A new characterisation of exponential stability

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***Abstract*— We present a new characterization of exponential stability for nonlinear systems in the form of Lyapunov functions which may be upper and lower bounded by monotonic functions satisfying a $growth$ order relationship rather than being polynomials of the state's norm. In particular, one may allow for Lyapunov functions with arbitrary weakly homogeneous bounds.**

I. Introduction

Consider the ordinary differential equation

$$\dot{x} = f(t, x) \tag{1}$$

where f is continuous in t and locally Lipschitz in x uniformly in t. Let $x = 0$ be an equilibrium point of the latter and denote the solutions with initial conditions $t_\circ \in \mathbb{R}_+$ and $x_\circ \in \mathbb{R}^n$, by $x(t, x_\circ, t_\circ)$. We recall that the trivial solution $x = 0$ is uniformly exponentially stable if there exist constants k, λ an r such that

$$|x_\circ| < r,\ t \in \mathbb{R}_+ \quad \Rightarrow \quad |x(t, x_\circ, t_\circ)| \le k|x_\circ|e^{-\lambda(t-t_\circ)}\ . \tag{2}$$

We say that the origin is uniformly globally exponentially stable if $r = \infty$.

Exponential stability of nonlinear systems described by ordinary differential equations dates back at least to Krasovskii's work in the late 1950s –*cf.* [4, Theorem 11.1]. The classical characterisation of uniform global exponential stability involves a Lyapunov function which satisfies upper and lower quadratic bounds of $|x|$. This has been extended to bounds that are polynomial of any order –*cf.* [2], [10], [3]:

Theorem 1 *Let $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$, where f is a locally Lipschitz function and $D \subset R^n$ be a domain containing $x = 0$. Then $x = 0$ is uniformly exponentially stable if and only if there exist a continuously differentiable function $V : [0, \infty) \times D \to R_+$ such that*

$$k_1|x|^p \le V(t, x) \le k_2|x|^p, \tag{3a}$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x) \le -k_3|x|^p \tag{3b}$$

for all $t \ge t_\circ \ge 0$ and all $x \in D$, where p and k_i, $i = 1, 2, 3$ are positive constants. If the assumptions hold globally, then $x = 0$ is uniformly globally exponentially stable.

Equivalent characterizations, in terms of a Lyapunov functions decreasing at sampling times have been reported in the context of adaptive control. See for instance [6], [3, Theorem 4.5] and [1]. For time-varying differential equations with locally Lipschitz right-hand side, the following is essentially contained in [5]:

[**Integral characterization of UGES**] For the dynamical system $\dot{x} = f(x, t)$ with f locally Lipschitz and $\sup_{x\neq 0} |f(x, t)|/|x| < \infty$ the origin is uniformly globally exponentially stable if and only if there exists $\gamma > 0$ and $p \ge 1$ such that

$$\int_0^\infty |x(t, x_\circ, t_\circ)|^p dt \le \gamma |x_\circ|^p\ . \tag{4}$$

In [9] exponential stability is characterized by integral conditions for systems described by differential inclusions (convex, upper semi-continuous). Such characterization is useful to establish UGES from inputoutput interconnection properties involving, *e.g.* L_2 bounds. In this short note we give new $differential$ characterisation of exponential stability equivalent to (3) but which does not rely on $explicit$ polynomial bounds but functions satisfying a growth-order relation. Further results are established for weakly homogeneous systems.

II. Main results

Consider the system

$$\dot{x} = f(t, x), \qquad f(t, 0) = 0 \tag{5}$$

Theorem 2 *Let $B_r := \{x \in R^n : |x| < r\}$ and suppose that $f(t, \cdot)$ is Lipschitz on B_r uniformly in t. Let $V : \mathbb{R}_+ \times B_r \to R_+$ be a continuously differentiable function such that for all $x \in B_r$, all $t \ge t_\circ$ and all $t_\circ \in \mathbb{R}_+$*

$$\alpha_1(|x|) \le V(t, x) \le \alpha_2(|x|) \tag{6}$$
$$\dot{V}(t, x) \le -\mu V(t, x), \tag{7}$$

where $\mu > 0$ is a constant and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ (functions $\mathbb{R}_+ \to \mathbb{R}_+$, strictly increasing, zero at zero and proper). Then, the origin of (5) is exponentially stable on B_r if and only if there exist constants $c > 0$, $c_1 > 0$ and $c_2 \in (0, 1)$ such that the following inequalities hold for all $s \ge 0$:

$$\alpha_1^{-1} \circ \alpha_2(s) \le cs \tag{8}$$
$$\alpha_1^{-1} \circ (c_1\alpha_2(s)) \le c_2 s\,. \tag{9}$$

If all the conditions of the theorem are satisfied globally (i.e.if $r = +\infty$ then the origin is uniformly globally exponentiable.

Proof of Theorem 2

I. Sufficiency.

Following the arguments of the proof of [3, Theorem 3.8] we can show that, for any given D and any constants $\rho > 0$ and $r > 0$ which satisfy $B_r \subset D$ and $\rho < \alpha_1(r)$, all the

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solutions of (5) starting from the set of initial conditions $x_0 \in \left\{x \in B_{\alpha_2^{-1}(\rho)}\right\}$ are well defined and moreover $x(x_0, t) \in B_r$ for all $t \geq 0$.

Now, let $T = \frac{1-c_1}{\mu c_1}$ if $c_1 < 1$ and $T = 1$ otherwise. From (7) it follows that $V(x(t)) \leq V(x(\tau))e^{-\mu(t-\tau)}$ for all $t \geq \tau \geq 0$ hence, $V(x(t)) \leq V(x(\tau))$ for all $\tau \in [0, t)$.

Integrating (6) from t to $t + T$ (with T defined above) we obtain

$$\begin{aligned} V(x(t+T)) - V(x(t)) &\leq -\mu \int_t^{t+T} V(x(\tau))d\tau \\ &\leq -\mu \int_t^{t+T} V(x(t+T))d\tau \\ &\leq -\mu T V(x(t+T)) \qquad \forall\, t \geq 0 \end{aligned} \tag{10}$$

where in the second step above we used the fact that $V(x(\tau)) \geq V(x(t+T))$ for all $\tau \in [t, t+T]$. Hence, $(1+\mu T)V(x(t+T)) \leq V(x(t))$ for all $t \geq 0$. From this we obtain

$$\begin{aligned} V(x(t+T)) &\leq \frac{1}{1+\mu T}V(x(t)) \\ &\leq c_1 V(x(t)) \quad \text{if } c_1 \leq 1 \\ V(x(t+T)) &\leq \frac{1}{1+\mu T}V(x(t)) \\ &\leq V(x(t)) \;\leq\; c_1 V(x(t)) \quad \text{if } c_1 \geq 1 \end{aligned} \tag{11}$$

Therefore, $V(x(t+T)) \leq c_1 V(x(t))$ for all $t \geq 0$. Using the bounds (11) and (6) we obtain

$$\alpha_1(|x(t+T)|) \leq c_1\alpha_2(|x(t)|) \qquad \forall\, t \geq 0\,.$$

Then, from (9) it follows that for all $t \geq 0$

$$|x(t+T)| \leq c_2|x(t)|. \tag{12}$$

The rest follows the proof guidelines of [3, Theorem 4.5]. For any $t \geq 0$ let $N = \lfloor \frac{t}{T} \rfloor$, where $\lfloor \cdot \rfloor$ stands for the lower integer part. Divide the interval $[t - NT, t]$ into N equal subintervals then,

$$\begin{aligned} |x(t)| &\leq c_2|x(t-T)| \\ &\leq c_2^2|x(t-2T)| \\ &\;\;\vdots \\ &\leq c_2^N|x(t-NT)|. \end{aligned} \tag{13}$$

Since for all $t \geq 0$ we have $V(x(t)) \leq V(x_0)$ therefore,

$$|x(t)| \leq \alpha_1^{-1}\alpha_2(|x_0|) \qquad \forall\; t \in [0, T] \tag{14}$$

consequently, from (8) it follows that $|x(t)| \leq c|x_0|$ for all $t \in [0, T]$. Combining the last bound with (13) we obtain

$$|x(x_0, t)| \leq c_2^N c|x_0| \leq c c_2^{t/T}|x_0| = c|x_0|e^{-bT},$$

where $b = \frac{1}{T}\ln\frac{1}{c_2}$. In case conditions (11)-(9) are satisfied globally we obtain that the system is UGES.

II. Necessity

Assume that the system (5) is (globally) exponentially stable for all $x \in D$ ($x \in R^n$) *i.e.*, there exist $k, \gamma > 0$ such that the trajectories of (5) satisfy

$$|x(t, x_0)| \leq k|x_0|e^{-\gamma t} \qquad \forall\; t \geq 0,\; x_0 \in D \quad (x_0 \in R^n).$$

Then, from the converse theorem on exponential stability (see for example [3]) it follows that there exists a Lyapunov function $V : D \to R_+$ ($V : R^n \to R_+$) that satisfies the inequalities

$$\begin{aligned} a_1|x|^2 \leq V(x) \leq a_2|x|^2 \\ \frac{dV}{dx}f(x) \leq -a_3|x|^2 \end{aligned}$$

for some positive constants a_i, $i = 1, 2, 3$.

Hence, the functions α_i ($i = 1, 2, 3$) in (11), (7) are given by $\alpha_i(s) = a_i s^2$. Simple calculations show that inequalities (8), (9) are satisfied. Indeed, $\alpha_1^{-1}(s) = \left(\frac{s}{a_1}\right)^{1/2}$ therefore, for arbitrary $c > 0$ we have

$$\alpha_1^{-1}(c_2\alpha_2(s)) = \left(\frac{c\alpha_2(s)}{a_1}\right)^{1/2} = \left(\frac{ca_2 s^2}{a_1}\right)^{1/2} = \sqrt{\frac{ca_2}{a_1}}\,s\,.$$

From this it follows trivially that (8) is satisfied with $c = \sqrt{a2/a_1}$ and (9) is satisfied for arbitrary $c_2 \in (0, 1)$ with $c_1 = \frac{a_1 c_2^2}{a_2}$. ■

Remark 1

- *If all conditions of the theorem 2 are satisfied except (8), then proceeding as before we obtain from (13) and (14) that*
$$|x(t, x_0)| \leq \alpha(|x_0|)e^{-bt}$$
- *The first part of the statement i.e., without requiring (8), (9) is equivalent to UGAS –cf. [8], [7].*

III. Exponential stability from weakly homogeneous bounds

Definition 1 *A real function $f : R^n \to R$ is said to be homogeneous of order k if for any constant $\alpha \geq 0$ the following inequality holds:*

$$f(\alpha x) = \alpha^k f(x).$$

Classical conditions imposed on the bounds of $V(x)$ and its derivative to insure exponential stability are formulated in terms of powers of x which are evidently homogeneous functions. In this section we show that this classical result can be extended to the class of systems with weakly homogeneous bounds on $V(x)$ and its derivative.

Our second theorem is stated in terms of Lyapunov functions satisfying weakly homogeneous bounds. We recall that real function $f : R_+ \to R_+$ is weakly homogeneous if for some function $L \in \mathcal{K}_\infty$ one has

$$f(\lambda x) \leq L(\lambda)f(x) \tag{15}$$

for all $\lambda > 0$.

Theorem 3 *The origin of $\dot{x} = f(x)$ is a uniformly exponentially stable equilibrium if and only if there exists continuously differentiable function $V : B_r \to R_+$, a weakly homogeneous function $\alpha \in \mathcal{K}_\infty$ such that for all $x \in D$ and all $t \geq 0$*

$$a_1\alpha(|x|) \leq V(x) \leq a_2\alpha(|x|) \tag{16}$$

$$\dot{V}(x) \leq -a_3 V, \tag{17}$$

for some positive constants a_1, a_2, a_3. If all the conditions of the theorem are satisfied globally (with $r = +\infty$) then the system (5) is uniformly globaly exponentially stable.

We wrap up this note with an example for which uniform global exponential stability is difficult to conclude from Theorem 1 yet, it may be concluded from our main results.

Example. Consider the system

$$\dot{x}_1 = -x_1 + \frac{x_2 g(x)}{1+x_1^2}$$

$$\dot{x}_2 = -x_2 + \frac{x_1 g(x)}{1+x_2^2}$$

where g is any locally Lipschitz function. It is easy to show that the origin is uniformly globally exponentially stable by invoking Theorem 3 and using the Lyapunov function

$$V(x) = \frac{1}{2}(x_1^4 + x_2^4) + x_1^2 + x_2^2\,.$$

Indeed the total time derivative of V yields $\dot{V} \leq -2V$. Note that this function does not have lower nor upper polynomial bounds however, the function $\alpha_1(s) := \frac{1}{2}V(s)$ is of the same growth order as $\alpha_2(s) := 2V(s)$. $\diamond$

Proof. I. Sufficiency.

From theorem 2 it follows that we need only to verify that conditions (8) and (9) are satisfied. To simplify the calculations let us take $a_1 = 1$ (what can be always done just by choosing e.g. $V_{new}(x) = V(x)/a_1$). Since $\alpha(s)$ is a weakly homogeneous function, then there exists $l \in \mathcal{K}_\infty$ such that $\alpha(\lambda s) \geq l(\lambda)\alpha(s)$ for all $\lambda > 0$. Then, for any $c_1 > 0$, defining $k = l^{-1}(c_1 a_2)$ we have

$$\alpha^{-1}(c_1 a_2 \alpha(s)) \leq \alpha^{-1}\left(l(k)\alpha(s)\right) \leq \alpha^{-1}\left(\alpha(ks)\right) = ks \tag{18}$$

Inequality (18) is valid for any $c_1 > 0$, therefore it is valid for $c_1 = 1$, hence (8) is satisfied. Moreover, we can always choose c_1 so that $k = l^{-1}(c_1 a_2) < 1$, so that (9) is satisfied as well. Therefore all the conditions of the theorem 2 are satisfied and therefore the system (5) is exponentially stable.

II. Necessity

The "only if" part of the proof follows directly the steps of the proof of Theorem 2. ■

IV. Conclusions

We have presented a new Lyapunov-like characterization of exponential stability for ordinary differential equations. Such characterization covers naturally sufficient and necessary conditions for particular cases such as $\mathcal{K}$-exponential stability and for homogeneous systems.

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