Port-Hamiltonian Systems on Open Graphs

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I. Abstract

In this talk we discuss how to define in an intrinsic manner port-Hamiltonian dynamics [3] on open graphs. Open graphs are graphs where some of the vertices are *boundary vertices* (terminals), which allow interconnection with other systems. We show that a directed graph carries two natural Dirac structures [3], called the Kirchhoff-Dirac structure and the vertex-edge Dirac structure. The port-Hamiltonian dynamics corresponding to the Kirchhoff-Dirac structure is exemplified by the dynamics of an RLC-circuit, see also [5], [4]. The port-Hamiltonian dynamics corresponding to the vertex-edge Dirac structure is illustrated by coordination control, in which case there is dynamics associated to every vertex and to every edge, and by standard consensus algorithms where there is dynamics associated to every vertex while every edge corresponds to a resistive relation.

II. Introduction

Recall that a *directed graph* consists of a finite set $\mathcal{V}$ of *vertices* and a finite set $\mathcal{E}$ of directed *edges*, together with a mapping from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$. A directed graph (from now on 'graph') is specified by its *incidence matrix* B, which is an $\bar{v} \times \bar{e}$ matrix, $\bar{v}$ being the number of vertices and $\bar{e}$ being the number of edges, with (i,j)-th element b_{ij} equal to 1 if the j-th edge is an edge towards vertex i, equal to -1 if the j-th edge is an edge originating from vertex i, and 0 otherwise.

Given a graph we define its *vertex space* Λ_0 as the real vector space of all functions from $\mathcal{V}$ to $\mathbb{R}$. Λ_0 can be identified with $\mathbb{R}^{\bar{v}}$. Furthermore, we define its *edge space* Λ_1 as the vector space of all functions from $\mathcal{E}$ to $\mathbb{R}$. Again, Λ_1 can be identified with $\mathbb{R}^{\bar{e}}$. In the context of an electrical circuit Λ_1 will be the vector space of currents *through* the edges in the circuit, and its dual space, denoted by Λ^1, defines the vector space of voltages *across* the edges. Similarly, the dual space of Λ_0 is denoted by Λ^0 and defines the vector space of potentials at the vertices.

The incidence matrix B can be regarded as the matrix representation of a linear map $B : \Lambda_1 \to \Lambda_0$, called the *incidence operator*. Its adjoint map, called the *co-incidence* operator, is denoted in matrix representation as $B^T : \Lambda^0 \to \Lambda^1$.

A useful extension of this set-up (e.g. for coordination control, where generally the motion is in 3-dimensional space) is to consider the vertex space and edge space to consist of functions to $\mathbb{R}^3$. In this case the incidence operator will be given in matrix representation by the Kronecker product $B \otimes I_3$, with I_3 the 3-dimensional identity matrix.

Although in Kirchhoff's original treatment of circuits external currents entering the vertices of the graph were an indispensable notion, this was not always articulated very well in subsequent formalizations of circuits and graphs. We will do so by extending the notion of graph to *open graph*. An open graph $\mathcal{G}$ is obtained from an ordinary graph with set of vertices $\mathcal{V}$ by identifying a subset $\mathcal{V}_b \subset \mathcal{V}$ of *boundary vertices*. The interpretation of $\mathcal{V}_b$ is that these are the vertices that are open to interconnection (i.e., with other open graphs). The remaining subset $\mathcal{V}_i := \mathcal{V} - \mathcal{V}_b$ are the *internal vertices* of the open graph.

Decomposing the incidence operator B as $\begin{bmatrix} B_i \\ B_b \end{bmatrix}$ with B_i the part of the incidence operator corresponding to the internal vertices, and B_b the part corresponding to the boundary vertices, Kirchhoff's current laws take the form

$$B_i I = 0, \quad B_b I = -I_b \tag{1}$$

Here the vector I_b of boundary currents belongs to the vector space Λ_b of functions from the boundary vertices $\mathcal{V}_b$ to $\mathbb{R}$ (which is identified with $\mathbb{R}^{\bar{v}_b}$, with $\bar{v}_b$ the number of boundary vertices). Kirchhoff's voltage laws are

$$V = B^T \psi = B_i^T \psi_i + B_b^T \psi_b, \tag{2}$$

where ψ_i denotes the vector of the potentials at the internal vertices and ψ_b the vector of potentials at the boundary vertices. Note that $\psi_b \in \Lambda^b$, where we define Λ^b to be the dual of the space of boundary currents Λ_b. This results in the following Dirac structure of allowed currents, voltages, boundary currents and boundary potentials, called the *Kirchhoff-Dirac structure*:

$$\begin{aligned} \mathcal{D}_K(\mathcal{G}) \quad := \quad & \{(I, V, I_b, \psi_b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_b \times \Lambda^b \mid \\ & B_i I = 0, B_b I = -I_b, \\ & \exists \psi_i \text{ s.t. } V = B_i^T \psi_i + B_b^T \psi_b\} \end{aligned} \tag{3}$$

The dynamics of an RLC-circuit is defined, on top of Kirchhoff's laws for its circuit graph, by the constitutive relations of its elements (capacitors, inductors and resistors). This yields a port-Hamiltonian system, cf. [5].

The incidence matrix B defines another Dirac structure, called the *vertex-edge Dirac structure*, as follows

$$\begin{aligned} \mathcal{D}_{ve}(\mathcal{G}) \quad := \quad & \{(I, V, \eta_i, \psi_i, I_b, \psi_b) \in \Lambda_1 \times \Lambda^1 \times \\ & \Lambda_0 \times \Lambda^0 \times \Lambda_b \times \Lambda^b \mid B_i I = -\eta_i, \\ & B_b I = -I_b, V = B_i^T \psi_i + B_b^T \psi_b\} \end{aligned} \tag{4}$$

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Using this vertex-edge Dirac structure one may define different forms of port-Hamiltonian dynamics on graphs.

As a first example we discuss *consensus algorithms*, in which case the vertices correspond to agents, and edges to the interactions between them. Associated to each agent v there is a vector $x_v \in \mathbb{R}$. In a standard set-up the vector x_v of each agent v satisfies the following dynamics

$$\dot{x}_v(t) = - \sum_{(v,w)\in E(\mathcal{G})} g_{(v,w)}(x_v(t) - x_w(t))$$

where $g_{(v,w)} > 0$ denotes a certain positive-definite *weight* associated to each edge (v, w) of the *undirected* graph.

Collecting all variables x_v into one vector $x \in \mathbb{R}^{\bar{v}}$, it can be readily checked that the dynamics can be written as

$$\dot{x} = -BGB^T x \tag{5}$$

with B the incidence matrix of the graph *endowed with an arbitrary orientation*, and G the diagonal matrix with elements $g_{(v,w)}$ for each edge (v, w). We will discuss its port-Hamiltonian structure, and show how this can be used to unify and generalize existing results. Also we will take a modular view on consensus dynamics by considering *leader* and *follower* agents, cf. [2], in which case the leader agents will correspond to boundary vertices of the graph. We discuss how a physical analogue for the above consensus dynamics is the dynamics of a number of unit masses (corresponding to each internal vertex), with linear dampers associated to the edges, and externally prescribed boundary velocities $u = e_b$ corresponding to the boundary vertices, with outputs $y = f_b$ being the boundary forces.

As a second example we will discuss *coordination control* of N dynamical systems, heavily inspired by [1]. In this case the N dynamical systems correspond to the vertices, while the dynamics associated with the edges corresponds to controller action. From a physical point of view the total system dynamics can be considered as a *mass-spring* system with masses corresponding to the vertices and springs corresponding to the edges.

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