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Primal and Dual Criteria for Robust Stability

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***Abstract*— Primal and dual formulations of stability criteria based on integral quadratic constraints (IQC) are discussed. The foundation for IQC based stability analysis is to use a convex cone of multipliers to characterize the uncertainty in a system. The primal and dual stability criteria are formulated as convex feasibility tests involving the nominal dynamics and multipliers from the cone and the polar cone, respectively. The motivation for introducing the dual is that it provides additional insight into the stability criterion and is sometimes easier to use than the primal.**

The case considered in this paper is when the uncertainty represents the interconnection of a complex network. The multipliers are used to describe characteristic properties of the network such as the spectral location or the structure of the underlying graph.

I. Introduction

In this paper we derive the dual of a class of primal stability criteria that involves point-wise in frequency multipliers from a convex cone. The multipliers are used to define integral quadratic constraints (IQCs) that characterize complicated or uncertain components in the system. In this paper we let the IQC characterize the network interconnection of a system where single-input single-output (SISO) linear systems are connected over the network. We will show that the dual criterion provides insight into the structure of the stability criterion which sometimes allow us to derive simpler and more explicit criteria.

The traditional point of view in large scale systems analysis is to characterize the various systems using integral quadratic constraints (IQC) and then combined these into an aggregate IQC that the interconnection operator must satisfy, see [8], [11] where dissipation theory was used and [7], [6] for general IQCs. Many applications of recent interest motivate the reverse point of view, i.e. to let the IQC characterize the network structure and then to verify that the subsystems jointly satisfy the complementary IQC. This is the motivation behind the examples in this paper.

In the next section we review some previous results that appeared in [4] and [2]. In section III we present the primal and dual stability criteria that will be used in Section IV where two new examples are considered.

A. Notation and Preliminaries

Let $\mathcal{A}^{n\times m}$ be the algebra of transfer functions obtained as the Laplace transforms of the impulse response functions (see [1])

$$h(t) = h_0(t)\theta(t) + \sum_{k=1}^{\infty} h_k\delta(t-t_k),$$

where $h_0(t) \in \mathbf{L}_1^{n\times m}[0,\infty)$, $h_k \in \mathbf{R}^{n\times m}$, $t_k \geq 0$, $\sum_{k=0}^{\infty}|h_k| < \infty$, and where $\theta(\cdot)$ is the unit step function and $\delta(\cdot)$ is the dirac distribution. Each $H \in \mathcal{A}^{n\times m}$ defines a stable system mapping $\mathbf{L}_2^n[0,\infty)$ to $\mathbf{L}_2^m[0,\infty)$ via the convolution integral

$$(Hw)(t) = \int_0^t h(t-\tau)v(\tau)d\tau + \sum_{k=0}^{\infty} h_k v(t-t_k).$$

and with induced norm $\|H\| = \sup_{\omega\in\mathbf{R}}(\bar{\sigma}(H(j\omega))$. The subclass of constantly proper systems $\mathcal{A}_{cp}^{n\times m}$ has $h_k = 0$ for $k \geq 1$ which implies that the transfer functions are continuous at infinity.

We let $S_{\mathbf{C}}^{m\times m} = \{X \in \mathbf{C}^{m\times m} : X = X^*\}$ be the vector space of Hermitian matrices equipped with the inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$ and the corresponding norm $\|X\| = \mathrm{tr}(X^2)^{1/2}$ (the Frobenius norm).

Suppose $K \subset S_{\mathbf{C}}^{m\times m}$ is a convex cone. The negative polar cone is the closed convex cone defined as

$$K^{\ominus} = \left\{Y \in \mathcal{S}_{\mathbf{C}}^{m\times m} : \langle X, Y\rangle \leq 0; \forall X \in K\right\}.$$

Finally, we will use the convex hull $\mathrm{co}\{w_1,\ldots,w_n\} := \{\sum_{i=1}^n \alpha_i w_i : \alpha_i \geq 0; \sum_{i=1}^n \alpha_i = 1\}$, the convex conic hull $\mathrm{cone}\{w_1,\ldots,w_n\} := \{\sum_{i=1}^n \alpha_i w_i : \alpha_i \geq 0\}$, and the direct sum of matrices $\oplus_{i=1}^n M_i = \mathrm{diag}(M_1,\ldots,M_n)$.

II. Introductory Examples

We consider the feedback interconnection of a transfer function $H \in \mathcal{A}_{cp}^{n\times n}$ with a network interconnection operator $\Gamma \in \mathcal{A}^{n\times n}$. The interconnection is called stable if

$$[H,\Gamma] := \begin{bmatrix} H \\ I \end{bmatrix} (I-\Gamma H)^{-1} \begin{bmatrix} -\Gamma & I \end{bmatrix} \in \mathcal{A}^{2n\times 2n}.$$

Our stability criteria will be formulated in terms of point-wise in frequency quadratic constraints on the transfer functions H and Γ. The starting point will be to find convex sets of multipliers that characterize the structural properties of Γ. Our characterizations will be frequency-wise and thus for any $\Gamma \in \mathbf{C}^{n\times n}$, we define

$$\Pi_\Gamma = \{\Pi \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \Pi_{11} \geq 0;\ \Pi_{22} \leq 0; \\ \Gamma^*\Pi_{11}\Gamma + \Gamma^*\Pi_{12} + \Pi_{12}^*\Gamma + \Pi_{22} \leq 0\},$$

which is a closed convex cone. The two conditions $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$ are included to simplify the discussion in this section. They convexify certain constraints and thus allow

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us to remove some technical assumptions. Less restrictive criteria will be discussed in the next section.

It is in general difficult to find the complete characterization and instead we need to restrict attention to subsets $\Pi_{j,\Gamma} \subset \Pi_\Gamma$, $j = 1, \ldots, N$, where each set is assumed to be a nonempty closed convex cone. Then we have the following primal and dual conditions for stability, see [4].

(a) **Primal condition:** For every $\omega \in \mathbf{R} \cup \{\infty\}$ there exists $\Pi \in \sum_{k=1}^{N} \Pi_{k,\Gamma(j\omega)}$ such that

$$(M_H \Pi)(j\omega) > 0 \tag{1}$$

(b) **Dual condition:** For every $\omega \in \mathbf{R} \cup \{\infty\}$ we have

$$M^{\times}_{H(j\omega)} Z \notin \cap_{k=1}^{N} \Pi^{\ominus}_{k,\Gamma(j\omega)} \tag{2}$$

for all $Z \in \mathcal{S}^{n\times n}_{\mathbf{C}}$ such that $Z \geq 0$ and $\mathrm{tr}(Z) = 1$.

Here the operator $M_H : \mathcal{S}^{2n\times 2n}_{\mathbf{C}} \to \mathcal{S}^{n\times n}_{\mathbf{C}}$ and its adjoint $M^{\times}_H : \mathcal{S}^{n\times n}_{\mathbf{C}} \to \mathcal{S}^{2n\times 2n}_{\mathbf{C}}$ are defined as

$$M_H \Pi = \begin{bmatrix} I \\ H \end{bmatrix}^* \Pi \begin{bmatrix} I \\ H \end{bmatrix}, \quad M^{\times}_H Z = \begin{bmatrix} I \\ H \end{bmatrix} Z \begin{bmatrix} I \\ H \end{bmatrix}^* \tag{3}$$

and the negative polar cone is the closed convex cone defined as

$$\Pi^{\ominus}_\Gamma = \left\{ W \in \mathcal{S}^{2n\times 2n}_{\mathbf{C}} : \langle W, \Pi \rangle \leq 0; \forall \Pi \in \Pi_\Gamma \right\}.$$

Note that at each frequency it is enough to find one cone $\Pi_{k_\omega,\Gamma(j\omega)}$ such that for every $Z \geq 0$ with $\mathrm{tr}(Z) = 1$ we have $M^{\times}_H Z \notin \Pi^{\ominus}_{k_\omega,\Gamma(j\omega)}$. Hence, the more multiplier descriptions we have the more likely it is to prove stability.

We provide two examples illustrating the two stability criteria.

Example 1: Consider the case when $H = \mathrm{diag}(H_1, \ldots, H_n)$, where each $H_k \in \mathcal{A}_{cp}$ and $\Gamma \in \mathbf{R}^{n\times n}$. The system represents a set of heterogeneous stable linear time-invariant (LTI) single-input single-output (SISO) systems interconnected over a network defined by the interconnection matrix Γ. One possibility is to use identical multipliers for the subsystems, i.e.

$$\Pi_\Gamma = \left\{ \begin{bmatrix} \pi_{11} I & \pi_{12} I \\ \bar{\pi}_{12} I & \pi_{22} I \end{bmatrix} : \pi_{11} \geq 0;\ \pi_{22} \leq 0; \right. \\ \left. \pi_{11}\Gamma^*\Gamma + \Gamma^*\pi_{12} + \pi_{12}^*\Gamma + \pi_{22} \leq 0 \right\}$$

Let us consider the case when Γ is a normal matrix, which means that Γ is unitarily diagonalizable. In this case the above characterization simplifies to $\Pi_\Gamma = \{\pi \otimes I_n : \pi \in \pi_\Lambda\}$, where

$$\pi_\Lambda = \left\{ \begin{bmatrix} \pi_{11} & \pi_{12} \\ \bar{\pi}_{12} & \pi_{22} \end{bmatrix} : \pi_{11} \geq 0;\ \pi_{22} \leq 0, \right. \\ \left. |\lambda_k|^2 \pi_{11} + 2\mathrm{Re}\,\bar{\lambda}_k \pi_{12} + \pi_{22} \leq 0;\ \forall \lambda_k \in \mathrm{eig}(\Gamma) \right\}.$$

In this case the primal condition becomes:

Primal condition: For every $\omega \in \mathbf{R} \cup \{\infty\}$ there exists $\pi \in \pi_\Lambda$ such that

$$\pi_{11} + 2\mathrm{Re}\,\pi_{12} H_k(j\omega) + \pi_{22}|H_k(j\omega)|^2 > 0, \tag{4}$$

for $k = 1, \ldots, n$.

By using [4] and [2] it can be shown that the dual criterion reduces to

Dual condition: For all $\omega \in \mathbf{R} \cup \{\infty\}$

$$\mathcal{N}[H_1, \ldots, H_n](\omega) \cap \Omega = \emptyset,$$

where the 3-dimensional *Nyquist polytope* is defined as

$$\mathcal{N}[H_1, \ldots, H_n](\omega) \\ = \mathrm{co}\{(\mathrm{Re}\, H_k(j\omega), \mathrm{Im}\, H_k(j\omega), |H_k(j\omega)|^2) : k = 1, \ldots, n\}$$

and the *instability region* is defined as

$$\Omega = \left\{ \alpha \cdot \mathrm{co}\left\{ \left(\mathrm{Re}\,\frac{1}{\lambda_k}, \mathrm{Im}\,\frac{1}{\lambda_k}, \frac{1}{|\lambda_k|^2} \right) : \lambda_k \in \{\lambda_1, \cdots, \lambda_n\}; \right. \right. \\ \left. \lambda_k \neq 0 \right\}, \alpha \geq 1 \right\} + (0, 0, \mathbf{R}_+)$$

This criterion can be visualized in a 3-dimensional Nyquist diagram. We refer to [2] for a more in-depth discussion about such criteria.

Example 2: Consider the feedback interconnection of

$$H = \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which corresponds to a negative feedback interconnection of a plant $G \in \mathcal{A}_{cp}$ with a compensator $K \in \mathcal{A}_{cp}$. It is easy to verify that we may use the multipliers

$$\Pi_\Gamma = \left\{ \begin{bmatrix} x_1 & & y & \\ & x_2 & & \bar{y} \\ \bar{y} & & -x_2 & \\ & y & & -x_1 \end{bmatrix} : x_1, x_2 \geq 0; y \in \mathbf{C} \right\} \tag{5}$$

In this case the primal condition becomes:

Primal condition: For every $\omega \in \mathbf{R} \cup \{\infty\}$ there exists $x_1, x_2 \geq 0$ and $y \in \mathbf{C}$ such that

$$x_1 - x_2|G(j\omega)|^2 + 2\mathrm{Re}\, yG(j\omega) > 0, \\ x_2 - x_1|K(j\omega)|^2 + 2\mathrm{Re}\, \bar{y}K(j\omega) > 0.$$

It is shown in [4] that the dual in this case reduces to the following simple criterion

Dual condition: For all $\omega \in \mathbf{R} \cup \{\infty\}$, $G(j\omega)K(j\omega) \notin (-\infty, -1]$.

The idea in this example can be generalized to systems interconnected over a bi-partite graph. The bipartite graph can be given a characterization analogous to (5) with a resulting stability criterion that takes both phase and gain of the loop gain into account. We discuss this in further detail in [4].

III. Primal and Dual Stability Criteria

We consider the feedback interconnection $[\mathbf{\Gamma}, \mathbf{H}]$ defined as

$$v = \mathbf{\Gamma} w + r_1, \\ w = \mathbf{H} v + r_2, \tag{6}$$

where $\mathbf{H}$ and $\mathbf{\Gamma}$ are linear causal operators on $\mathbf{L}_2^n[0, \infty)$. This interconnection is called stable if there exists $c > 0$ such that $\|v\|_{\mathbf{L}_2} + \|w\|_{\mathbf{L}_2} \leq c\|r\|_{\mathbf{L}_2}$, for all $r \in \mathbf{L}_2^{2n}[0, \infty)$. In this paper we restrict attention to the case when $\mathbf{H}$ is the convolution operator corresponding to a transfer function $H \in \mathcal{A}_{cp}^{n\times n}$ and $\mathbf{\Gamma}$ either is the convolution operator corresponding to a transfer function $\Gamma \in \mathcal{A}^{n\times n}$ or a constant real or complex matrix $\Gamma \in \mathbf{C}^{n\times n}$. Thus we assume both operators to be bounded.

In the stability results below we use the notation $[\Gamma, H]$ to denote the feedback interconnection in (6) in the case when $\mathbf{H}$ is defined by a transfer function $H \in \mathcal{A}_{cp}$ and $\mathbf{\Gamma}$ either is a transfer function $\Gamma \in \mathcal{A}^{n\times n}$ or a real matrix. We further use the operators $M_H : \mathcal{S}_{\mathbf{C}}^{2n\times 2n} \to \mathcal{S}_{\mathbf{C}}^{n\times n}$ and $M_H^{\times} : \mathcal{S}_{\mathbf{C}}^{n\times n} \to \mathcal{S}_{\mathbf{C}}^{2n\times 2n}$ defined in (3).

Assumption 1 (Assumptions under known Γ):

(a) There exists sets of multipliers of the form

$$\Pi_{k,\Gamma} \subset \{\Pi \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \Pi_{22} \leq 0; \\ \Gamma^*\Pi_{11}\Gamma + \Gamma^*\Pi_{12} + \Pi_{12}^*\Gamma + \Pi_{22} \leq 0\},$$

$k = 1, \ldots, N$, which are assumed to be closed convex cones.

(b) There exists a transfer function $H_0 \in \mathcal{A}_{cp}^{n\times n}$ such that $[\Gamma, H_0]$ is stable.

Theorem 1: Under Assumption 1 the system (6) is stable if either of the following equivalent conditions are satisfied

(a) **Primal condition:** For every $\omega \in \mathbf{R} \cup \{\infty\}$ there exists $\Pi \in \sum_{k=1}^{N} \Pi_{k,\Gamma(j\omega)}$ such that

$$(M_H\Pi)(j\omega) > 0 \quad \text{and} \quad (M_{H_0}\Pi)(j\omega) > 0. \qquad (7)$$

(b) **Dual condition:** For every $\omega \in \mathbf{R} \cup \{\infty\}$

$$(M_H^{\times}Z_1)(j\omega) + (M_{H_0}^{\times}Z_2)(j\omega) \notin \cap_{k=1}^{N} \Pi_{k,\Gamma}^{\ominus} \qquad (8)$$

for all $(Z_1, Z_2) \in \mathcal{Z}$ where

$$\mathcal{Z} = \{(Z_1, Z_2) \in (\mathcal{S}_{\mathbf{C}}^{n\times n} \times \mathcal{S}_{\mathbf{C}}^{n\times n}) : Z_1, Z_2 \geq 0; \\ \mathrm{tr}(Z_1) + \mathrm{tr}(Z_2) = 1\}.$$

It is often the case that the network interconnection is not exactly specified or known. Assume that $\Gamma \in \mathcal{S}_{\Gamma}$, where $\mathcal{S}_{\Gamma}$ is a set of network interconnection operators such that following assumption holds:

Assumption 2 (Assumptions under uncertain Γ):

(a) There exists sets of multipliers of the form

$$\Pi_{k,\Gamma} \subset \{\Pi \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \\ \Gamma^*\Pi_{11}\Gamma + \Gamma^*\Pi_{12} + \Pi_{12}^*\Gamma + \Pi_{22} \leq 0,\ \forall \Gamma \in \mathcal{S}_{\Gamma}\},$$

$k = 1, \ldots, N$, which are assumed to be closed convex cones. Note that the constraints $\Pi_{22} \leq 0$ are no longer necessary to include.

(b) the set $\mathcal{S}_{\Gamma}$ is connected in the induced norm topology

(c) there exists $\Gamma_0 \in \mathcal{S}_{\Gamma}$ such that the interconnection $[\Gamma_0, H]$ is stable.

Given these assumptions we get the alternative result

Corollary 1: Under Assumption 2 the system (6) is stable if either of the following equivalent conditions are satisfied

(a) **Primal condition:** For $\omega \in \mathbf{R} \cup \{\infty\}$ there exists $\Pi \in \sum_{k=1}^{N} \Pi_{k,\Gamma(j\omega)}$ such that

$$(M_H\Pi)(j\omega) > 0 \qquad (9)$$

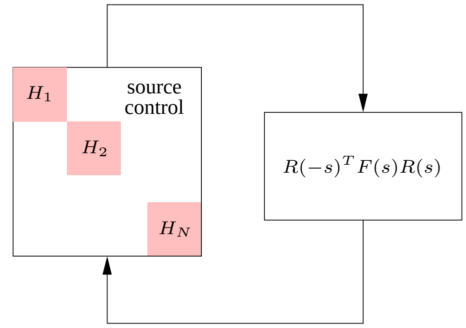

Fig. 1. Equilibrium dynamics of Internet congestion control.

(b) **Dual condition:** For every $\omega \in \mathbf{R} \cup \{\infty\}$

$$(M_H^{\times}Z)(j\omega) \notin \cap_{k=1}^{N} \Pi_{k,\Gamma}^{\ominus} \qquad (10)$$

for all $Z \in \mathcal{Z}$ where

$$\mathcal{Z} = \{Z \in (\mathcal{S}_{\mathbf{C}}^{n\times n} : Z \geq 0;\ \mathrm{tr}(Z) = 1\}.$$

Remark 1: Note that the primal and dual conditions in the two results are identical in the case when $H_0 = 0$. We consider this case in the applications below.

Remark 2: A proof of Theorem 1 can be found in [4]. Some generalizations of these results can be found in [3].

IV. Applications

In this section we consider a few examples illustrating the primal and dual stability criterion. We consider the case when a set of linear single-input single-output dynamics $\{H_k : k = 1, \ldots, n\}$ are interconnected over a network described by Γ. This implies that $H = \mathrm{diag}(H_1, \ldots, H_n)$ in the results in the previous section.

A. Bi-Partite Interconnections

In the next example we consider a model of Internet congestion control where the the routing matrices and the link dynamics have been combined into one block, see e.g. [5] for modeling details. The block diagram is illustrated in Figure 1, where $F = \oplus_{l=1}^{L} F_l$ and $\rho(R(j\omega)^* R(j\omega)) \leq 1$. Here $R(s)$ is a routing matrix whose only dynamics are delays.

Then the system is an interconnection on the form (6) with[1]

$$\Gamma(s) = R(-s)^T F(s) R(s)$$

For simplicity assume that $G_k, F_k \in \mathcal{A}_{cp}$ so that we can use Corollary 1 with $H_0 = 0$. Let us use frequency-wise multipliers of the form

$$\Pi_{\Gamma} = \left\{ \begin{bmatrix} \pi_{11} I_n & \pi_{12} I_n \\ \pi_{12}^* I_n & \pi_{22} I_n \end{bmatrix} \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \pi_{22} \leq 0; \\ \Gamma^*\Gamma\pi_{11} + \Gamma^*\pi_{12} + \pi_{12}^*\Gamma + \pi_{22} \leq 0 \right\}, \quad (11)$$

where $\Gamma := R(j\omega)^* F(j\omega) R(j\omega))$. This means that at each frequency the H_k must agree on one multiplier defined by the

[1]Note that $\Gamma \notin \mathcal{A}$ but the system can be transformed in such a way so the criterion in this example is a valid stability criterion. See [4].

π_{kl} variables, for $k,l=1,2$. The primal can be formulated as

Primal condition: For every $\omega\in\mathbf{R}\cup\{\infty\}$ there exists $\pi\in\pi_{\Gamma(j\omega)}$ such that

$$\pi_{11}+2\mathrm{Re}\,\pi_{12}H_k(j\omega)+\pi_{22}|H_k(j\omega)|^2>0,$$

where for $\Gamma=\Gamma(j\omega)$ we define

$$\begin{aligned}\pi_\Gamma=&\{\pi\in\mathcal{S}_{\mathbf{C}}^{2\times 2}:\pi_{22}\le 0;\\ &\qquad\Gamma^*\Gamma\pi_{11}+\Gamma^*\pi_{12}+\pi_{12}^*\Gamma+\pi_{22}\le 0\}\\ =&\{\pi\in\mathcal{S}_{\mathbf{C}}^{2\times 2}:\langle\pi,\,V_0\rangle\le 0;\\ &\qquad\langle\pi,\,V_1(v)\rangle\le 0,\ v\in\mathbf{C}^n;|v|=1\}\end{aligned}$$

where

$$V_0=\begin{bmatrix}0&0\\0&1\end{bmatrix},\qquad V_1(v)=\begin{bmatrix}v^*\Gamma^*\Gamma v&v^*\Gamma v\\ v^*\Gamma^* v&v^*v\end{bmatrix}.$$

Next we derive the dual condition. The condition $W\in\Pi_\Gamma^\ominus$ is equivalent to

$$\langle\Phi,\,W\rangle=\left\langle\begin{bmatrix}\pi_{11}&\pi_{12}\\ \bar\pi_{12}&\pi_{22}\end{bmatrix},\begin{bmatrix}\mathrm{tr}(W_{11})&\mathrm{tr}(W_{12})\\ \overline{\mathrm{tr}(W_{12})}&\mathrm{tr}(W_{22})\end{bmatrix}\right\rangle\le 0,$$

for all $\pi\in\pi_\Gamma$. It can be shown that

$$\begin{aligned}\pi_\Gamma^\ominus&=\mathrm{cl\,cone}\{V_0,V_1(v):v\in\mathbf{C}^n;|v|=1\}\\ &=\mathrm{cl\,cone}\left\{\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}v^*\Gamma^*\Gamma v&v^*\Gamma v\\ v^*\Gamma^*v&v^*v\end{bmatrix},|v|=1\right\},\end{aligned}$$

see, e.g. Chapter 14 of [10]. It follows that the polar cone becomes

$$\Pi_\Gamma^\ominus=\left\{\begin{bmatrix}W_{11}&W_{12}\\ W_{12}^*&W_{22}\end{bmatrix}\in\mathcal{S}_{\mathbf{C}}^{2n\times 2n}:\begin{bmatrix}\mathrm{tr}(W_{11})&\mathrm{tr}(W_{12})\\ \mathrm{tr}(W_{12}^*)&\mathrm{tr}(W_{22})\end{bmatrix}\not\in\pi_\Gamma^\ominus\right\}.$$

Hence, the dual condition $M_HZ\not\in\Pi_\Gamma^\ominus$ becomes

$$\begin{bmatrix}\mathrm{tr}(Z)&\mathrm{tr}(ZH^*)\\ \mathrm{tr}(HZ)&\mathrm{tr}(HZH^*)\end{bmatrix}\not\in\pi_\Gamma^\ominus$$

Since H is diagonal it is no restriction to let Z be diagonal with nonnegative diagonal elements satisfying $\sum_{k=1}^n z_k=1$. This gives the following dual:

Dual condition: For every $\omega\in\mathbf{R}\cup\{\infty\}$

$$\mathrm{co}\left\{\begin{bmatrix}1&H_k(j\omega)^*\\ H_k(j\omega)&|H_k(j\omega)|^2\end{bmatrix}:k=1,\dots,n\right\}\cap\pi_{\Gamma(j\omega)}^\ominus=\emptyset.$$

The primal and dual criterion provide little insight as they are formulated. They must be verified using computations. However, it turns out that a rather simple criterion can be derived from the dual if we accept some additional conservatism. To this end we notice that the change of variables $\hat z_k=z_k|H_k(j\omega)|^2$ in (note that the criterion is satisfied if $H_k(j\omega)=0$)

$$\sum_{k=1}^N z_k\begin{bmatrix}1\\ H_k(j\omega)\end{bmatrix}\begin{bmatrix}1\\ H_k(j\omega)\end{bmatrix}^*\not\in\pi_\Gamma^\ominus$$

gives the alternative formulation

$$\sum_{k=1}^N\hat z_k\begin{bmatrix}H_k(j\omega)^{-1}\\ 1\end{bmatrix}\begin{bmatrix}H_k(j\omega)^{-1}\\ 1\end{bmatrix}^*\not\in\pi_\Gamma^\ominus,\qquad(12)$$

where we may assume $\hat z_k\ge 0$ satisfies $\sum_{k=1}^N\hat z_k=1$. This follows since it is possible to re-scale by multiplying with a positive number on both sides of the equation. It is then no loss of generality to assume that the elements in $\pi_\Gamma^\ominus$ are scaled such that the $(2,2)$ element satisfies

$$1-\alpha+\alpha|v|^2=1,\ \alpha\in[0,1],\ |v|=1,$$

which implies that (12) equivalently can be stated as

$$\begin{aligned}\mathrm{co}\left\{\left(\frac{1}{H_k(j\omega)},\frac{1}{|H_k(j\omega)|^2}\right):H_k(j\omega)\neq 0\right\}\\ \not\in\mathrm{cl\,cone}\left\{(v^*\Gamma v,v^*\Gamma^*\Gamma^*v):|v|\le 1\right\}\end{aligned}$$

To simplify this criterion we use an idea which we adopt from [12], [5] to overestimate the set on the right hand side. Let R_l denote the l^{th} row of R. Since $\rho(R^*R)\le 1$ it follows $\sum_{l=1}^L|R_lv|^2\le 1$ and thus

$$\begin{aligned}(v^*\Gamma v,v^*\Gamma^*\Gamma v)&=(v^*R^*FRv,v^*R^*F^*RR^*FRv)\\ &\subset\{(v^*R^*FRv,\alpha v^*R^*F^*FRv):\alpha\in[0,1]\}\\ &=\left\{\sum_{l=1}^L(F_l,\alpha|F_l|^2)|R_lv|^2:\alpha\in[0,1]\right\}\\ &=\left\{(\sum_{l=1}^L|R_lv|^2)\sum_{l=1}^L(F_l,\alpha|F_l|^2)\frac{|R_lv|^2}{\sum_{l=1}^L|R_lv|^2}:\alpha\in[0,1]\right\}\\ &\subset\left\{\mathrm{co}\{(0,0),(F_l,\alpha|F_l|^2):l=1,\dots,L\}:\alpha\in[0,1]\right\}\\ &=\mathrm{co}\{(0,0),(F_l,0),(F_l,|F_l|^2):l=1,\dots,L\}\end{aligned}$$

It follows that

$$\begin{aligned}&\mathrm{cl\,cone}\left\{(v^*\Gamma v,v^*\Gamma^*\Gamma v):|v|\le 1\right\}\\ &\qquad\subset\mathrm{co}\{(0,0),(F_l,0),(F_l,|F_l|^2):l=1,\dots,L\}\end{aligned}$$

and hence the dual condition is satisfied if

$$\begin{aligned}&\mathrm{co}\left\{\left(\frac{1}{H_k(j\omega)},\frac{1}{|H_k(j\omega)|^2}\right):H_k(j\omega)\neq 0\right\}\cap\\ &\quad\mathrm{co}\{(0,0),(F_l,0),(F_l,|F_l|^2)(j\omega):l=1,\dots,L\}=\emptyset\end{aligned}$$

If we project to the complex plane we get the following simple dual.

Simple dual condition: For every $\omega\in\mathbf{R}\cup\{\infty\}$

$$\mathrm{co}\{H_1^{-1},\dots,H_N^{-1}\}(j\omega)\cap\mathrm{co}\{0,F_1,\dots,F_L\}(j\omega)=\emptyset.$$

B. Sparse Graphs

Consider the case when the interconnection $\Gamma=[\gamma_{ij}]_{i,j=1}^n$ is defined in terms of the adjacency matrix of a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with vertex set $\mathcal{V}=\{1,2,\dots,n\}$ and edge set $\mathcal{E}$. Then γ_{ij} is the number of edges from vertex j to vertex i and γ_{ii} is the number of loops from vertex i to itself. The dynamics in node i can be represented as

$$w_i=\mathbf{H}_i(\sum_{j=1}^n\gamma_{ij}w_j+r_{1,i})+r_{2,j}$$

If the graph is sparse then this sum only contains a few terms. We will next discuss how to use the detailed graph structure to define multipliers. Weighted graphs are also covered by our results and we may consider $\Gamma \in \mathcal{A}^{n\times n}$ by doing frequency-wise analysis.

Let

$$\Gamma = \begin{bmatrix}\gamma_1^* \\ \vdots \\ \gamma_n^*\end{bmatrix}$$

where $\gamma_i^* = \begin{bmatrix}\gamma_{i1} & \dots & \gamma_{in}\end{bmatrix}$. We may then use the multipliers

$$\Pi_k = \left\{ \begin{bmatrix}\pi_{11}e_k e_k^T & \pi_{12}e_k\gamma_k^* \\ \bar{\pi}_{12}\gamma_k e_k^T & \pi_{22}\gamma_k\gamma_k^*\end{bmatrix} : \pi \in \pi_{set} \right\}$$

where e_k is the k^{th} unit vector in $\mathbf{R}^n$ and where π_{set} constrains the multipliers appropriately. We will here consider the case when

$$\pi_{set} = \Big\{ \pi = \begin{bmatrix}\pi_{11} & \pi_{12} \\ \bar{\pi}_{12} & \pi_{22}\end{bmatrix} : \psi_k^*\pi\psi_k \leq 0,\ k = 1,2; \\ \psi_3^*\pi\psi_3 \geq 0 \Big\}.$$

where $\psi_1^T = \begin{bmatrix}1 & 1\end{bmatrix}$ and either

$$\psi_2^T = \begin{bmatrix}0 & 1\end{bmatrix} \text{ and } \psi_3^T = \begin{bmatrix}0 & 0\end{bmatrix} \tag{13}$$

or

$$\psi_2^T = \begin{bmatrix}0 & 1\end{bmatrix} \text{ and } \psi_3^T = \begin{bmatrix}1 & 0\end{bmatrix}. \tag{14}$$

The first inequality involving ψ_1 ensures that the multiplier characterization is valid while either of the last two could be included in order to enforce convexity of the multipliers to apply the results in the previous section. Indeed, in the case (13) we may apply Theorem 1 and in the case (14) we may apply Corollary 1 with $\Gamma_0 = 0$. A special case of (14) that will considered here is

$$\pi_{set} = \left\{ \pi = \begin{bmatrix}x & iy \\ -iy & -x\end{bmatrix} : x \geq 0;\ y \in \mathbf{R} \right\}. \tag{15}$$

Primal Stability Criterion: First, note that the multipliers in Π_k define valid characterizations of the graph since

$$\Gamma^*\Pi_{11}\Gamma + \Gamma^*\Pi_{12} + \Pi_{12}^*\Gamma + \Pi_{22} = \gamma_k\gamma_k^*\psi_1^*\pi\psi_1 \leq 0$$

since $\pi \in \pi_{set}$.

Primal condition: For every $\omega \in [0,\infty]$ there exists $\pi_k \in \pi_{set}$, $k = 1,\dots,n$, such that

$$\Psi = \sum_{k=1}^{n} M_H\Pi_k = \sum_{k=1}^{n}\begin{bmatrix}I \\ H(j\omega)\end{bmatrix}\begin{bmatrix}\pi_{k,11}e_k e_k^T & \pi_{k,12}e_k\gamma_k^* \\ \bar{\pi}_{k,12}\gamma_k e_k^T & \pi_{k,22}\gamma_k\gamma_k^*\end{bmatrix}\begin{bmatrix}I \\ H(j\omega)\end{bmatrix}^* > 0$$

This criterion has some attractive scalability properties: 1) the number of parameters to optimize grows linearly as 4n with the dimension of the graph, 2) if one more dynamics enter the network or some system is modified, feasibility of the primal problem can be tested using an inexpensive sufficient test. To see this, let us introduce the notation

$$\check{\Gamma} = \begin{bmatrix}\check{\gamma}_1^* \\ \vdots \\ \check{\gamma}_n^* \\ \check{\gamma}_{n+1}^*\end{bmatrix} = \left[\begin{array}{c|c}\gamma_1^* & \gamma_{1,n+1} \\ \vdots & \vdots \\ \gamma_n^* & \gamma_{n,n+1} \\ \hline \gamma_{n+1}^* & \gamma_{n+1,n+1}\end{array}\right]$$

$$\check{H} = \oplus_{k=1}^{n+1}H_k = \left[\begin{array}{c|c}H & 0 \\ \hline 0 & H_{n+1}\end{array}\right]$$

$$\check{\Pi}_k = \begin{bmatrix}\pi_{k,11}\check{e}_k\check{e}_k^T & \pi_{k,12}\check{e}_k\check{\gamma}_k^* \\ \bar{\pi}_{k,12}\check{\gamma}_k\check{e}_k^T & \pi_{k,22}\check{\gamma}_k\check{\gamma}_k^*\end{bmatrix}$$

where $\check{e}_k$ is the k^{th} unit vector in $\mathbf{R}^{n+1}$ and $\pi_k \in \pi_{set}$.

The primal condition becomes

$$\check{\Psi} = \sum_{k=1}^{n+1} M_{\check{H}}\check{\Pi}_k = \begin{bmatrix}\Psi + \check{\Psi}_{11} & \check{\Psi}_{12} \\ \check{\Psi}_{12}^* & \check{\Psi}_{22}\end{bmatrix} > 0$$

where the blocks are defined in terms of the multipliers as

$$\begin{aligned}
\Psi &= \sum_{k=1}^{n} M_H\Pi_k \\
\check{\Psi}_{11} &= \pi_{n+1,22}H^*\gamma_{n+1}\gamma_{n+1}^*H \\
\check{\Psi}_{12} &= \sum_{k=1}^{n}\pi_{k,12}e_k\gamma_{k,n+1}H_{n+1} + \bar{\pi}_{n+1,12}H^*\gamma_{n+1} \\
&\quad + \pi_{n+1,22}\gamma_{n+1,n+1}H^*\gamma_{n+1} \\
\check{\Psi}_{22} &= M_{H_{n+1}}\Pi_{n+1} + \sum_{k=1}^{n+1}\pi_{k,22}|\gamma_{k,n+1}|^2|H_k|^2
\end{aligned}$$

where

$$\Pi_{n+1} = \begin{bmatrix}\pi_{n+1,11} & \pi_{n+1,12}\gamma_{n+1,n+1} \\ \bar{\pi}_{n+1,12}\bar{\gamma}_{n+1,n+1} & \pi_{n+1,22}|\gamma_{n+1,n+1}|^2\end{bmatrix}$$

We will discuss a low complexity and low dimensional test to verify that the system remains stable when the H_{n+1} enters the network (or is perturbed). We assume that

$$\Psi(j\omega) = \sum_{k=1}^{n}(M_H\Pi_k)(j\omega) \geq \phi(j\omega)I$$

for some strictly positive function ϕ. Assume further that the dynamics in node $n+1$ only communicates with the dynamics in nodes $\mathcal{N} \subset \{1,\dots,n\}$, i.e. $k \in \mathcal{N}$ if and only if either $\gamma_{k,n+1} \neq 0$ or/and $\gamma_{n+1,k} \neq 0$. Let

$$E = [\ e_k : \ \ k \in \mathcal{N}\ ]$$

Then the system remains stable if there exists[2] $\pi_{n+1} \in \pi_{set}$ such that

$$\begin{bmatrix}\phi I + E^T\check{\Psi}_{11}E & E^T\check{\Psi}_{12} \\ \check{\Psi}_{12}^*E & \check{\Psi}_{22}\end{bmatrix}(j\omega) > 0$$

for all $\omega \in \mathbf{R}\cup\{\infty\}$. This "greedy" approach of accommodating new dynamics into the stability test will not always work but is simple to use and inexpensive. It works analogously to verify that the system remains stable if some of the dynamics is perturbed.

[2] All other multipliers are kept fixed.

Dual Stability Criterion: In order to derive the dual stability criterion we first notice that

$$\Pi_k^{\ominus} = \left\{ W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^* & W_{22} \end{bmatrix} : \begin{bmatrix} e_k^T W_{11} e_k & e_k^T W_{12}\gamma_k \\ \gamma_k^* W_{12}^* e_k & \gamma_k^* W_{22}\gamma_k \end{bmatrix} \in \pi_{set}^{\ominus} \right\}$$

which follows since it is easy to show that for any $\Pi \in \Pi_k$ we have

$$\langle W, \Pi \rangle = \left\langle \begin{bmatrix} e_k^T W_{11} e_k & e_k^T W_{12}\gamma_k \\ \gamma_k^* W_{12}^* e_k & \gamma_k^* W_{22}\gamma_k \end{bmatrix}, \begin{bmatrix} \pi_{11} & \pi_{12} \\ \bar{\pi}_{12}^* & \pi_{22} \end{bmatrix} \right\rangle$$

which is non-positive if and only if

$$\begin{bmatrix} e_k^T W_{11} e_k & e_k^T W_{12}\gamma_k \\ \gamma_k^* W_{12}^* e_k & \gamma_k^* W_{22}\gamma_k \end{bmatrix} \in \pi_{set}^{\ominus}$$

We will first focus on the case (15) in which

$$\pi_{set}^{\ominus} = \left\{ W = \begin{bmatrix} w_{11} & w_{12} \\ \bar{w}_{12} & w_{22} \end{bmatrix} : \text{Im}\, w_{12} = 0;\ w_{11} - w_{22} \le 0 \right\}$$

Since the condition $M_H Z \in \Pi_k^{\ominus}$ equivalently can be written as

$$\begin{aligned} \text{Im}\, \gamma_k^* H Z e_k &= 0 \\ e_k^T Z e_k - \gamma_k^* H Z H^* \gamma_k &\le 0 \end{aligned} \tag{16}$$

we get the dual:

Dual condition: For every $\omega \in \mathbf{R} \cup \{\infty\}$ and all $Z \in \mathcal{Z}$ there exists $k \in \{1, \ldots, n\}$ such that the equation system

$$\begin{aligned} \text{Im}\, \gamma_k^* H Z e_k &= 0 \\ e_k^T Z e_k - \gamma_k^* H Z H^* \gamma_k &\le 0 \end{aligned}$$

is violated.

Next consider the case when π_{set} is defined by (13). In this case

$$\pi_{set}^{\ominus} = \text{cone}\{\psi_1\psi_1^T, \psi_2\psi_2^T\}.$$

Then the condition $M_H Z \in \Pi_k^{\ominus}$ is equivalent to

$$\begin{bmatrix} e_k^T Z e_k & e_k^T Z H \gamma_k \\ \gamma_k^* H Z e_k & \gamma_k^* H Z H^* \gamma_k \end{bmatrix} = \begin{bmatrix} \nu_1 & \nu_1 \\ \nu_1 & \nu_1 + \nu_2 \end{bmatrix}$$

for some $\nu_1, \nu_2 \ge 0$. Using the upper left equation in the lower right equation and the fact that ψ_1 is a real number gives the equations

$$\begin{aligned} \text{Im}\, e_k^T Z H \gamma_k &= 0 \\ \text{Re}\, e_k^T Z H \gamma_k &= e_k^T Z e_k \\ e_k^T Z e_k - \gamma_k^* H Z H^* \gamma_k &\le 0 \end{aligned} \tag{17}$$

We get the dual:

Dual condition: For every $\omega \in \mathbf{R} \cup \{\infty\}$ and all $Z \in \mathcal{Z}$ there exists $k \in \{1, \ldots, n\}$ such that the equation system

$$\begin{aligned} \text{Im}\, \gamma_k^* H Z e_k &= 0 \\ \text{Re}\, e_k^T Z H \gamma_k &= e_k^T Z e_k \\ e_k^T Z e_k - \gamma_k^* H Z H^* \gamma_k &\le 0 \end{aligned}$$

is violated.

To get more insight into these dual criteria consider the first equation in (16). It can be shown using Lemma 5 in [9] that $Z = \sum_{l=1}^n z_l z_l^*$, where

$$\gamma_k^* H z_l z_l^* e_k - e_k^T z_l z_l^* H^* \gamma_k = 0 \tag{18}$$

If we use this in the last equation of (16) we get

$$\sum_{l=1}^{n} (|e_k^T z_l|^2 - |\gamma_k^* H z_l|^2) \le 0$$

At least one of these terms must be negative. Hence, we have established the existence of a nonzero vector $z \in \mathbf{C}^n$ such that

$$\begin{aligned} \gamma_k^* H z z^* e_k - e_k^T z z^* H^* \gamma_k &= 0 \\ e_k^T z z^* e_k - \gamma_k^* H z z^* H^* \gamma_k &\le 0 \end{aligned}$$

Vice versa, if the above two equations hold then $Z = zz^*$ satisfies $M_H Z \in \Pi_k^{\ominus}$. Note that the two equations are trivial unless $e_k^T z = z_k \neq 0$ in which case they are equivalent to the condition

$$\frac{\gamma_k^* H z}{e_k^T z} \in (-\infty, -1] \cup [1, \infty).$$

Let us now consider the condition: $\exists Z \in \mathcal{Z}$ such that $M_H Z \in \cap_{k=1}^n \Pi_k^{\ominus}$. In order for (18) to hold for all $k = 1, \ldots, n$ we must let the representation $Z = \sum_{l=1}^n z_l^k (z_l^k)^*$ depend on the index k. It is easy to see that $z_l^k = Z^{1/2} u_l^k$, $l = 1, \ldots, n$ where u_l^k are orthonormal. This gives the following alternative formulation of the first dual condition in this section

Alternative dual: For every $\omega \in \mathbf{R} \cup \{\infty\}$ and $Z \in \mathcal{Z}$ there exists $k \in 1, \ldots, n$ such that for all orthonormal u_l, $l = 1, \ldots, n$, either

$$\gamma_k^* H z_l z_l^* e_k - e_k^T z_l z_l^* H^* \gamma_k = 0$$

is violated for some $z_l = Z^{1/2} u_l$ or

$$\sum_{l=1}^{n} (|e_k^T z_l|^2 - |\gamma_k^* H z_l|^2) > 0.$$

We note that a necessary condition for the dual to hold is obtained as follows: Let $z_{unit} \stackrel{\text{def}}{=} \{z \in \mathbf{C}^n : |z| = 1\}$. If the dual above is satisfied then

$$\forall z \in z_{unit},\ \exists k \quad s.t. \quad \frac{\gamma_k^* H z}{e_k^T z} \notin (-\infty, -1] \cup [1, \infty).$$

This necessary condition is straightforward to interpret. It implies that the equation

$$(\lambda I - \Gamma H(j\omega)) z = 0$$

has no nontrivial solution for any $|\lambda| \ge 1$. Since it holds for any $\omega \in \mathbf{R} \cup \{\infty\}$ it follows that

1) the closed loop system is well-posed, i.e. $I - \Gamma H(\infty)$ is an invertible matrix,
2) $\phi(j\omega) = \det(I - \tau\Gamma H(j\omega)) \neq 0$, for all $\omega \in \mathbf{R} \cup \{\infty\}$ and any $\tau \in [0, 1]$.

The second conditions is a zero exclusion property which allow us to conclude stability of the network. The alternative dual requires more: For every subspace, there exists k such that either $\mathrm{Im}\, e_k^T z_l z_l^* H^* \gamma_k = 0$ is violated for some z_l or $\sum_{l=1}^n (|e_k^T z_l|^2 - |\gamma_k^* H z_l|^2) > 0$. This gives an indication of the conservatism of the criterion.

REFERENCES

[1] C.A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, New York, 1975.

[2] U. T. Jönsson and C.-Y. Kao. A scalable robust stability criterion for systems with heterogeneous lti components. *IEEE Transactions on Automatic Control*, 2010. Accepted to be published. The electric preprint is available via IEEEXplore.

[3] Ulf T. Jönsson. Primal and dual stability criteria for systems interconnected over uncertain and time-varying networks. Draft paper: Spring 2010.

[4] U.T. Jönsson. Primal and dual stability criteria for robust stability and their application to systems interconnected over a bi-partite graph. In *Proceedings of the American Control Conference*, Baltimore, MD, USA, 2010.

[5] I. Lestas and G. Vinnicombe. Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems. *IEEE Transactions on Automatic Control*, 51(10):1613–1625, 2006.

[6] A Megretski, C. Kao, U. Jönsson, and A. Rantzer. *A Guide To IQC-beta: Software for Robustness Analysis*. Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, 1997.

[7] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 42(6):819–830, June 1997.

[8] P. Moylan and D. Hill. Stability criteria for large-scale systems. *IEEE Transactions on Automatic Control*, 23(2):143–149, 1978.

[9] A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters*, 28(1):7–10, June 1996.

[10] R.T Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[11] M. Vidyasagar. *Input-Output Analysis of Large Scale Interconnecred Systems.* Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1981.

[12] G. Vinnicombe. Robust congestion control for the internet. 2002.