A Novel Discontinuous Lyapunov Functional Approach to Networked-Based Stabilization

Kun Liu, Vladimir Suplin and Emilia Fridman

Abstract— **This paper presents a new stability analysis of Networked Control Systems (NCSs), where the sampling and the constant network-induced delays are taken into account. The new method is inspired by discontinuous Lyapunov functionals that were recently introduced for sampled-data systems in [1] (in the framework of impulsive system representation) and in [2] (in the framework of input delay approach). However, extensions of the above Lyapunov constructions to NCSs lead to complicated conditions, which become conservative if the network-induced delays is not small. In the present paper a novel discontinuous Lyapunov functional is introduced, which is based on the application of the Wirtinger type inequality. This functional leads to efficient stability conditions in terms of Linear Matrix Inequality (LMIs). The new stability analysis is applied to sampled-data stabilization by using artificial delay.**

Keywords: time-varying delay, networked control systems, sampled-data systems, Lyapunov functional, LMI.

I. Introduction

Control systems in which control loops are closed through a communication network are called NCSs [3]. NCSs have received increasing attention because of their advantages in the practical applications. Network-based stabilization has become a hot research area [4]-[6].

As in sampled-data control, two main approaches have been used to uncertain NCSs leading to conditions in terms of LMIs [7]. The first one is time-delay approach, where the system is modeled as a continuous-time system with the delayed input/output [8]-[10]. The time-delay approach became popular in NCSs, being applied via *time-independent* Lyapunov-Krasovskii functionals [5]. Recently the time-delay approach to sampled-data systems was revised by using the scaled small gain theorem and a tighter upper bound on the L_2-induced norm of the uncertain term [11]. The second approach is based on the representation of the sampled-data system in the form of impulsive model [12], [13].

Recently the impulsive model approach was extended to the case of variable sampling with a known upper bound [1] and to NCSs [6], where a discontinuous Lyapunov function method was introduced. This method improved the existing time-independent Lyapunov-based results and it inspired a piecewise-continuous (in time) Lyapunov functional approach to sampled-data systems [2]. Extensions of the above discontinuous Lyapunov constructions to NCSs lead to complicated conditions [6], [14]. Moreover, this conditions become conservative if the network-induced delay is not small.

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In the present paper a novel discontinuous Lyapunov functional is introduced, which is based on the application of the Wirtinger type inequality [15]. Being applied to sampled-data systems, the new method recovers conditions of [11] and, thus, is more conservative than the one of [2]. However, for the positive values of network-induced delays the new method leads to efficient sufficient conditions. The new stability analysis is applied to the problem of sampled-data stabilization by using artificial delay [16].

Notation: Throughout the paper $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. The space of functions $\phi:[a,b]\to\mathcal{R}^n$, which are absolutely continuous on $[a,b)$, have a finite $\lim_{\theta\to b^-}\phi(\theta)$ and have square integrable first order derivatives is denoted by $W[a,b]$ with the norm

$$\|\phi\|_W=\max_{\theta\in[a,b]}|\phi(\theta)|+\left[\int_a^b|\dot\phi(s)|^2ds\right]^{\frac{1}{2}}.$$

We also denote $x_t(\theta)=x(t+\theta)$, $\dot x_t(\theta)=\dot x(t+\theta)$, $(\theta\in[-\tau_M,0])$.

II. Problem Formulation

Consider the following system controlled through a network:

$$\dot x(t)=Ax(t)+Bu(t), \tag{1}$$

where $x(t)\in\mathcal{R}^n$ is the state vector, $u(t)\in\mathcal{R}^{n_u}$ is the control input, A and B are system matrices with appropriate dimensions. We let s_k denote the sampling time and t_k the updating instant time, i.e. the time instant at which the k-th sample arrives to the destination.We assume the following:

- The sampler is time-driven, and the controller and Zero-Order Hold (ZOH) are event-driven.
- The network-induced delay is constant, denoted by h.

Thus we have $t_k:=s_k+h$. We take into account data packet dropouts by allowing the sampling to be nonuniform. As in [5], [6], we assume that

$$t_{k+1}-s_k\le\tau_M,\ k=0,1,2,... \tag{2}$$

where τ_M denotes the maximum time span between the time s_k at which the state is sampled and the time t_{k+1} at which next update arrives at the controller.

The state feedback controller has a form $u(t_k) = Kx(t_k - h)$, where K is the controller gain. Thus, considering the behavior of the ZOH, we have

$$u(t) = Kx(t_k - h),\ t_k \le t < t_{k+1},\ k = 0,1,2,... \quad (3)$$

with t_{k+1} being the next updating instant time of the ZOH after t_k. The closed-loop system (1), (3) has a form

$$\dot{x}(t) = Ax(t) + A_1 x(t_k - h),\ t_k \le t < t_{k+1}, \quad (4)$$

where $k = 0,1,2,...$ and $A_1 = BK$.

The objective of the present paper is to derive efficient LMI conditions via novel discontinuous Lyapunov functionals. The following lemma will be useful:

Lemma 1: [2] Let there exist positive numbers α, β, γ and a functional $V : \mathcal{R} \times W[-\tau_M, 0] \times L_2[-\tau_M, 0] \to \mathcal{R}$ such that

$$\alpha|\phi(0)|^2 \le V(t, \phi, \dot{\phi}) \le \beta\|\phi\|_W^2. \quad (5)$$

Let the function $\bar{V}(t) = V(t, x_t, \dot{x}_t)$ is continuous from the right for $x(t)$ satisfying (4), absolutely continuous for $t \ne t_k$ and satisfies

$$\lim_{t \to t_k^-} \bar{V}(t) \ge \bar{V}(t_k). \quad (6)$$

If along (4)

$$\dot{\bar{V}}(t) < -\gamma|x(t)|^2, \qquad t \ne t_k, \quad (7)$$

then (4) is asymptotically stable.

A novel Lyapunov functional construction will be based on the Wirtinger type inequality [15], which can be extended to the vector case:

Lemma 2: Let $z(t) \in W[a, b]$ and $z(a) = 0$ or $z(b) = 0$. Then for any $n \times n$-matrix $R > 0$ the following inequality holds:

$$\int_a^b z^T(\xi)Rz(\xi)d\xi \le \frac{4(b-a)^2}{\pi^2}\int_a^b \dot{z}^T(\xi)R\dot{z}(\xi)d\xi. \quad (8)$$

III. Stability Conditions

By using the time-delay approach, (4) can be represented as a continuous-time system

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)) \quad (9)$$

with a non-small or interval delay $t_k - h = t - \tau(t) \in [h, \tau_M]$. The stability of the latter system can be analyzed via time-independent functionals of the form [17]:

$$V(x_t, \dot{x}_t) = V_n(x_t, \dot{x}_t) + V_Z(x_t, \dot{x}_t), \quad (10)$$

where V_n is a "nominal" functional for stability analysis of the 'nominal" system with constant delay

$$\dot{x}(t) = Ax(t) + A_1 x(t - h) \quad (11)$$

and where [18]

$$\begin{aligned} &V_Z(x_t, \dot{x}_t) = \int_{t-\tau_M}^{t-h} x^T(s)Z_1 x(s)ds \\ &+(\tau_M - h)\int_{-\tau_M}^{-h}\int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta, Z_1 > 0,\ Z_2 > 0. \end{aligned} \quad (12)$$

We suggest the following discontinuous Lyapunov functional

$$\begin{aligned} V_d(t, x_t, \dot{x}_t) &= \bar{V}_1(t) \\ &= V_n(x_t, \dot{x}_t) + V_Z(x_t, \dot{x}_t) + V_W(t, x_t, \dot{x}_t) \end{aligned} \quad (13)$$

with the discontinuous term

$$\begin{aligned} &V_W(t, x_t, \dot{x}_t) = (\tau_M - h)^2 \int_{t_k - h}^{t} \dot{x}^T(s)W\dot{x}(s)ds \\ &-\tfrac{\pi^2}{4}\int_{t_k-h}^{t-h}[x(s) - x(t_k - h)]^T W[x(s) - x(t_k - h)]ds, \\ &W > 0,\ t_k \le t < t_{k+1},\ k = 0,1,2,... \end{aligned} \quad (14)$$

By Wirtinger-type inequality (8), $V_W \ge 0$. Moreover, V_W can be represented as a sum of the continuous in time term $(\tau_M - h)^2 \int_{t-h}^{t} \dot{x}^T(s)W\dot{x}(s)ds$ with the discontinuous one, where the latter vanishes at $t = t_k$. Hence, the condition $\lim_{t\to t_k^-} \bar{V}(t) \ge \bar{V}(t_k)$ holds.

Differentiating V_W we have

$$\begin{aligned} &\tfrac{d}{dt}V_W = (\tau_M - h)^2\dot{x}^T(t)W\dot{x}(t) \\ &-\tfrac{\pi^2}{4}[x(t-h) - x(t_k - h)]^T W[x(t-h) - x(t_k - h)]. \end{aligned} \quad (15)$$

Nominal functional V_n will be chosen in two forms:
1) a discretized Lyapunov functional [19]

$$\begin{aligned} &V_{n1}(t, x_t, \dot{x}_t) = x^T(t)P_1 x(t) \\ &+2x^T(t)\int_{-h}^{0} Q(s)x(t+s)ds \\ &+\int_{-h}^{0}\int_{-h}^{0} x^T(t+s)R(s,\theta)dsx(t+\theta)d\theta \\ &+\int_{-h}^{0} x^T(t+s)S(s)x(t+s)ds,\ P_1 > 0 \end{aligned} \quad (16)$$

with continuous and piecewise-linear functions $Q(s), S(s)$ and $R(s, \theta)$,

2) a simple Lyapunov functional:

$$\begin{aligned} &V_{n2}(t, x_t, \dot{x}_t) = x^T(t)P_1 x(t) + \int_{t-h}^{t} x^T(s)R_1 x(s)ds \\ &+h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta,\ P_1 > 0, R_1 > 0, R_2 > 0. \end{aligned} \quad (17)$$

We start with the stability conditions via V_d of (13) with $V_n = V_{n1}$. Following [20], we divide the delay interval $[-h, 0]$ into N segments $[\theta_p, \theta_{p-1}]$, $p = 1,...,N$ of equal length $r = h/N$, where $\theta_p = -pr$. This divides the square $[-h, 0] \times [-h, 0]$ into $N \times N$ small squares $[\theta_p, \theta_{p-1}] \times [\theta_q, \theta_{q-1}]$. Each small square is further divided into two triangles.

The continuous matrix functions $Q(s)$ and $S(s)$ are chosen to be linear within each segment and the continuous matrix function $R(s, \theta)$ is chosen to be linear within each triangular:

$$\begin{aligned} &Q(\theta_p + \alpha r) = (1-\alpha)Q_p + \alpha Q_{p-1}, \\ &S(\theta_p + \alpha r) = (1-\alpha)S_p + \alpha S_{p-1},\ \alpha \in [0,1], \\ &R(\theta_p + \alpha r, \theta_q + \beta r) = \\ &\begin{cases} (1-\alpha)R_{pq} + \beta R_{p-1,q-1} + (\alpha - \beta)R_{p-1,q},\ \alpha \ge \beta, \\ (1-\beta)R_{pq} + \alpha R_{p-1,q-1} + (\beta - \alpha)R_{p,q-1},\ \alpha < \beta. \end{cases} \end{aligned} \quad (18)$$

Thus, the Lyapunov functional is completely determined by $P_1, Q_p, S_p, R_{pq}, p, q = 0,1,...,N$.

Theorem 1: The system (4) is asymptotically stable if there exist $n \times n$ matrices $P_1 > 0$, P_2, P_3, T, $S_p = S_p^T$, Q_p, $R_{pq} = R_{qp}^T$, $p = 0,1,...,N$, $q = 0,1,...,N$, $W > 0$,

$Z_i > 0$, $i = 1, 2$ and $n \times 2n$-matrices Y, M such that following LMIs hold:

$$\begin{bmatrix} P_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0, \tag{19}$$

$$\begin{bmatrix} \Xi_{di} & \begin{bmatrix} D^s \\ 0 \end{bmatrix} & \begin{bmatrix} D^a \\ 0 \end{bmatrix} \\ * & -R_d - S_d & 0 \\ * & * & -3S_d \end{bmatrix} < 0, \ i = 1, 2, \tag{20}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \tag{21}$$

$$\tilde{Q} = [Q_0 \ Q_1 \ ...Q_N], \ \tilde{S} = diag\{1/rS_0, 1/rS_1, ..., 1/rS_N\},$$
$$\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & ... & R_{0N} \\ R_{10} & R_{11} & ... & R_{1N} \\ ... & ... & ... & ... \\ R_{N0} & R_{N1} & ... & R_{NN} \end{bmatrix}. \tag{22}$$

$$\Xi_{d1} = \begin{bmatrix} \Psi_d & \begin{bmatrix} -Q_N \\ 0 \end{bmatrix} - Y^T & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - M^T + Y^T & M^T & (\tau_M - h) M^T \\ * & -S_N + Z_1 - \frac{\pi^2}{4} W & \frac{\pi^2}{4} W - T & 0 & 0 \\ * & * & -\frac{\pi^2}{4} W + T + T^T & 0 & 0 \\ * & * & * & -Z_1 & 0 \\ * & * & * & * & -(\tau_M - h)^2 Z_2 \end{bmatrix}, \tag{23}$$

$$\Xi_{d2} = \begin{bmatrix} \Psi_d & \begin{bmatrix} -Q_N \\ 0 \end{bmatrix} - Y^T & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - M^T + Y^T & M^T & (\tau_M - h) M^T \\ * & -S_N + Z_1 - \frac{\pi^2}{4} W & \frac{\pi^2}{4} W - T & 0 & 0 \\ * & * & -\frac{\pi^2}{4} W + T + T^T & 0 & (\tau_M - h) T^T \\ * & * & * & -Z_1 & 0 \\ * & * & * & * & -(\tau_M - h)^2 Z_2 \end{bmatrix}, \tag{24}$$

$$\begin{aligned} \Psi_d &= P^T \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} P \\ &+ \begin{bmatrix} Q_0 + Q_0^T + S_0 & 0 \\ 0 & (\tau_M - h)^2 (Z_2 + W) \end{bmatrix}, \\ S_d &= diag\{S_0 - S_1, S_1 - S_2, ..., S_{N-1} - S_N\}, \\ R_d &= \begin{bmatrix} R_{d11} & R_{d12} & ... & R_{d1N} \\ R_{d21} & R_{d22} & ... & R_{d2N} \\ ... & ... & ... & ... \\ R_{dN1} & R_{dN2} & ... & R_{dNN} \end{bmatrix}, \\ R_{dpq} &= r(R_{p-1,q-1} - R_{pq}), \\ D^s &= [D_1^s \ D_2^s \ ... \ D_N^s], \ D^a = [D_1^a \ D_2^a \ ... \ D_N^a], \\ D_p^s &= \begin{bmatrix} r/2(R_{0,p-1} + R_{0p}) - (Q_{p-1} - Q_p) \\ r/2(Q_{p-1} + Q_p) \\ -r/2(R_{N,p-1} + R_{Np}) \end{bmatrix}, \\ D_p^a &= \begin{bmatrix} -r/2(R_{0,p-1} - R_{0p}) \\ -r/2(Q_{p-1} - Q_p) \\ r/2(R_{N,p-1} - R_{Np}) \end{bmatrix} \end{aligned} \tag{25}$$

Proof: First, from [19], the Lyapunov functional condition $\bar{V}_1(t) \geq \varepsilon \|x(t)\|^2$, $\varepsilon > 0$ is satisfied if $S_p > 0$, $p = 0, 1, ..., N$ and (19) hold. Secondly, differentiating the Lyapunov functionals V_d of (13) with $V_n = V_{n1}$, and applying the Jensen's inequality [19], the free-weighting matrices techniques [18], descriptor model transformation [21] , and the convex method [22], we obtain sufficient LMI conditions (20)($i = 1, 2$) for the stability. Moreover, LMI (20) ($i = 1, 2$) imply that $S_0 > S_1 > ...S_N > 0$ (see Proposition 5.22 of [19]). ■

If we use the simple form of $V_n = V_{n2}$, by similar arguments we arrive to

Corollary 1: The system (1) is asymptotically stable if there exist $n \times n$ matrices $P_1 > 0$, P_2, P_3, T, $W > 0$, $Z_i > 0$,$R_i > 0$, $i = 1, 2$ and $n \times 2n$-matrices Y, M such that the following LMIs are feasible:

$$\Xi_{s1} = \begin{bmatrix} \Psi_s & \begin{bmatrix} R_2 \\ 0 \end{bmatrix} - Y^T & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - M^T + Y^T & M^T & (\tau_M - h) M^T \\ * & -R_1 - R_2 + Z_1 - \frac{\pi^2}{4} W & \frac{\pi^2}{4} W - T & 0 & 0 \\ * & * & -\frac{\pi^2}{4} W + T + T^T & 0 & 0 \\ * & * & * & -Z_1 & 0 \\ * & * & * & * & -(\tau_M - h)^2 Z_2 \end{bmatrix}, \tag{26}$$

$$\Xi_{s2} = \begin{bmatrix} \Psi_s & \begin{bmatrix} R_2 \\ 0 \end{bmatrix} - Y^T & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - M^T + Y^T & M^T & (\tau_M - h) Y^T \\ * & -R_1 - R_2 + Z_1 - \frac{\pi^2}{4} W & \frac{\pi^2}{4} W - T & 0 & 0 \\ * & * & -\frac{\pi^2}{4} W + T + T^T & 0 & (\tau_M - h) T^T \\ * & * & * & -Z_1 & 0 \\ * & * & * & * & -(\tau_M - h)^2 Z_2 \end{bmatrix}. \tag{27}$$

where

$$\begin{aligned} P &= \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \\ \Psi_s &= P^T \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} P \\ &+ \begin{bmatrix} R_1 - R_2 & 0 \\ 0 & h^2 R_2 + (\tau_M - h)^2 (Z_2 + W) \end{bmatrix}. \end{aligned} \tag{28}$$

Remark 1: Results of Theorem 1 and Corollary 1 with $W = 0$ guarantee the network-based stabilization of (1) under assumption (2), where the network-induced delay $\eta_k \geq h$ is variable.

IV. Examples

Example 1: Consider the system from [4][3]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t), \tag{29}$$

where $u(t) = -[3.75 \ 11.5] x(t_k - h), \ t_k \leq t < t_{k+1}$.

For the values of h given in Table I, by applying Theorem 1 with $N = 1$ and Corollary 1, we obtain the same maximum values of τ_M, that preserve the stability (see Table I). We see that discontinuous terms of Lyapunov functionals improve the performance.

Example 2: We consider the following simple and much-studied problem (see [24] and the references therein):

$$\dot{x}(t) = -x(t_k - h), \quad t_k \leq t < t_{k+1}, \ k = 0, 1, 2 \cdots \tag{30}$$

For the values of h given in Table II, by applying Theorem 1 with $N = 1$ and Corollary 1, we obtain the maximum

TABLE I
EXAMPLE 1: MAX. VALUE OF τ_M FOR DIFFERENT h

$\tau_M \setminus h$	0	0.1	0.2	0.4	0.6	0.9
$Th1, Cor1$	1.41	1.37	1.33	1.25	1.18	1.11
[14]	1.68	1.33	1.26	1.18	1.14	1.07
$Th1, Cor1, (W=0)$	1.04	1.05	1.06	1.07	1.07	1.07

values of τ_M, that preserve the stability (see Table II). Also in this Example the discontinuous terms of LKFs improve the performance.

TABLE II
EXAMPLE 2: MAX. VALUE OF τ_M FOR DIFFERENT h

$\tau_M \setminus h$	0	0.1	0.2	0.4	0.9	1.2
$Th1, Cor1$	1.67	1.64	1.60	1.54	1.44	1.42
[14]	1.99	1.64	1.57	1.50	1.42	1.36
$Th1, Cor1(W=0)$	1.33	1.35	1.36	1.37	1.37	1.39

Example 3: Consider the system in [19]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t_k - h).$$

This system is unstable for non-delay case, so Corollary 1 cannot be applied to this system. For different $h > 0$, we obtain the maximum values of τ_M that preserve the stability (see Table III). Also in this Example the discontinuous terms of Lyapunov functional improve the performance.

TABLE III
EXAMPLE 3: MAX. VALUE OF τ_M FOR DIFFERENT h

$\tau_M \setminus h$		0.5	0.65	0.8
$N=1$	$Th1$	1.03	1.27	1.36
	$Th1(W=0)$	0.84	1.05	1.16
$N=2$	$Th1$	1.07	1.39	1.65
	$Th1(W=0)$	0.86	1.12	1.34

V. SAMPLED-DATA STABILIZATION BY USING ARTIFICIAL DELAY

It is well-known, that using artificial delay in the static output feedback can stabilize the systems, which are not stabilizable without delay [16]. Thus, the double integrator

$$\ddot{x}(t) = u(t), \; y(t) = x(t) \tag{31}$$

can be stabilized by using a control action of the form $u(t) = -k_1 x(t-h_1) - k_2 x(t-h_2)$, where h_1 and h_2 are constant delays and $0 \leq h_1 < h_2$. The main criticism of the above method, that it has no advantages over the dynamic output-feedback and that its implementation needs buffer for all the measurements $y(t+\theta), \; \theta \in [-h_2, 0]$.

For the sampled-data control of systems with uncertainties in the matrices, the observer-based design is complicated and may lead to conservative results. From the other side, a simple static output feedback using the previous measurements can be easily designed and implemented. Consider the system (1) with the known and constant sampling period $h > 0$ and with the measured output

$$y(t_k) = Cx(t_k), \; k = 0, 1, 2, \ldots \tag{32}$$

where $y(t) \in \mathcal{R}^{n_l}$, C is constant matrix of appropriate dimensions. The control signal is assumed to be generated by a ZOH function with a sequence of hold times $0 = t_0 < t_1 < \ldots < t_k < \ldots$

$$u(t) = u_d(t_k), \; t_k \leq t < t_{k+1}, \tag{33}$$

where $\lim_{k\to\infty} t_k = \infty$ and u_d is a discrete-time control signal.

We consider the following static output feedback controller:

$$\begin{aligned} u(t) &= K_1 C x(t_k) + K_2 C x(t_{k-m}) \\ &= K_1 C x(t_k) + K_2 C x(t_k - mh), \\ & m = 1, 2, \ldots, \quad t_k \leq t < t_{k+1}. \end{aligned} \tag{34}$$

The closed-loop system (1) , (34) has the form

$$\dot{x}(t) = Ax(t) + A_1 x(t_k) + A_2 x(t_k - mh), \tag{35}$$

where

$$A_1 = BK_1C, \; A_2 = BK_2C. \tag{36}$$

We extend stability analysis of Part III to the system of (35). Consider the following discontinuous Lyapunov functional:

$$\begin{aligned} & V_{sam}(t, x_t, \dot{x}_t) = \bar{V}_2(t) \\ & = V_{n1}(t, x_t, \dot{x}_t) + V_U(t, x_t, \dot{x}_t) + V_W(t, x_t, \dot{x}_t), \\ & P_1 > 0, U > 0, W > 0, \; t \in [t_k, t_{k+1}), \end{aligned} \tag{37}$$

where

$$\begin{aligned} & V_{n1}(t, x_t, \dot{x}_t) = x^T(t) P_1 x(t) \\ & + 2x^T(t) \int_{-mh}^{0} Q(s) x(t+s) ds \\ & + \int_{-mh}^{0} \int_{-mh}^{0} x^T(t+s) R(s,\theta) ds x(t+\theta) d\theta \\ & + \int_{-mh}^{0} x^T(t+s) S(s) x(t+s) ds, \end{aligned} \tag{38}$$

and

$$V_U(t, \dot{x}_t) = (h - t + t_k) \int_{t_k}^{t} \dot{x}^T(s) U \dot{x}(s) ds, \tag{39}$$

$$\begin{aligned} & V_W(t, x_t, \dot{x}_t) = h^2 \int_{t_k - mh}^{t} \dot{x}^T(s) W \dot{x}(s) ds \\ & - \tfrac{\pi^2}{4} \int_{t_k - mh}^{t - mh} [x(s) - x(t_k - mh)]^T W [x(s) - x(t_k - mh)] ds. \end{aligned} \tag{40}$$

The term V_U was introduced in [2]for sampled-data control. We note that V_U term vanishes before t_k $(h - t_k + t_{k+1} = h - h = 0)$ and after t_k $(\int_{t_k}^{t_k} \dot{x}^T(s) U \dot{x}(s) ds = 0)$. By Wirtinger-type inequality (8) we verify that $V_W \geq 0$, and V_W does not grow in the jumps t_k. So the condition $\lim_{t \to t_k^-} \bar{V}_2(t) \geq \bar{V}(t_k)$ holds.

Similar to Theorem 1, we arrive to the following:

Theorem 2: The system (35) is asymptotically stable if there exist $n \times n$ matrices $P_1 > 0$, P_2, P_3, T, Y_1, Y_2, $S_p = S_p^T$, Q_p, $R_{pq} = R_{qp}^T$, $p = 0, 1, \ldots, N$, $q = 0, 1, \ldots, N$, and $U > 0$, $W > 0$ such that LMI (19) and the following LMIs hold:

$$\begin{bmatrix} \bar{\Xi}_{di} & \begin{bmatrix} D^s \\ 0 \end{bmatrix} & \begin{bmatrix} D^a \\ 0 \end{bmatrix} \\ * & -R_d - S_d & 0 \\ * & * & -3S_d \end{bmatrix} < 0, \; i = 1, 2, \tag{41}$$

where P is defined in (21), $\tilde{Q}, \tilde{S}, \tilde{R}$ are defined in (22) ($\bar{r} = mr$ instead of r), S_d, R_d, D^s, D^a are defined in (25) ($\bar{r} = mr$ instead of r), and

$$\bar{\Xi}_{d1} = \begin{bmatrix} \bar{\Psi}_d + \begin{bmatrix} 0 & 0 \\ 0 & hU \end{bmatrix} & \begin{bmatrix} -Q_N \\ 0 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - \begin{bmatrix} Y_1^T - T \\ Y_2^T \end{bmatrix} \\ * & -S_N - \frac{\pi^2}{4} W & \frac{\pi^2}{4} W & 0 \\ * & * & -\frac{\pi^2}{4} W & 0 \\ * & * & * & T + T^T \end{bmatrix}, \quad (42)$$

$$\bar{\Xi}_{d2} = \begin{bmatrix} \bar{\Psi}_d & \begin{bmatrix} -Q_N \\ 0 \end{bmatrix} & h \begin{bmatrix} Y_1^T \\ Y_2^T \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - \begin{bmatrix} Y_1^T - T \\ Y_2^T \end{bmatrix} \\ * & -S_N - \frac{\pi^2}{4} W & 0 & \frac{\pi^2}{4} W & 0 \\ * & * & -hU & 0 & hT \\ * & * & * & -\frac{\pi^2}{4} W & 0 \\ * & * & * & * & T + T^T \end{bmatrix}, \quad (43)$$

$$\bar{\Psi}_d = P^T \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} P + \begin{bmatrix} Q_0 + Q_0^T + S_0 - Y_1 - Y_1^T & -Y_2 \\ 0 & h^2 W \end{bmatrix}.$$

Remark 2: Assuming that B is of full rank, the unknown gains K_1 and K_2 in (34) that stabilize (1) can be found as follows. Without loss of generality, B can be taken in the form: $B^T = [0 \ B_2^T]$, where $B_2 \in \mathcal{R}^{n_u \times n_u}$ is non-singular. Consider the slack variables P_2, P_3 of the following form:

$$P_2 = \begin{bmatrix} G_{21} & G_{22} \\ G & \varepsilon_2 I_{n_u} \end{bmatrix}, \ P_3 = \begin{bmatrix} G_{31} & G_{32} \\ G & \varepsilon_3 I_{n_u} \end{bmatrix}, \quad (44)$$

where ε_2 and ε_3 are scalar tuning parameters, G is some given constant matrix and $G_{2i}, G_{3i}, i = 1, 2$ are arbitrary matrices of appropriate dimensions. Then K_1, K_2 can be found from LMIs of Theorem 2.

Example 4: Consider the system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t_k) = [1 \ 0] x(t_k), \ t_k \le t < t_{k+1}, \ x(t) \in \mathcal{R}^2.$$

It is known that this system cannot be stabilizable by the non-delayed static output-feedback $u(t) = Ky(t_k), \ t_k \le t < t_{k+1}$. We take $m = 1$ and we search for the static output-feedback of the form

$$u(t) = Ky(t_{k-1}) = Ky(t_k - h), \quad t_k \le t < t_{k+1}, \ t_{k+1} - t_k = h. \quad (45)$$

Given the controller gain K, by applying Theorem 2 with $m = 1$ to the closed-loop system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} x(t_k - h) \quad (46)$$

we obtain the value of sampling period h that preserves the stability (see Table IV).

TABLE IV
EXAMPLE 4: THE VALUE OF SAMPLING PERIOD h FOR DIFFERENT CONTROLLER GAIN K

$N = 1$	$K = 1.0$	$h \in [0.320 \ \ 0.625]$
	$K = 1.5$	$h \in [0.189 \ \ 0.545]$
	$K = 1.8$	$h \in [0.154 \ \ 0.345]$
$N = 2$	$K = 1.0$	$h \in [0.293 \ \ 0.856]$
	$K = 1.5$	$h \in [0.100 \ \ 0.827]$
	$K = 1.8$	$h \in [0.153 \ \ 0.685]$

Example 5: Consider the second-order integrator [23]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t_k) = [1 \ 0] x(t_k), \ t_k \le t < t_{k+1}, \ x(t) \in \mathcal{R}^2.$$

This system also cannot be stabilizable by the non-delayed static output-feedback $u(t) = Ky(t_k), \ t_k \le t < t_{k+1}$. We take $m = 3$ and we look for the following static output-feedback:

$$\begin{aligned} u(t) &= K_1 y(t_k) + K_2 y(t_{k-3}) \\ &= K_1 y(t_k) + K_2 K y(t_k - 3h), \\ t_k &\le t < t_{k+1}, \ t_{k+1} - t_k = h. \end{aligned} \quad (47)$$

In order to illustrate the efficiency of our method, given the controller gains K_1 and K_2, we apply Theorem 2 with $m = 3$ to the closed-loop system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ K_1 & 0 \end{bmatrix} x(t_k) + \begin{bmatrix} 0 & 0 \\ K_2 & 0 \end{bmatrix} x(t_k - 3h). \quad (48)$$

The value of sampling period h that preserve the stability are given in Table V.

TABLE V
EXAMPLE 5: THE VALUE OF SAMPLING PERIOD h FOR DIFFERENT CONTROLLER GAINS K_1 AND K_2

$N = 1$	$K_1 = -0.350, \quad K_2 = 0.100$	$h \in [10^{-6} \ \ 0.436]$
	$K_1 = -0.7947, K_2 = 0.3067$	$h \in [10^{-6} \ \ 0.372]$
$N = 2$	$K_1 = -0.350, \quad K_2 = 0.100$	$h \in [10^{-5} \ \ 0.563]$
	$K_1 = -0.7947, K_2 = 0.3067$	$h \in [10^{-6} \ \ 0.464]$

VI. CONCLUSIONS

A novel discontinuous Lyapunov functional has been introduced for analysis of networked control systems under assumption of constant network-induced delay and of a bounded number of packet dropouts. The construction of the functional is based on the Wirtinger type inequality. The main result is applied to sample-data stabilization by using artificial delay. Numerical examples illustrate the efficiency of the new method.

VII. ACKNOWLEDGEMENT

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