. Note that one side of the surface of the lens is transparent to the reflected radiation, hence this side has no requirements on surface accuracy. The other side of the lens is coated with a reflecting surface, e.g., a very thin aluminum coating. This surface will then be used to focus the radiation we want to observe (e.g., the light of a star) to the sensor array, which is mounted to the satellite. But in order to ensure that the radiation is focused exactly on the sensor array the reflecting surface has to have a specific surface accuracy. To achieve the required surface accuracy we need active control.

To be able to change the shape of the reflector one can use several hundred actuators patches, made of piezoelectric polymers, which are bonded to the actual reflective shell (here after we call the shell of the reflector also the base layer). Moreover, in order to be able to change the shape of the reflecting surface locally, the actuators are spread out over the whole surface. If one applies a current to the actuators, the piezoelectric material will change its length and due to the bonding to the shell of the reflector the reflecting surface will bend locally. The final goal is then to develop a control algorithm which uses the piezoelectric effect to remove disturbances on the surface. But to be able to do this one needs a mathematical model which describes the dynamics of the real world object to be controlled.

In this paper we show how to develop a model for a linear plate with piezo actuation in the port-Hamiltonian (pH) modeling framework [13], [7]. Note that we choose as modeling framework the pH framework due to its excellent properties. For example a system in pH form is automatically passive, and hence it does not generate any energy. Note that passive systems with dissipation and no external influences have the property that they converge to their equilibrium. One can then exploit the interconnection and the equilibrium property of pH systems to design a controller. One of the most important properties of pH systems is that the interconnection of pH models is done by describing the energy flow between the two systems. This makes the pH modeling framework very useful if one wants to model multi-physics phenomena and/or very complex systems. Hence, one can model one simple domain/subsystem of the total system, e.g., model the mechanical domain of an electromechanical system. Then one determines the energy exchange of that subsystem, e.g., via constitutive equations. From the description of the energy

ISBN 978-963-311-370-7

exchange between the systems one can easily determine an interconnection law to interconnect all subsystems. Note that, since one only uses the description of energy flow between the systems, it is also possible to model mixed finite-infinite dimensional systems like flexible link manipulators. This leads to a very efficient way to model complex systems in a rather simple way, by dividing the complex model in several simple models which are then interconnected in an energy conserving manner.

The framework of pH systems has been successfully used to model mechanical, electrical, chemical, and electromechanical systems for both finite dimensional and infinite dimensional phenomena, see [12], [6], [1]. Of course a model in pH form will be equivalent to models in other frameworks, but they do not share the properties which make models in pH framework so suitable for control.

In this paper we show how to model a purely piezoelectric plate. In the past we also developed a model for a nonlinear Euler-Bernoulli beam [16] and a nonlinear Timoshenko beam [15]. Although the beam models were a good starting point due to their lower complexity, these models were just examples to show that our modeling and control strategy works in practice. In [5] a model for a Mindlin plate with linear deformations was derived. The approach proposed here differs from the one presented in [5], since we derive a model which uses the piezoelectric effect for actuation. Additionally we derive the equations of motion by using the generalized Hamiltonian's principle, see [8].

The paper is organized as follows. In Section II we introduce shortly the physics that we use to derive the 2-D model of a piezoelectric composite in the linear plate framework. Next in Section III we show how to derive a model of a linear piezoelectric plate with a dynamic electrical field in the pH framework.

Note that the proposed model can also be used for modeling other structures, namely any flexible structure with piezo actuation, e.g., for vibration control in civil engineering.

II. Background on Continuum dynamics And the piezoelectric effect

In this section we briefly introduce the physics to be used in the following sections. We focus on linear materials and large deformations [11]. The reason why we consider large deformations in combination with linear materials is that we want to derive a model which can be used for control design. If one designs a controller it is desired that the controller is robust. Robustness (in the area of control) means that the controller is able to achieve the goal with a certain accuracy even if the mathematical model which was used to design the controller is not an exact representation of the real world situation. Hence, the usage of linear material properties is a valid assumption for modeling in the case of control design, since the controller will be able to handle the modeling errors which are caused by using linear material properties. For the sake of simplicity of notation we will omit the spatial and time dependency if it is clear from the context.

We start by defining the strain ε of the plate which is a measure of deformation of the plate. The strain ε in the plate is related to the deformation $\mathbf{u}$ of the plate. The deformation of the plate is described as a vector which gives the deformation in the z_1 direction ($\mathbf{u}_1$), z_2 direction ($\mathbf{u}_2$) and z_3 direction ($\mathbf{u}_3$). Moreover, the deformation is defined for any point in the plate, hence $\mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]^\top$ depends on the position $\mathbf{z} = [z_1,\, z_2,\, z_3]^\top$ in the plate. The strain of the plate is

$$\varepsilon_{ij} \quad = \quad \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial z_j} + \frac{\partial \mathbf{u}_j}{\partial z_i} + \sum_{k=1}^{3} \frac{\partial \mathbf{u}_k}{\partial z_i}\frac{\partial \mathbf{u}_k}{\partial z_j}\right) \tag{1}$$

where $i, j \in \{1, 2, 3\}$. Hence, the strain at a given point consists of 9 components but only 6 of the components are unique since it holds that $\varepsilon_{31} = \varepsilon_{13}$, $\varepsilon_{21} = \varepsilon_{12}$ and $\varepsilon_{23} = \varepsilon_{32}$. So, we are able to write the strain in a 6 dimensional vector.

$$\varepsilon = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{13}, \varepsilon_{12}, \varepsilon_{23}]^\top$$

where ε_{11}, ε_{22}, and ε_{33} are the normal strains in z_1, z_2 and z_3 direction respectively and ε_{13}, ε_{12}, and ε_{23} are the shear strains. Every element of the strain vector is a continuous scalar function of space, i.e. $\varepsilon_{ij} \in \mathcal{C}^0 : \mathbb{R}^3 \to \mathbb{R}$.

At any point where the plate is deformed ($\varepsilon \neq 0$) a stress will be present. For example if the plate is stretched in the z_1 direction the strain ε_{11} will be positive. This stretching yields a stress also in the z_1-direction which we denote as σ_{11}. Hence, there exists a relation between stress and strain. The stress in the plate at a certain position is described in the same way as the strain, so it consists of 6 components where the first 3 describe the normal stresses and the last 3 describe the shear stresses at a given point $\left(\sigma_{ij} \in \mathcal{C}^0 : \mathbb{R}^3 \to \mathbb{R}\right)$. Here we use linear material properties (Hooke's Law) for the relation between stress and strain which can be stated as

$$\sigma = \mathbf{C}\varepsilon$$

where $\mathbf{C} \in \mathbb{R}^{6\times 6}$ is the material stiffness matrix which relates stress and strain. In this paper we assume only homogeneous materials, so the matrix $\mathbf{C}$ is a constant matrix.

For piezoelectric materials the piezoelectric effect can induce an additional stress in the material which is caused by an electrical field (actuation property). Similarly the deformation of the piezoelectric element also changes the electrical field in the element (sensing property). These properties result into coupled constitutive relations for piezoelectric materials [10] and can be described as

$$\begin{bmatrix} \sigma \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & -\mathbf{h}^\top \\ \mathbf{h} & \epsilon^e \end{bmatrix} \begin{bmatrix} \varepsilon \\ \mathbf{D} \end{bmatrix} \tag{2}$$

here $\mathbf{D} = [\mathbf{D}_1,\, \mathbf{D}_2,\, \mathbf{D}_3]^\top$ is the electrical displacement and $\mathbf{E} = [\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3]^\top$ is the electrical field in the piezo element at a specific point in space. Each element of the electrical displacement and the electrical field is an element of $\mathcal{C}^0 : \mathbb{R}^3 \to \mathbb{R}$. The parameter $\epsilon^e \in \mathbb{R}^{3\times 3}$ is the electrical permittivity matrix and describes the relation between the electrical displacement and the electrical field. The piezoelectric constant matrix $\mathbf{h} \in \mathbb{R}^{3\times 6}$ of the material describes the relation between the electrical displacement $\mathbf{D}$ and the stress σ. We assume that all material property matrices are constant and so spatially independent.

In this paper we derive the equations of motion for the model via the generalized Hamilton's principle [8]. This principle states that for a piezoelectric material it must hold that

$$\delta \int_{t_0}^{t_1} (K - P + W) dt = 0 \tag{3}$$

where K is the kinetic energy, P is the potential energy and W is the external energy of the plate. We will simplify (3) until we obtain the equations of motions in a form which is similar to the one which is used in classical mechanical literature, e.g. [9]. For all these energies V is the volume of the structure and B expresses the surface. We denote by $\int_V \circ dV$ the volume integral and by $\oint_B \circ dB$ the surface integral of the given structure.

III. pH MODELING OF A LINEAR PIEZOELECTRIC PLATE WITH A DYNAMIC ELECTRICAL FIELD

In this section we derive a linear infinite dimensional pH model of a piezoelectric plate. The model that we derive is different than the one derived in [5] as follows. In [5] the authors treat a classical mindlin plate which is the simplest plate model. The model that we derive here can be easily extended to very complex plate models. Additionally, the authors of [5] treat a purely mechanical plate with boundary actuation. Although boundary actuation is sufficient for several applications, we are focusing on a piezoelectric plate with an in-plane actuation, due to the actuation limitations in space. The addition of piezoelectric elements then introduces nontrivial electromechanical coupling issues.

The derivation of the pH model is performed similar to the derivation of the beam models presented in [15] and can be split in four parts as follows: we first derive the strain and the geometry of the piezoelectric plate, secondly we define the stored energy, thirdly we derive the equations of motion, and, finally, we derive the interconnection structure. Due to the fact that these four steps are quite evolving for the 2-D case we will first derive a model for a plate without a piezoelectric effect and then add the piezo effect.

Now we determine the strain in the plate which is caused by its deformation. This is the first step that has to be taken in order to determine a distributed pH model for a plate. The strain will then be used to calculate the strain energy (potential energy) stored in the plate which is induced due to deformation. For a plate it is in general assumed that the following displacement takes place

$$\mathbf{u} = \begin{bmatrix} u_0(z_1, z_2) - z_3\phi_u(z_1, z_2) \\ v_0(z_1, z_2) - z_3\phi_v(z_1, z_2) \\ w(z_1, z_2) \end{bmatrix}$$

where $u_0(z_1, z_2)$ and $v_0(z_1, z_2)$ are the displacements of a material point at the neutral line of the plate in z_1 and z_2 direction, respectively. The deformation in z_3 direction is given by $w(z_1, z_2)$. The rotation of the cross section in z_1 and z_2 direction is given by $\phi_u(z_1, z_2)$ and $\phi_v(z_1, z_2)$, respectively.

Next we apply (1) to our deformation vector. Because we treat a linear plate, we assume that $\mathbf{u}_{i,j} \ll 1$ for all $i, j \in \{1, 2, 3\}$, we neglect all quadratic terms in our strain. Hence, we obtain the following simplified strains

$$\begin{aligned} \varepsilon_{11} &= u_{0,1} - z_3\phi_{u,1} \\ \varepsilon_{22} &= v_{0,2} - z_3\phi_{v,2} \\ \varepsilon_{12} &= u_{0,2} - z_3\phi_{u,2} + v_{0,1} - z_3\phi_{v,1} \\ \varepsilon_{13} &= \frac{1}{2}(w_1 - \phi_u) \\ \varepsilon_{23} &= \frac{1}{2}(w_2 - \phi_v) \end{aligned} \tag{4}$$

where the subscript ,1 and ,2 are used to denote a spatial derivative in z_1 and z_2 coordinate respectively, e.g. $u_{0,1} = \frac{\partial}{\partial z_1} u_0$. We define the geometry of the plate as follows (see also Figure 2). Let L_1 be the length of the piezoelectric plate in the direction z_1, so, $z_1 \in [0, L_1]$. Let L_2 be the length of the piezoelectric plate in the direction z_2, so, $z_2 \in [0, L_2]$. Finally, let h be the height of the plate, so, we assume that $z_3 \in [0, h]$. Furthermore, we define the upper surface of the plate as $A_u = \int_0^{L_1} \int_0^{L_1} dxdy = L_1 \cdot L_2$.

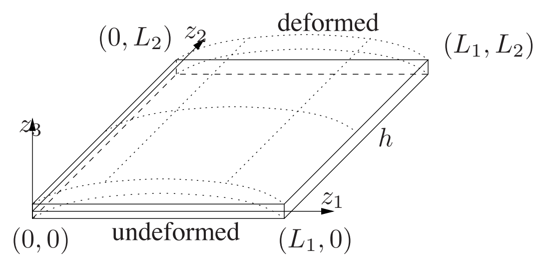

Fig. 2. Geometry of the considered plate

A. Hamiltonian of a plate

The general Hamiltonian of a purely mechanical plate is the same as the Hamiltonian of the beam models [15] and has the following general form

$$H(\mathbf{u}, \varepsilon) = \frac{1}{2} \int_V \rho \|\dot{\mathbf{u}}\|^2 + \sigma^\top \varepsilon dV.$$

The Hamiltonian is defined as a volume integral, but since we want to describe a plate it is clear that the z_3 dimension is obsolete. The reduction to 2-D can be performed by integrating over the coordinate that the strain variables are not depending on. Hence, we integrate over the z_3 coordinate. We split the derivation of the simplified version in two parts.

1) Kinetic energy: The Kinetic energy of a plate is given by

$$K = \frac{1}{2} \int_V \rho \|\dot{\mathbf{u}}\|_2^2 dV \tag{5}$$

here ρ is the density of the used material and $\dot{\mathbf{u}}$ is the velocity of a specific atom. By integrating (5) over the height of the beam we obtain

$$\begin{aligned} K = \frac{1}{2} \int_{A_u} \rho \Big(& h\dot{u}_0^2 - 2I_0\dot{u}_0\dot{\phi}_u + I\dot{\phi}_u^2 + h\dot{v}_0^2 \\ & -2I_0\dot{v}_0\dot{\phi}_u + I\dot{\phi}_u^2 + h\dot{w}^2 \Big) dA_u \end{aligned}$$

with $I = \int_0^h z_3^2 dz$, $I_0 = \int_0^h z_3 dz$. Now we can redefine the kinetic energy in a quadratic form

$$K(\mathbf{p}) = \frac{1}{2}\int_{A_u} \mathbf{p}^\top \mathbf{M}^{-1}\mathbf{p} dA_u \quad (6)$$

where

$$\mathbf{p} := \mathbf{M}\frac{\partial}{\partial t}\dot{\tilde{\mathbf{u}}} = \rho \begin{bmatrix} h & 0 & 0 & -I_0 & 0 \\ 0 & h & 0 & 0 & -I_0 \\ 0 & 0 & h & 0 & 0 \\ -I_0 & 0 & 0 & I & 0 \\ 0 & -I_0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \dot{u}_0 \\ \dot{v}_0 \\ \dot{w}_0 \\ \dot{\phi}_u \\ \dot{\phi}_v \end{bmatrix}.$$

The matrix $\mathbf{M}$ is also called the mass matrix and $\dot{\tilde{\mathbf{u}}}$ is the vector of velocity parameters defined as $\dot{\tilde{\mathbf{u}}} = \left[\dot{u}_0, \dot{v}_0, \dot{w}, \dot{\phi}_u, \dot{\phi}_v\right]$. The momenta vector $\mathbf{p}$ consists of the following elements: $\mathbf{p}_1$ which is the moment in the z_1 direction, $\mathbf{p}_2$ which is the moment in the z_2 direction, $\mathbf{p}_3$ which is the moment in the z_3 direction, $\mathbf{p}_4$ which is the angular moment of the cross section around the z_1 axis, and $\mathbf{p}_5$ which is the angular moment of the cross section around the z_2-axis.

2) Potential energy: The potential energy stored in the plate can be described as

$$P = \frac{1}{2}\int_{A_u}\int_0^h \sigma^\top \varepsilon dz_3 dA_u.$$

The constitutive equations (2) relate the stresses and fluxes in the plate as follows

$$\begin{aligned} \sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} \\ \sigma_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} \\ \sigma_{12} &= G\varepsilon_{12}, \quad \sigma_{13} = G\varepsilon_{13} \\ \sigma_{23} &= G\varepsilon_{23}. \end{aligned}$$

Hence, the potential energy is given by

$$\begin{aligned} P = \frac{1}{2}\int_{A_u}\int_0^h & (\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + 2\sigma_{12}\varepsilon_{12} + \quad (7) \\ & 2\sigma_{13}\varepsilon_{13} + 2\sigma_{23}\varepsilon_{23})\, dz_3 dA_u. \end{aligned}$$

Recall $\varepsilon_{ij} = \varepsilon_{ji}$, $\forall i \neq j$. Now we are able to state the total stored energy in the mechanical plate.

3) Simplified Hamiltonian of a plate: Combining the two results (6) and (7) yields the simplified Hamiltonian

$$\begin{aligned} H = \frac{1}{2}\int_{A_u} & \mathbf{p}^\top \mathbf{M}^{-1}\mathbf{p} + \int_0^h \sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} \quad (8) \\ & + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{13}\varepsilon_{13} + 2\sigma_{23}\varepsilon_{23} dz_3 dA_u \end{aligned}$$

The expression of the here defined Hamiltonian will be used in Section III-C for the definition of the pH model.

4) Variational derivative of the Hamiltonian: Next we have to derive the variational derivative with respect to the momenta $\mathbf{p}$ and the chosen strain parameters $\tilde{\varepsilon}$. This step is necessary since we want to define the pH model of the plate by using the generalized Hamiltonian's principle (3). We are using the following strain parameters

$$\tilde{\varepsilon} = [u_{0,1}, u_{0,2}, v_{0,1}, v_{0,2}, w_{,1}, w_{,2}, \phi_u, \phi_{u,1}, \phi_{u,2}, \phi_v, \phi_{v,1}, \phi_{v,2}]^\top$$

Note that since these calculations are quite evolving we are splitting them into several parts. We start by deriving the variational derivative with respect to the momenta. To this aim, we first define

$$\tilde{\mathbf{p}} = \mathbf{p} + \delta\mathbf{p} = \mathbf{p} + \xi\varphi$$

where $\xi > 0$ and φ is an arbitrary function which is zero at the boundary of A_u. Substituting this into (6) and using a Taylor expansion yields then

$$\begin{aligned} K(\tilde{\mathbf{p}}) &= \frac{1}{2}\int_0^L \tilde{\mathbf{p}}^\top M^{-1}\tilde{\mathbf{p}} dz_1 = \int_0^L F_k(\tilde{\mathbf{p}}) dz_1 \\ &= \int_0^L F_k(\mathbf{p}) + \frac{\partial F_k}{\partial \mathbf{p}}\xi\varphi + \mathcal{O}(\xi^2) dz_1 \\ &= K(p) + \delta K(p). \end{aligned}$$

So, we obtain that

$$\begin{aligned} \delta K(\mathbf{p}) &= \frac{1}{2}\int_0^L \frac{\partial}{\partial \mathbf{p}}\left(\mathbf{p}^\top M^{-1}\mathbf{p}\right)\delta\mathbf{p} dz_1 \\ &= \int_0^L M^{-1}\mathbf{p}\delta\mathbf{p} dz_1. \end{aligned}$$

Furthermore, since it holds that $\delta K = \int_0^L \frac{\delta K}{\delta \mathbf{p}}\delta\mathbf{p} dz_1$ we obtain that the variational derivative of K with respect to $\mathbf{p}$ is given by

$$\frac{\delta K}{\delta \mathbf{p}} = M^{-1}\mathbf{p} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{\phi}_u \\ \dot{\phi}_v \end{bmatrix}. \quad (9)$$

The next step is to derive the variational derivatives with respect to the strain parameters $\tilde{\varepsilon}$. The derivation of the variational derivative of the potential energy is exactly the same as for the kinetic energy, so we skip the derivation here. Also, due to the complexity of the variational derivative we present here only the variational derivative with respect to $u_{0,1}$

$$\frac{\delta P}{\delta u_{0,1}} = \int_0^h \sigma_{11} dz_3 \quad (10)$$

and leave the rest to the interested reader. This concludes the computation of the variational derivative of the Hamiltonian.

B. Equations of motion of a 2-D plate

The equations of motion for a plate which is pressurized from below are derived with the method of extended Hamiltonian principle [8]. Note that since the principle of least action must hold the variation of the stored energy has to be equal to zero. So,

$$\delta(K - P + W) = 0. \quad (11)$$

Next we have to calculate the variational derivatives of the kinetic (6) and the potential energy (7) of the plate with respect to displacements parameters $\tilde{\mathbf{u}} = [u_0, v_0, w, \phi_u, \phi_v]^\top$. Since the Hamiltonian (8) corresponding to a 2-D plate has a complex expression, for the sake of simplicity of explanation we split the derivation of its variational derivatives into several parts. We start by deriving the variational derivative of the kinetic energy (6).

1) Variation of the kinetic energy: To be able to derive the variation of the kinetic energy we have to define variations of the velocities, e.g. for the velocity in the z_1-direction

$$\dot{\tilde{u}}_0 = \dot{u}_0 + \delta\dot{u}_0 = \dot{u}_0 + \xi\varphi_u$$

where $\xi > 0$ and φ_u is an arbitrary function which is assumed to be equal to zero at the boundaries of A_u. All other variations are defined similarly. By substituting the variations into the kinetic energy equation and by using the Taylor expansion we obtain

$$\begin{aligned} K &= \int_{A_u} F_K(\dot{\tilde{u}}_0, \dot{\tilde{v}}_0, \dot{\tilde{w}}, \dot{\tilde{\phi}}_u, \dot{\tilde{\phi}}_v) dA_u \\ &= \int_{A_u} F_K(\dot{u}_0, \dot{v}_0, \dot{w}, \dot{\phi}_u, \dot{\phi}_v) + \frac{\partial F_K}{\partial \dot{u}_0}\xi\varphi_u + \frac{\partial F_K}{\partial \dot{v}_0}\xi\varphi_v \\ &\quad + \frac{\partial F_K}{\partial \dot{\phi}_u}\xi\varphi_{\phi_u} + \frac{\partial F_K}{\partial \dot{\phi}_v}\xi\varphi_{\phi_v} + \frac{\partial F_K}{\partial \dot{w}}\xi\varphi_w + \mathcal{O}(\xi^2) dA_u. \end{aligned}$$

Hence, the variational derivative is given by

$$\delta K = M\dot{\tilde{\mathbf{u}}}\delta\dot{\tilde{\mathbf{u}}}.$$

Moreover, in order to be able to derive the equations of motions we have to derive the variational derivative with respect to the displacements. This can be accomplished when using integration by parts and when using the fact that $\delta\tilde{\mathbf{u}} = 0$ at the boundary of A_u as follows

$$\int_{t_0}^{t_1} \delta K dt = \int_{t_0}^{t_1} \mathbf{p}\delta\dot{\tilde{\mathbf{u}}} dt = -\int_{t_0}^{t_1} \dot{\mathbf{p}}\delta\tilde{\mathbf{u}} dt.$$

This is the first step towards deriving the equations of motion for a purely mechanical plate. The next step is to derive the variational derivative of the potential energy.

2) Variation of the potential energy: Now derive the variational derivative of the potential energy with respect to the deformation so that later on we will be able to derive the equations of motion. However, since the variation of the potential energy is quite complex, but straightforward, and is essentially the same as for the kinetic energy we just present here the results.

The variation of the potential energy with respect to the deformations is given by

$$\begin{aligned} \delta P = \int_{A_u} & \left(-\partial_1 \int_0^h \sigma_{11} dz_3 - \partial_2 \int_0^h \sigma_{12} dz_3\right) \delta u_0 \\ & + \left(-\partial_2 \int_0^h \sigma_{22} dz_3 - \partial_1 \int_0^h \sigma_{12} dz_3\right) \delta v_0 \\ & + \left(-\partial_1 \int_0^h w_{,2}\sigma_{12} dz_3 - \partial_2 \int_0^h w_{,1}\sigma_{12} dz_3\right. \\ & \left. -\partial_1 \int_0^h \sigma_{13} dz_3 - \partial_2 \int_0^h \sigma_{23} dz_3\right) \delta w \\ & + \left(-\partial_1 \int_0^h -z_3\sigma_{11} dz_3 - \partial_2 \int_0^h -z_3\sigma_{12} dz_3\right. \\ & \left. - \int_0^h \sigma_{13} dz_3\right) \delta\phi_u + \left(-\partial_2 \int_0^h -z_3\sigma_{22} dz_3\right. \\ & \left. -\partial_1 \int_0^h -z_3\sigma_{12} dz_3 - \int_0^h \sigma_{13} dz_3\right) \delta\phi_v dA_u. \end{aligned}$$

Note that we used ∂_i with $i \in \{1, 2\}$ to express $\frac{\partial}{\partial z_i}$. The last step before we can state the equations of motion is to calculate the variation of the external influences. We assume here that the plate is pressurized from below and all other sides have no external influences. Then we obtain

$$\begin{aligned} \delta W &= \int_{A_u} \mathbf{f}_u \; \delta\mathbf{u}_1|_{z_3=0} + \mathbf{f}_v \; \delta\mathbf{u}_2|_{z_3=0} + \mathbf{f}_w \; \delta\mathbf{u}_3|_{z_3=0} \, dA_u \\ & \int_{A_u} \mathbf{f}_u\delta u_0 + \mathbf{f}_v\delta u_0 + \mathbf{f}_w\delta w dA_u. \end{aligned}$$

To fulfill (11) the variation of the total energy has to be zero at any point (z_1, z_2) in the plate. So, the terms in the integrand have to be zero at every point $(z_1, z_2) \in A_u$ which yields the following equations of motion for a plate

$$\begin{aligned} \dot{\mathbf{p}}_1 &= \partial_1 \int_0^h \sigma_{11} dz_3 + \partial_2 \int_0^h \sigma_{12} dz_3 + \mathbf{f}_u \qquad (12) \\ \dot{\mathbf{p}}_2 &= \partial_1 \int_0^h \sigma_{12} dz_3 + \partial_2 \int_0^h \sigma_{22} dz_3 + \mathbf{f}_v \\ \dot{\mathbf{p}}_3 &= \partial_1 \int_0^h w_{,2}\sigma_{12} dz_3 + \partial_2 \int_0^h w_{,1}\sigma_{12} dz_3 \\ &\quad + \partial_1 \int_0^h \sigma_{13} dz_3 + \partial_2 \int_0^h \sigma_{23} dz_3 + \mathbf{f}_w \\ \dot{\mathbf{p}}_4 &= \partial_1 \int_0^h -z_3\sigma_{11} dz_3 + \partial_2 \int_0^h -z_3\sigma_{12} dz_3 + \int_0^h \sigma_{13} dz_3 \\ \dot{\mathbf{p}}_5 &= \partial_2 \int_0^h -z_3\sigma_{22} dz_3 + \partial_1 \int_0^h -z_3\sigma_{12} dz_3 + \int_0^h \sigma_{13} dz_3 \end{aligned}$$

These equations describe the dynamics of the mechanical plate. However we also have to define the equations of motion corresponding to the strain parameters. But, since their derivation is straightforward we just state the equation of motion for $u_{0,1}$

$$\dot{u}_{0,1} = \frac{\partial}{\partial z_1}\dot{u}_0 \qquad (13)$$

and leave the rest to the interested reader. These equations of motion are used in the next section to define the interconnection structure for the pH model of the plate.

C. Interconnection structure of a 2-D plate

Now we use the results of the last subsections to derive an interconnection structure which is able to represent the equations of motion of the system in pH form.

As state of the pH system we choose the momenta and the strain parameters, so $\mathbf{x} = [\mathbf{p}, \tilde{\varepsilon}]^\top$. The variational derivative of the Hamiltonian (8) with respect to the state is given by

$$\delta_{\mathbf{x}} H = \begin{bmatrix} \delta_{\mathbf{p}} H \\ \delta_{\tilde{\varepsilon}} H \end{bmatrix}$$

where $\delta_{\mathbf{p}} H$ is the variational derivative of the Hamiltonian with respect to the momenta $\mathbf{p}$, defined by (9) and $\delta_{\tilde{\varepsilon}}$ is the variational derivative with respect to the strain parameters $\tilde{\varepsilon}$, defined by (10).

So, using the state variables and the variation of the Hamiltonian we can rewrite (12) and (13) in terms of $\delta_{\mathbf{x}} H$. For example for the equation of motion of $\mathbf{p}_1$ we obtain

$$\dot{\mathbf{p}}_1 = \partial_1 \frac{\delta P}{\delta u_{0,1}} + \partial_2 \frac{\delta P}{\delta u_{0,2}} + \mathbf{f}_u$$

We do this for all equations of motions. Then we can write the following pH model

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\tilde{\varepsilon}} \end{bmatrix} &= \begin{bmatrix} 0 & \mathbf{J}_m \\ -\mathbf{J}_m^* & 0 \end{bmatrix} \delta_{\mathbf{x}} H + B_m \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_v \\ \mathbf{f}_w \end{bmatrix} \\ \mathbf{y} &= B_m^\top \nabla H \end{aligned} \quad (14)$$

where

$$\begin{aligned} \mathbf{J}_m &= \begin{bmatrix} d & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_2 & D_1 & D_2 \end{bmatrix} \\ B_m^\top &= \begin{bmatrix} I_3 & 0 \end{bmatrix} \\ d &= \begin{bmatrix} \partial_1 & \partial_2 \end{bmatrix} \\ D_1 &= \operatorname{diag}(\partial_1, \partial_2) \\ D_1 &= \operatorname{diag}(\partial_2, \partial_1) \end{aligned}$$

Now that we have derived the equations of motion for a purely mechanical plate. Next we add the piezoelectric effect so that later on we can to actuate the system.

D. Adding the piezoelectric effect with a dynamic electrical field

Before we can determine the piezoelectric effect in the material we have to define the geometry of the piezoelectric element. This has to be done so that afterwards we are able to describe the electromagnetic field which is used to actuate the system. We assume that the piezoelectric layer that we consider here has two electrodes bonded to it. These electrodes are then used to induce the electromagnetic field. Note that since we have assumed that our plate is laying in the $z_1 z_2$ plane, the obvious choice is to assume that the electrodes are also in the $z_1 z_2$ plane. This means that we can assume, similar to a plate capacitor, that the electrical field E has only the z_3 component. Hence, $E_1 = E_2 = 0$. Then we derive the dynamics of the electromagnetic field by using 2-D Maxwell's equations, see e.g. [4]. For the interaction between the mechanical and the electrical domain we assume that the constitutive equations (2) hold. Different to the standard definition of piezoelectric material [10] we do not neglect the magnetic field by assuming a quasi static electrical field. The reason for this is that we can show for the case of a piezoelectric beam [**?**], that neglecting the dynamics of the magnetic field yields a system which cannot be stabilized by means of piezoelectric actuation.

1) 2-D version of Maxwell's equations: To derive the dynamics of the electromagnetic field which will be used to actuate the system we use as base Maxwell's equations. But because we assumed that $E_1 = E_2 = 0$, we start with a simplification of Maxwell's equations. Due to this assumption we can transform Maxwell's equations given by

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{D} &= \nabla \times \mathbf{H}_m \\ -\frac{\partial}{\partial t} \mathbf{B}_m &= \nabla \times \mathbf{E} \end{aligned}$$

into a simpler form

$$\begin{aligned} -\dot{B}_{m,1} &= \frac{\partial}{\partial z_2} E_3 \\ -\dot{B}_{m,2} &= -\frac{\partial}{\partial z_1} E_3 \\ \dot{D}_3 &= \frac{\partial}{\partial z_1} H_{m,2} - \frac{\partial}{\partial z_2} H_{m,1}. \end{aligned}$$

Hence, the magnetic field has only a z_1 and z_2 component $(H_3 = 0)$. It holds that $B_{m,1} = \mu H_{m,1}$, $B_{m,2} = \mu H_{m,2}$, and $D_3 = \epsilon^e E_3$, with μ the permeability and ϵ^e the permittivity of the piezoelectric material. These equations must still hold for any point in the plate. But since we treat a 2-D plate we want to express the Maxwell's equations in a two dimensional way by reducing one dimension. To simplify the calculations we neglect for the moment the coupling between the mechanical and electrical domain and treat the independent Maxwell's equations. We start with calculating the charge and the flux induced by the $\mathbf{D}$ and $\mathbf{B}_m$ field on the cross sectional areas which are penetrated by the field as follows:

$$\begin{aligned} q &= \int_{A_u} \mathbf{D} \cdot dA_u \\ \phi_1^e &= \int_{A_1} \mathbf{B}_m \cdot dA_1 \\ \phi_2^e &= \int_{A_2} \mathbf{B}_m \cdot dA_2 \end{aligned}$$

where A_1 is the cross section laying in the $z_1 z_3$ plane and A_2 is the cross section laying in the $z_2 z_3$ plane. If we integrate over the penetrated surfaces we obtain

$$\begin{aligned} q &= L_1 \cdot L_2 \cdot D_3 \\ \phi_1^e &= -h \cdot L_1 \cdot B_{m,1} \\ \phi_2^e &= -h \cdot L_2 \cdot B_{m,2}. \end{aligned}$$

From this result it follows that the charge and the flux densities on the surface A_u can be described as follows

$$q = D_3, \quad \phi_1^e = -hB_{m,1}, \quad \phi_2^e = -hB_{m,2}.$$

Then we can rewrite the energy function in terms of charge and flux distribution

$$\begin{aligned} H &= \frac{1}{2}\int_{A_u}\int_0^h H_{m,1}B_{m,1} + H_{m,2}B_{m,2} + D_3E_3 dz_3 dA_u \\ &= \frac{1}{2}\int_{A_u}\frac{1}{\mu h}\phi_1^{e^2} + \frac{1}{\mu h}\phi_2^{e^2} + \frac{h}{\epsilon^e}q^2 dA_u. \end{aligned}$$

The variational derivative of the transformed energy function is given by

$$\begin{aligned} \frac{\delta H}{\delta \phi_1^e} &= \frac{1}{\mu h}\phi_1^e, \quad \frac{\delta H}{\delta \phi_2^e} = \frac{1}{\mu h}\phi_2^e \\ \frac{\delta H}{\delta q} &= \frac{h}{\epsilon^e}q. \end{aligned}$$

Next we have to express the Maxwell's equations in the new coordinates (ϕ_1^e, ϕ_2^e, q). To be able to do this we integrate the first two equations of motions over the height and substitute the variational derivatives

$$\begin{aligned} -h\dot{B}_{m,1} &= \frac{\partial}{\partial z_2}hE_3 \Rightarrow \dot{\phi}_1^e = \frac{\partial}{\partial z_2}\frac{\delta H}{\delta q} \\ -h\dot{B}_{m,2} &= -\frac{\partial}{\partial z_1}hE_3 \Rightarrow \dot{\phi}_2^e = -\frac{\partial}{\partial z_1}\frac{\delta H}{\delta q} \\ \dot{D}_3 &= \frac{\partial}{\partial z_1}H_{m,2} - \frac{\partial}{\partial z_2}H_{m,1} \\ \Rightarrow \dot{q} &= -\frac{\partial}{\partial z_1}\frac{1}{\mu h}(-hB_{m,2}) + \frac{\partial}{\partial z_2}\frac{1}{\mu h}(-hB_{m,1}) \\ &= -\frac{\partial}{\partial z_1}\frac{\delta H}{\delta \phi_2} + \frac{\partial}{\partial z_2}\frac{\delta H}{\delta \phi_1} \end{aligned}$$

Then we recast the equations of motion of the 2-D Maxwell's equations in the following pH system

$$\begin{bmatrix} \dot{\phi}_1^e \\ \dot{\phi}_2^e \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \partial_2 \\ 0 & 0 & -\partial_1 \\ \partial_2 & -\partial_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta \phi_1^e} \\ \frac{\delta H}{\delta \phi_2^e} \\ \frac{\delta H}{\delta q} \end{bmatrix}.$$

We will use the new derived coordinates to update the energy function of the coupled system.

2) Hamiltonian of the coupled system: The stored energy of the 2-D piezoelectric plate can be stated as follows

$$H = \int_V \rho\,\|\mathbf{u}\|_2^2 + \sigma^\top\varepsilon + \mathbf{B}_m\mathbf{H}_m + DEdV.$$

The energy function describes the energy stored in the whole volume. So, because we want to describe the plate as a 2-D object we will integrate over the thickness of the plate and also replace the electromagnetic fields with the derived charge and flux densities. For simplicity we neglect the kinetic energy $\int_V \mathbf{p}^\top M^{-1}\mathbf{p} + \mathbf{B}_m\mathbf{H}_m dV$ which is unchanged by the coupling of the two PDEs.

$$\begin{aligned} P = &\frac{1}{2}\int_V C_{11}\left(u_{0,1} - z\phi_{u,1} + \frac{1}{2}w_{,1}^2\right)^2 \\ &+C_{22}\left(v_{0,2} - z\phi_{v,2} + \frac{1}{2}w_{,2}^2\right)^2 \\ &+2C_{12}\left(v_{0,2} - z\phi_{v,2} + \frac{1}{2}w_{,2}^2\right)\left(u_{0,1} - z\phi_{u,1} + \frac{1}{2}w_{,1}^2\right) \\ &+2G\left(\frac{1}{2}(u_{0,2} - z\phi_{u,2} + v_{0,1} - z\phi_{v,1} + w_{,1}w_{,2})\right)^2 \\ &+2G\left(\frac{1}{2}(w_{,1} - \phi_u)\right)^2 + 2G\left(\frac{1}{2}(w_{,2} - \phi_v)\right)^2 dV \\ &-2h_{31}q\int_0^h\left(u_{0,1} - z\phi_{u,1} + \frac{1}{2}w_{,1}^2\right)dz_3 \\ &-2h_{32}q\int_0^h\left(v_{0,2} - z\phi_{v,2} + \frac{1}{2}w_{,2}^2\right)dz_3 + \frac{1}{\epsilon^e}q^2 dB \end{aligned} \quad (15)$$

The variational derivatives we can calculate in the same way as for the purely mechanical plate (10). So we are only going to state here the first variational derivative with respect to $u_{0,1}$

$$\frac{\delta H}{\delta u_{0,1}} = \int_0^h C_{11}\varepsilon_{11} + C_{12}\varepsilon_{12}dz_3 - h_{31}hq$$

and leave the other variational derivatives to the interested reader. Note that the variational derivative of the kinetic energy is unchanged.

3) Equations of motion and pH model: Now we derive the equations of motion. We first define the mechanical potential energy

$$\begin{aligned} P_{\text{mech}} = &\frac{1}{2}\int_V (C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} - h_{31}q)\varepsilon_{11} \\ &+ (C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} - h_{32}q)\varepsilon_{22} \\ &+\sigma_{23}\varepsilon_{23} + \sigma_{13}\varepsilon_{13} + \sigma_{12}\varepsilon_{12}dV. \end{aligned}$$

Next, to derive the equations of motions, we have to recalculate the variations of the potential energy for every direction separately. But since the piezoelectric effect influences only the strains ε_{11} and ε_{22} it is sufficient to update only these.

a) Stress-strain z_1 direction:

$$\begin{aligned} \delta P_{11} = &\int_{A_u} -\frac{\partial}{\partial z_1}\int_0^h C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} - h_{31}q dz_3\delta u_0 \\ &- \frac{\partial}{\partial z_1}\int_0^h -z_3(C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} - h_{31}q)dz_3\delta\phi_u dA_u \end{aligned}$$

b) Stress-strain z_2 direction:

$$\begin{aligned} \delta P_{22} = &\int_B -\frac{\partial}{\partial z_2}\int_0^h (C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} - h_{32}q)dz_3\delta v_0 \\ &- \frac{\partial}{\partial z_2}\int_0^h -z_3(C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} - h_{32}q)dz_3\delta\phi_v dA_u \end{aligned}$$

Using this result as we did for the derivation of (12) we obtain the following equations of motion of a piezoelectric plate with a dynamic electromagnetic field

$$\begin{aligned}
\dot{\mathbf{p}}_1 &= \frac{\partial}{\partial z_1}\frac{\delta P}{\delta u_{0,1}} + \frac{\partial}{\partial z_2}\frac{\delta P}{\delta u_{0,2}} + \mathbf{f}_u \\
\dot{\mathbf{p}}_2 &= \frac{\partial}{\partial z_2}\frac{\delta P}{\delta v_{0,2}} + \frac{\partial}{\partial z_1}\frac{\delta P}{\delta v_{0,1}} + \mathbf{f}_v \\
\dot{\mathbf{p}}_3 &= \frac{\partial}{\partial z_1}\frac{\delta P}{\delta w_{,1}} + \frac{\partial}{\partial z_2}\frac{\delta P}{\delta w_{,2}} + \mathbf{f}_w \\
\dot{\mathbf{p}}_4 &= \frac{\partial}{\partial z_1}\frac{\delta P}{\delta \phi_{u,1}} - \frac{\delta P}{\delta \phi_u} + \frac{\partial}{\partial z_2}\frac{\delta P}{\delta \phi_{u,2}} + h_{\min}\mathbf{f}_u \\
\dot{\mathbf{p}}_5 &= \frac{\partial}{\partial z_2}\frac{\delta P}{\delta \phi_{v,2}} - \frac{\delta P}{\delta \phi_v} + \frac{\partial}{\partial z_1}\frac{\delta P}{\delta \phi_{v,1}} + h_{\min}\mathbf{f}_v
\end{aligned}$$

Note that the equations of motions for the strain are unchanged. However, we still have to derive the equations of motion for the electromagnetic field. To this aim we substitute the constitutive equations into the Maxwell's equations and use the defined flux and charge density to describe them

$$\begin{aligned}
\dot{\phi}_1^e &= \frac{\partial}{\partial z_2}\left(-h_{31}\int_0^h \varepsilon_{11}dz_3 - h_{32}\int_0^h \varepsilon_{22}dz_3 + \frac{h}{\epsilon^e}q\right) \\
&= \frac{\partial}{\partial z_2}\frac{\delta H}{\delta q} \\
\dot{\phi}_2^e &= -\frac{\partial}{\partial z_1}\left(-h_{31}\int_0^h \varepsilon_{11}dz_3 - h_{32}\int_0^h \varepsilon_{22}dz_3 + \frac{h}{\epsilon^e}q\right) \\
&= -\frac{\partial}{\partial z_1}\frac{\delta H}{\delta q} \\
\dot{q} &= -\frac{\partial}{\partial z_1}\frac{1}{\mu h}(-hB_{m,2}) + \frac{\partial}{\partial z_2}\frac{1}{\mu h}(-hB_{m,1}) \\
&= -\frac{\partial}{\partial z_1}\frac{\delta H}{\delta \phi_2} + \frac{\partial}{\partial z_1}\frac{\delta H}{\delta \phi_1}
\end{aligned}$$

Hence, we can now write the system in the following pH form

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & J_m & 0 & 0 \\ -J_m^* & 0 & 0 & 0 \\ 0 & 0 & 0 & J_e \\ 0 & 0 & -J_e^* & 0 \end{bmatrix}\delta_{\mathbf{x}}H + \begin{bmatrix} I_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}\mathbf{u} \qquad (16) \\
\mathbf{y}_m &= \begin{bmatrix} I_3 & 0 & 0 & 0 \end{bmatrix}\delta_{\mathbf{x}}H
\end{aligned}$$

where B_m and J_m are the same matrices as for the purely mechanical model and

$$J_e = \begin{bmatrix} 0 & 0 & \partial_2 \\ 0 & 0 & -\partial_1 \\ \partial_2 & -\partial_1 & 0 \end{bmatrix}, \; \phi^e = \begin{bmatrix} \phi_y^e \\ \phi_x^e \end{bmatrix}.$$

Note that the coupling between the mechanical and the electromagnetic domain is done only via the Hamiltonian and not via the interconnection structure.

IV. Conclusion

In this paper we have derived the port-Hamitonian (pH) model for a linear plate, both purely mechanical and with piezo actuation. This model can now be used to derive a shape controller for an inflatable space structure. We choose the early lumping approach — control design based on a spatial discretized model. We have used a spatial discretization scheme [2] which preserves the pH structure of the model. We have implemented one of the so derived finite dimensional system which approximates the dynamics of the plate on one finite element. Further, we will use an inteconnection of several finite dimensional systems to derive a discretization of a full plate. This finite dimensional model is then the basis for the design of a shape controller.

V. Acknowledgments

We would like to thank the MicroNed programme for the funding of the research. We also would like to thank Arjan van der Schaft from the University of Groningen for his help in deriving the model.

References

[1] F. Couennea, C. Jalluta, B. Maschkea, M. Tayakouta, and P. Breedveldb. Bond graph for dynamic modelling in chemical engineering. *Chemical Engineering and Processing: Process Intensification*, 47(11):1994–2003, October 2008.

[2] G. Golo, V. Talasila, A.J. van der Schaft, and B.M. Maschke. Hamiltonian discretization of boundary control systems. *Automatica*, 40(757-771), 2004.

[3] C.H.M. Jenkins. *Gossamer Spacecraft: Membrane and Inflatable structures technology for space applications*, volume 191. Progress in Astronautics and Aeronautics, 2001.

[4] A. Macchelli. *Port Hamiltonian systems. A unified approach for modeling and control finite and infinite dimensional physical systems*. PhD thesis, University of Bologna, DEIS, 2003.

[5] A. Macchelli, C. Melchiorri, and L. Bassi. Port-based modelling and control of the mindlin plate. In *44th IEEE Conference on Decision and Control and European Control Conference ECC*, 2005.

[6] B. Maschke and A. J. van der Schaft. Port controlled hamiltonian representation of distributed parameter sytems. In N.E. Leonard and R. Ortega, editors, *Proceedings IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, pages 28–38, 2000.

[7] B. Maschke and A.J. van der Schaft. Port-controlled Hamiltonian systems: modelling origins and system theoretic properties. In M. Fliess, editor, *Proceedings 2nd IFAC Symposium on Nonlinear Control Systems*, 1992.

[8] Leonard Meirovitch. *Fundamentals of Vibrations*. McGraw-Hill Companies, 2000.

[9] R.D. Mindlin. *An Introduction to the Mathematical Theory of Vibrations of Elastic Plates*. World Scientific, 2007.

[10] The Institute of Electrical and Electronics Engineers, Inc. *An American National Standard IEEE Standard on Piezoelectricity*, ieee standards board edition, 1987.

[11] S.P. Timoschenko and J.N. Goodier. *Theory of Elasticity*. McGraw-Hill International Editions, thrid edition edition, 1970.

[12] A. J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*. Springer London, 2nd edition edition, 2000.

[13] A. J. van der Schaft and B. M. Maschke. The Hamiltonian formulation of energy conserving physical systems with external ports. *Archiv für Elektronik und Übertragungstechnik*, 49:362–371, 1995.

[14] A. Vinogradov, S. Ji, C. Jenkins, and Y. Bar-Cohen. State-of-the-art developments in the field of electro active polymers. In *Materials Research Society Fall Meeting*, November 2005.

[15] T. Voß and J.M.A. Scherpen. Modeling for control of a nonlinear Timoshenko beam with piezo actuation. Submitted to *SIAM Journal on Control and Optimization*, 2009.

[16] T. Voß J.M.A. Scherpen, and A.J. van der Schaft. Modeling for control of an inflatable space reflector, the linear 1-D case. In *MTNS*, 2008.