A State-Space Solution of Bilateral Diophantine Equations over $\mathcal{RH}_\infty$

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***Abstract*— This paper studies a class of real-rational matrix bilateral Diophantine equations (BDE) arising in numerous control problems. A necessary and sufficient solvability condition is derived in terms of state-space realizations of rational matrices involved in the equation. This condition is given in terms of a constrained matrix Sylvester equation and is numerically tractable. An explicit state-space parametrization of all solutions is also derived. This parameterization effectively includes two parameters: one is a "standard" $\mathcal{RH}_\infty$ parameter and another one arises if the Sylvester equation is non-uniquely solvable. A condition, in terms of zeros of rational matrices involved in the BDE, is found under which the Sylvester equation has a unique solution and, hence, the parametrization is affine in a single $\mathcal{RH}_\infty$ parameter.**

I. Introduction

Linear (polynomial or real-rational) equation in X and Y of the form $PX + YM = T$, for given P, M and T, is referred to as the *bilateral Diophantine equation* (BDE). This equation is closely related to the notion of skew-primeness, introduced in [19], and plays an important role in various control and estimation problems. For example, in [1], [4], [20], [11] different versions of the regulation problem with internal stability are reduced to a single polynomial BDE. Coupled polynomial BDEs are involved in the solution of deterministic least square and stochastic minimal variance control problems in [13]. Similar equations also appear in the solution of the dead-beat control problem [14], the standard $\mathcal{H}_2$ control [3], and the MMSE state estimation [5] problems. In the rational setting, BDE is involved in various tracking and regulation problems in [17], [6], [21] and plays a key role in input / output stabilization of two-side model matching problem [12].

Generally speaking, BDE, in either polynomial or rational setting, can be associated with problems, where some exogenous non-decaying signals have to be asymptotically tracked or rejected. It is then not a surprise that this equation arises in various approaches. For example, in the context of the geometric approach, the BDE is related to the existence of complementary invariant subspaces, [10], [9], while in the behavioral setting it can be associated with the direct sum behavior decomposition [2].

There are several solutions of BDE available in the literature. The existing solutions, however, appear to be either numerically involved or difficult for the analysis. In [13] a solvability condition for a polynomial BDE is derived in terms of the equivalence of two composite matrices[1] and a procedure for constructing a particular solution is provided. The latter relies on the McMillan form and, as a result, might be difficult for numerical implementation. An alternative approach proposed in [19] is based on nontrivial operations, like finding common divisors of polynomial matrices, which are also not quite reliable numerically. Moreover, neither of these approaches provides a parameterization of all solutions, which is required for example in [4] and [12]. In the rational setting, a complete solution of BDE can be inferred from [7]. This solution makes use of the Kronecker product and, consequently, blurs the problem structure and results in cumbersome expressions, which are difficult to analyze. In [22] and [9] the possibility of solving rational BDEs using state-space techniques was mentioned. Yet, to the best of our knowledge, there are no state-space solutions currently available in the literature.

In this paper we consider a special case of the rational BDE, in which M is square and invertible (see the notation paragraph below for the definition of invertibility). This version of BDE commonly appears in applications, such as regulation problem with internal stability [4], [20] and two-side model matching stabilization [12]. In the past, favorable properties of BDEs with invertible M were perceived, for example, in [19] and [8]. Yet, to the best of our knowledge, even in this special case no explicit and numerically tractable solutions are available in the literature. In this work we demonstrate that once M is invertible, the state-space technique provides an efficient tool for finding a transparent and numerically feasible solution. In particular, necessary and sufficient solvability conditions are derived in terms of a constrained matrix Sylvester equation and an explicit state-space parameterization of all existing solutions is provided. In addition, we find an intuitively clear sufficient condition, similar to that discussed in [19], for the existence of an affine parameterization of all solutions in terms of a single stable, but otherwise arbitrary, parameter. We believe that the solution derived in this work can be useful in the context of various optimization problems in the model matching setup and relevant control / estimation applications. In particular, it can be exploited in extending the results of [16] to the general four-block setting, see [12] for more details.

It should be mentioned that the current work is inspired by [15], where a state-space solution of the comprehensive stabilization problem, intimately related to BDE, was derived.

The paper is organized as follows. In Section II the problem

This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 1238/08).

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[1]This is an extension of solvability condition for algebraic Sylvester equation, see [18].

ISBN 978-963-311-370-7

is formulated and its solution is presented. The proof of the main result can be found in Section III. In Section IV we present two simple numerical examples, illustrating our results. Concluding remarks are given in Section V.

a) Notation: Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$ be a full column rank matrix , then $A^+ \in \mathbb{R}^{n\times m}$ and $A^\perp \in \mathbb{R}^{(m-n)\times m}$ denote a left inverse of A and its complement, i.e., any matrices such that $A^+A = I$, $A^\perp A = 0$, and $\left[\begin{smallmatrix} A^+ \\ A^\perp \end{smallmatrix}\right] \in \mathbb{R}^{m\times m}$ is nonsingular. By $\mathcal{RH}_\infty^{m\times n}$ we denote the set of $m \times n$ real-rational stable and proper transfer matrices. When dimensions are irrelevant or clear from the context, we omit the superscript and write $\mathcal{RH}_\infty$. A square proper transfer matrix $G(s)$ is said to be invertible if $G^{-1}(s)$ exists and is proper. In the rational case, the invertibility of $G(s)$ is equivalent to that of $G(\infty)$.

II. Problem Formulation and Solution

Consider transfer functions $P \in \mathcal{RH}_\infty^{q\times n}$, $M \in \mathcal{RH}_\infty^{m\times m}$, and $T \in \mathcal{RH}_\infty^{q\times m}$ given by their *minimal* state-space realizations

$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right], \tag{1a}$$

$$M(s) = \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & D_m \end{array}\right], \tag{1b}$$

$$T(s) = \left[\begin{array}{c|c} A_t & B_t \\ \hline C_t & D_t \end{array}\right]. \tag{1c}$$

The problem addressed in this paper is formulated as follows.

EQ: Given $\mathcal{RH}_\infty$ transfer matrices $P(s), M(s), T(s)$ in (1) with square and invertible $M(s)$, find whether the BDE

$$P(s)X(s) + Y(s)M(s) = T(s) \tag{2}$$

is solvable in $X(s) \in \mathcal{RH}_\infty^{n\times m}$ and $Y(s) \in \mathcal{RH}_\infty^{q\times m}$ and characterize all solutions if one exists.

Throughout, we make the following assumptions about the problem data:

$\boldsymbol{\mathcal{A}_1}$: $P(\infty) = D_p$ has full column rank,

$\boldsymbol{\mathcal{A}_2}$: $M(\infty) = D_m = I$,

$\boldsymbol{\mathcal{A}_3}$: $M(s)$ has no zeros in the open left-hand plane.

Assumption $\boldsymbol{\mathcal{A}_1}$ is rather technical. It facilitates the derivation of the Sylvester equation, involved in the main result of this paper, yet can be omitted by replacing this equation with an implicit one, see Remark 2 for details. Assumptions $\boldsymbol{\mathcal{A}_2}$ and $\boldsymbol{\mathcal{A}_3}$ impose no loss of generality, since any stable and stably invertible part of $M(s)$ can be absorbed into $Y(s)$ and the non singularity of D_m is guaranteed by invertibility of $M(s)$.

Remark 1: Since $P(s)$ and $M(s)$ can always be exchanged, by duality, the scope of the current work is effectively confined to equations in which either $M(s)$ or $P(s)$ are square and invertible. As explained in Remark 3, which follows Theorem 1 below, the scope can be further extended to the equations in which either $M(s)$ or $P(s)$ has a full rank at infinity.

To formulate the solution of **EQ**, define matrices

$$A_p^\times := A_p - B_pD_p^+C_p \quad \text{and} \quad A_m^\times := A_m - B_mC_m.$$

The minimality of the realization of $M(s)$ in (1) implies that the eigenvalues of $A_m^\times$ are the transmission zeros of $M(s)$. Therefore, according to $\boldsymbol{\mathcal{A}_3}$, the matrix $A_m^\times$ is anti-stable. The following theorem, the proof of which is given in Section III, constitutes the main result of this paper.

Theorem 1: Let $\boldsymbol{\mathcal{A}_{1\text{–}3}}$ hold. Then **EQ** is solvable iff there exist matrices Z_1 and Z_2 satisfying

$$\begin{bmatrix} A_t & 0 \\ -B_pD_p^+C_t & A_p^\times \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} - \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}A_m^\times = \begin{bmatrix} -B_t \\ B_pD_p^+D_t \end{bmatrix}C_m, \tag{3a}$$

$$D_p^\perp\left(D_tC_m + C_tZ_1 + C_pZ_2\right) = 0 \tag{3b}$$

and then all pairs solving (2) can be characterized as

$$X(s) = X_p(s) + Q(s)M(s), Y(s) = Y_p(s) - P(s)Q(s), \tag{4}$$

where $Q \in \mathcal{RH}_\infty$ but otherwise arbitrary and $X_p(s)$, $Y_p(s)$ are given by

$$X_p(s) = -\left[\begin{array}{c|c} A_m & B_m \\ \hline D_p^+(D_tC_m + C_tZ_1 + C_pZ_2) & 0 \end{array}\right]$$

$$Y_p(s) = \left[\begin{array}{cc|c} A_t & 0 & B_t + Z_1B_m \\ 0 & A_p & Z_2B_m \\ \hline C_t & C_p & D_t \end{array}\right],$$

where Z_1 and (not necessarily unique) Z_2 are solutions of (3).

Remark 2: It is apparent from the proof of Theorem 1 in Section III that the only use of $\boldsymbol{\mathcal{A}_1}$ is in deriving the constrained Sylvester equation (3) in Lemma 4. It is then readily verifiable that if $\boldsymbol{\mathcal{A}_1}$ does not hold, the result of Theorem 1 is still valid modulo the replacement of (3) with the implicit equation

$$\begin{bmatrix} A_t & 0 & B_t & 0 \\ 0 & A_p & 0 & B_p \\ C_t & C_p & D_t & D_p \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ C_m \\ Z_3 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ C_m \\ Z_3 \end{bmatrix}A_m^\times. \tag{5}$$

Once D_p has full column rank, this equation is equivalent to (3). To show this, premultiply the third row of (5) by $\left[\begin{smallmatrix} D_p^\perp \\ D_p^+ \end{smallmatrix}\right]$ to obtain:

$$D_p^\perp(C_tZ_1 + C_pZ_2 + D_tC_m) = 0, \tag{6}$$

$$Z_3 = -D_p^+(C_tZ_1 + C_pZ_2 + D_tC_m). \tag{7}$$

The substitution of (7) into the second row of (5) yields

$$(A_p - B_pD_p^+C_p)Z_2 - Z_2A_m^\times = B_pD_p^+(C_tZ_1 + D_tC_m). \tag{8}$$

It is clear now that the first row of (5) together with (8) are equivalent to (3a), while (6) coincides with (3b).

Remark 3: The result and the proof of Theorem 1 rely heavily upon the assumption that $M(s)$ is square and invertible. Nonetheless, the proposed approach can be extended to problems in which $M(s)$ is not necessarily square (although still has a full rank). Indeed, if $M(s)$ is "tall", we can always find a bi-stable transformation of $Y(s)$ that brings the equation to a form

$$P(s)X(s) + \begin{bmatrix} Y_1(s) & Y_2(s) \end{bmatrix} \begin{bmatrix} M_1(s) \\ 0 \end{bmatrix} = T(s),$$

for some $M_1(s)$ satisfying $\boldsymbol{\mathcal{A}_2}$. Obviously, $Y_2(s)$ is not constrained by the equation and the problem reduces to the BDE

$$P(s)X(s) + Y_1(s)M_1(s) = T(s),$$

satisfying $\boldsymbol{\mathcal{A}_2}$. If $M(s)$ is "fat", the BDE can always be postmultiplied by a bi-stable transfer matrix transforming it into the form

$$P(s)\begin{bmatrix} X_1(s) & X_2(s) \end{bmatrix} + Y(s)\begin{bmatrix} M_1(s) & 0 \end{bmatrix} = \begin{bmatrix} T_1(s) & T_2(s) \end{bmatrix},$$

where $M_1(s)$ again satisfies $\boldsymbol{\mathcal{A}_2}$. This equation can now be split into

$$P(s)X_1(s) + Y(s)M_1(s) = T_1(s),\ \ P(s)X_2(s) = T_2(s),$$

where the first equation is a BDE satisfying $\boldsymbol{\mathcal{A}_2}$ and the second one is a standard unilateral Diophantine equation independent on the first equation.

Theorem 1 reduces the solvability of the BDE (2) to that of the matrix Sylvester equation (3a) constrained by (3b) and provides a complete parameterization of all BDE solutions. It is worth emphasizing that Q is not the only free parameter in this parametrization. Another parameter is Z_2, which might not be uniquely defined by (3). Indeed, Z_1, satisfying the Sylvester equation $A_t Z_1 - Z_1 A_m^\times = -B_t C_m$, is unique (because A_t and $-A_m^\times$ are both Hurwitz). Then Z_2, satisfying $A_p^\times Z_2 - Z_2 A_m^\times = B_p D_p^+ (C_t Z_1 + D_t C_m)$, might not be unique as $A_p^\times$ and $-A_m^\times$ might have common eigenvalues. The freedom we have in the choice of Z_2, once it exists, is essential for the completeness of parameterizations (4), i.e., Z_2 can not be frozen or absorbed into $Q \in \mathcal{RH}_\infty$. In fact, it can be shown that in general there is no affine parameterization of all solutions of (2) given in terms of a single $\mathcal{RH}_\infty$ parameter, see the second example in Section IV for more details.

The presence of *two parameters*, having different kinds of constraints imposed on them, renders the usage of parametrizations (4) less convenient. It is therefore of interest to see, under what conditions (4) are "standard" affine parametrizations in a single $\mathcal{RH}_\infty$ parameter. To this end, examine equations (3), which play a key role in the solution of **EQ**. Our immediate goal is to rewrite (3a) in a form, which is more convenient for the analysis of (3). The idea is that any linear combination of (3b) can be added to the second line of (3a) without altering the equations. Formally, this is formulated as follows:

Lemma 1: Matrices Z_1 and Z_2 satisfying (3b) solve (3a) iff they satisfy

$$\begin{bmatrix} A_t & 0 \\ -B_L C_t & A_p^\times - L_p D_p^\perp C_p \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} - \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} A_m^\times = \begin{bmatrix} -B_t \\ (B_p D_p^+ + L_p D_p^\perp) D_t \end{bmatrix} C_m \quad (9)$$

for an arbitrary L_p, where $B_L := B_p D_p^+ + L_p D_p^\perp$.

According to the discussion preceding Lemma 9, Z_2 might be non-unique only if $A_p^\times - L_p D_p^\perp C_p$ and $A_m^\times$ have common eigenvalues. We may thus try to use the degree of freedom we have in (9), i.e., the matrix L_p, to shift all such modes. This, however, is impossible for the modes unobservable from $D_p^\perp C_p$. Taking into account the result of Lemma 5 in the Appendix, these are exactly transmission zeros of $P(s)$. This, together with the fact that the eigenvalues of $A_m^\times$ are the zeros of $M(s)$, suggests that a possible workaround is to assume that the transmission zeros of $P(s)$ and $M(s)$ do not coincide. If this assumption holds, we can always choose L_p for which the Sylvester equation (9) is guaranteed to have a unique solution (e.g., to stabilize all observable modes of $A_p^\times$). With such a choice, **EQ** reduces to solving (9), checking whether its solution satisfies (3b), and substituting the solution into (4). The following corollary can thus be formulated.

Corollary 1: If $P(s)$ and $M(s)$ have no common transmission zeros, L_p can be chosen such that $A_p^\times - L_p D_p^\perp C_p$ and $A_m^\times$ have no common eigenvalues. In this case, **EQ** is solvable iff the unique solution of (9) satisfies (3b) and the parameterization of all solutions of **EQ**, given by (4), is affine in its single parameter $Q(s)$.

It is worth mentioning that the existence of a simplified affine parameterization is of importance in numerous application, where the condition of Corollary 1 is generically satisfied, see [12] for the details.

Remark 4: Note that the condition in Corollary 1 is sufficient but not necessary for the existence of affine parameterization in terms of a single $\mathcal{RH}_\infty$ parameter. For example, consider the scalar BDE

$$\frac{s-1}{s+1} \cdot X(s) + Y(s) \cdot \left(\frac{s-1}{s+1}\right)^2 = \frac{s-1}{s+1}.$$

Clearly, this equation does not satisfy the condition of Corollary 1. Moreover, the result of Theorem 1 yields the following parameterization of all solutions

$$X(s) = \frac{2}{s+1} + q \cdot \frac{2(s-1)}{(s+1)^2} + Q(s) \cdot \left(\frac{s-1}{s+1}\right)^2,$$
$$Y(s) = 1 - \frac{2}{s+1} \cdot q - \frac{s-1}{s+1} \cdot Q(s),$$

which is given in terms of two parameters: $q \in \mathbb{R}$ and $Q \in \mathcal{RH}_\infty$. On the other hand, it can be shown that the set of all solutions of this equation can be also characterized by

$$X(s) = 1 + Q(s) \cdot \frac{s-1}{s+1} \quad \text{and} \quad Y(s) = -Q(s),$$

which is affine on a single $\mathcal{RH}_\infty$ parameter.

A. Solvability of BDE—interpretation

Having derived necessary and sufficient solvability conditions for **EQ**, in Theorem 1, we now explore some of its properties. In particular, we are interested in gaining an insight into underlying reasons that BDE might become unsolvable. Intuitively, the solvability of (2) is closely related to geometric properties of $P(s)$ at the points where $M(s)$ has transmission zeros. Indeed, if λ_i is a transmission zero of $M(s)$ with the corresponding input direction v_i, (2) imposes the following interpolation constraint on $X(s)$ at $s=\lambda_i$:

$$P(\lambda_i)X(\lambda_i)v_i = T(\lambda_i)v_i.$$

This immediately implies that BDE is solvable only if for any transmission zero λ_i of $M(s)$ and any of its input directions v_i

$$T(\lambda_i)v_i \in \operatorname{Im} P(\lambda_i). \tag{10}$$

It turns out that in some cases condition (10) is not only necessary, but also sufficient for solvability. In particular, this holds for the special case of the problem, in which all transmission zeros of $M(s)$ are simple, i.e., under the assumption that

$\boldsymbol{\mathcal{A}_4}$: geometric multiplicity of any transmission zero of $M(s)$ equals its algebraic multiplicity.

In this case, the realization of $M(s)$ can always be chosen so that

$$A_m^\times = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{bmatrix},$$

where $\lambda_i,\ i=1,\dots,m$, are the transmission zeros of $M(s)$. Consequently, equation (5), whose solvability is equivalent to that of **EQ**, can be rewritten as

$$\begin{bmatrix} A_t-\lambda_i I & 0 & B_t & 0 \\ 0 & A_p-\lambda_i I & 0 & B_p \\ C_t & C_p & D_t & D_p \end{bmatrix} \begin{bmatrix} Z_{1i} \\ Z_{2i} \\ C_{m,i} \\ Z_{3i} \end{bmatrix} = 0 \tag{11}$$

for $i=1,\dots,m$, where Z_{1i}, Z_{2i}, Z_{3i}, and $C_{m,i}$ are the ith columns of Z_1, Z_2, Z_3, and C_m, respectively. The left hand side of (11) involves the Rosenbrock system matrix of

$$\begin{bmatrix} T(s) & P(s) \end{bmatrix} = \left[\begin{array}{cc|cc} A_t & 0 & B_t & 0 \\ 0 & A_p & 0 & B_p \\ \hline C_t & C_p & D_t & D_p \end{array}\right]$$

and, since A_t and A_p are Hurwitz, the poles of this system do not coincide with λ_i, $i=1,\dots,m$. Therefore, according to Lemma 6 in the Appendix, the existence of Z_1, Z_2, and Z_3 satisfying (11) is equivalent to the existence of Z_3 satisfying

$$\begin{bmatrix} T(\lambda_i) & P(\lambda_i) \end{bmatrix} \begin{bmatrix} C_{m,i} \\ Z_{3i} \end{bmatrix} = 0$$

for every $i=1,\dots,m$. The latter can be rewritten as

$$T(\lambda_i)C_{m,i} = -P(\lambda_i)Z_{3i},$$

implying that if assumption $\boldsymbol{\mathcal{A}_4}$ holds, the problem is solvable iff

$$T(\lambda_i)C_{m,i} \in \operatorname{Im} P(\lambda_i), \quad i=1,\dots,m.$$

To interpret the last statement, the meaning of the vectors $C_{m,i}$ should be clarified. To this end, denote by e_i the standard basis in $\mathbb{R}^n$ and note that

$$(A_m - B_mC_m - \lambda_i I)e_i = 0 \quad \text{and} \quad C_m e_i = C_{m,i}.$$

These equations can be rewritten as

$$\begin{bmatrix} A_m-\lambda_i I & B_m \\ C_m & I \end{bmatrix} \begin{bmatrix} e_i \\ -C_{m,i} \end{bmatrix} = 0,$$

showing that $C_{m,i}$ may be interpreted as the input direction (see [23] for the definition) of the ith transmission zero of $M(s)$. Thus, we just proved the following result:

Proposition 1: Let $\boldsymbol{\mathcal{A}_4}$ hold. Then (2) is solvable iff condition (10) is satisfied for any transmission zero λ_i of $M(s)$ and any of its input directions v_i.

Remark 5: If $\boldsymbol{\mathcal{A}_4}$ does not hold, interpolation constraints on $X(s)$ should involve the zero structure of derivatives of $M(s)$. Detailed accounting of this situation goes beyond the scope of this paper. Below we just show that (10) might no longer be sufficient for the solvability of (2) in this case. To this end, consider the following BDE:

$$\begin{bmatrix} \left(\frac{s-1}{s+1}\right)^2 \\ 1 \end{bmatrix} X(s) + Y(s) \begin{bmatrix} \left(\frac{s-1}{s+1}\right)^2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s+1} & 1 \\ 1 & 0 \end{bmatrix},$$

which obviously does not satisfy $\boldsymbol{\mathcal{A}_4}$. In this example

$$m=1, \quad \lambda_1 = 1 \quad \text{and} \quad v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}'.$$

It can be seen that

$$T(\lambda_1)v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \operatorname{Im} P(\lambda_1) = \operatorname{Im} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

i.e, the condition of Proposition 1 does hold. At the same time, this equation is not solvable. To see this, it is sufficient to consider its $(1,1)$ sub-block,

$$\left(\frac{s-1}{s+1}\right)^2 X_{11}(s) + Y_{11}(s)\left(\frac{s-1}{s+1}\right)^2 = \frac{s-1}{s+1},$$

which effectively reads $X_{11}(s)+Y_{11}(s) = \frac{s+1}{s-1}$ and obviously has no $\mathcal{RH}_\infty$ solutions.

III. Proof of Theorem 1

The invertibility of $M(s)$ enables us to reformulate **EQ** as a stabilization problem. Namely, this is the problem of finding an $X \in \mathcal{RH}_\infty$ for which

$$Y(s) = (T(s) - P(s)X(s))M(s)^{-1} \tag{12}$$

belongs to $\mathcal{RH}_\infty$ as well. Such an $X(s)$ will hereafter be referred to as a stabilizing solution and the goal in **EQ** is to characterize all stabilizing solutions and all resulting stable $Y(s)$. The following technical result, which might appear an overparametrization, represents any stable $X(s)$ in a form convenient for the analysis of (12).

Lemma 2: Any $X \in \mathcal{RH}_\infty$ can be represented as

$$X(s) = X_0(s) + Q(s)M(s), \tag{13}$$

for some $Q \in \mathcal{RH}_\infty$ and $X_0 \in \mathcal{RH}_\infty$ given by the following state-space realization:

$$X_0(s) = \left[\begin{array}{c|c} A_m & B_m \\ \hline C_0 & 0 \end{array}\right], \tag{14}$$

for some C_0.

Proof: Let X be an arbitrary transfer matrix from $\mathcal{RH}_\infty$, given by its state-space realization

$$X(s) = \left[\begin{array}{c|c} A_x & B_x \\ \hline C_x & D_x \end{array}\right]. \tag{15}$$

If (13) holds, then $Q = (X - X_0)M^{-1}$, and its state-space realization is given by

$$Q(s) = \left[\begin{array}{cc|c} A_x & -B_x C_m & B_x \\ 0 & A_m^\times & B_m \\ \hline C_x & -C_0 - D_x C_m & D_x \end{array}\right]. \tag{16}$$

Because A_x is Hurwitz and $A_m^\times$ is anti-stable (according to $\mathcal{A}_3$), there exists a unique solution of the Sylvester equation

$$-A_x Z_x + Z_x A_m^\times = B_x C_m.$$

Applying the state transformation $\left[\begin{smallmatrix} I & Z_x \\ 0 & I \end{smallmatrix}\right]$ to (16), we get:

$$Q(s) = \left[\begin{array}{cc|c} A_x & 0 & B_x + Z_x B_m \\ 0 & A_m^\times & B_m \\ \hline C_x & -C_x Z_x - C_0 - D_x C_m & D_x \end{array}\right].$$

It is now readily seen that the choice

$$C_0 = -C_x Z_x - D_x C_m \tag{17}$$

yields that

$$Q(s) = \left[\begin{array}{c|c} A_x & B_x + Z_x B_m \\ \hline C_x & D_x \end{array}\right] \in \mathcal{RH}_\infty. \tag{18}$$

This completes the proof by showing that an arbitrary $X \in \mathcal{RH}_\infty$ admits (13) with C_0 and $Q \in \mathcal{RH}_\infty$ given by (17) and (18), respectively. ■

The result above provides a parameterization of all $X \in \mathcal{RH}_\infty$ in terms of a stable but otherwise arbitrary rational matrix $Q(s)$ and a matrix C_0. The following result will help to clarify the meaning of Lemma 2 in the context of the stabilization of (12):

Lemma 3: There is a stabilizing solution $X \in \mathcal{RH}_\infty$ of (12) iff X_0 given by (14) satisfies $(T - PX_0)M^{-1} \in \mathcal{RH}_\infty$ for some matrix C_0.

Proof: The "if" part is trivial. To show the "only if" part, assume that X is a stabilizing solution of (12). By Lemma 2 this implies that

$$Y = (T - PX_0)M^{-1} - PQ$$
$$\Updownarrow$$
$$(T - PX_0)M^{-1} = Y + PQ \in \mathcal{RH}_\infty$$

for some $Q \in \mathcal{RH}_\infty$ and C_0. This completes the proof. ■

Lemma 3 effectively implies that (12) is stabilizable iff there exists a stabilizing solution of the form (14). Moreover, all $X \in \mathcal{RH}_\infty$ stabilizing (12) can be characterized by (13), where $Q(s)$ is an arbitrary stable rational matrix and $X_0(s)$ is an arbitrary stabilizing solution, having the form (14). This reduces **EQ** to the problem of finding whether there exists C_0 for which X_0, given by (14), stabilizes (12), and characterizing all such matrices. The following result can be formulated.

Lemma 4: X_0 given by (14) is a stabilizing solution of (12) iff

$$C_0 = -D_p^+(D_t C_m + C_t Z_1 + C_p Z_2), \tag{19}$$

where Z_1 and Z_2 satisfy (3).

Proof: Substituting the state-space structure of X_0 into (12) yields that

$$Y = \left[\begin{array}{ccc|c} A_t & 0 & -B_t C_m & B_t \\ 0 & A_p & -B_p C_0 & 0 \\ 0 & 0 & A_m^\times & B_m \\ \hline C_t & C_p & -D_p C_0 - D_t C_m & D_t \end{array}\right]. \tag{20}$$

Since A_t and A_p are Hurwitz and $A_m^\times$ is anti-stable, the following Sylvester equation has a unique solution:

$$-\begin{bmatrix} A_t & 0 \\ 0 & A_p \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} A_m^\times = \begin{bmatrix} B_t C_m \\ B_p C_0 \end{bmatrix}. \tag{21}$$

The application of the state transformation

$$\begin{bmatrix} I & 0 & Z_1 \\ 0 & I & Z_2 \\ 0 & 0 & I \end{bmatrix}$$

to (20) yields

$$Y = \left[\begin{array}{ccc|c} A_t & 0 & 0 & B_t + Z_1 B_m \\ 0 & A_p & 0 & Z_2 B_m \\ 0 & 0 & A_m^\times & B_m \\ \hline C_t & C_p & \hat{C} & D_t \end{array}\right], \tag{22}$$

where

$$\hat{C} := -C_t Z_1 - C_p Z_2 - D_p C_0 - D_t C_m.$$

By the minimality of the realization of M in (1), the pair $(A_m^\times, B_m)$ is controllable. Thus, Y is stable iff all eigenvalues of $A_m^\times$ are unobservable in (22), namely, iff $\hat{C} = 0$. Premultiplying the latter equality by the nonsingular

$$\begin{bmatrix} D_p^+ \\ D_p^\perp \end{bmatrix}$$

yields that it is satisfied iff

$$D_p^+(-C_t Z_1 - C_p Z_2 - D_t C_m) - C_0 = 0, \tag{23}$$
$$D_p^\perp(-C_t Z_1 - C_p Z_2 - D_t C_m) = 0. \tag{24}$$

So far, we proved that X_0 is a stabilizing solution of (12) iff (21), (23) and (24) are satisfied simultaneously. Now, (19) follows from (23), and (3) follows from substituting (23) into (21) and gathering the resulting equation with (24). ■

The proof of Theorem 1 can be completed now by verifying that for the choice of C_0, specified in Lemma 4, X_0 and the resulting stabilized Y equal X_p and Y_p respectively. The parameterizations in (4) follow then from (13) and from substituting (13) into (12).

IV. ILLUSTRATIVE EXAMPLES

Example 1

Consider the following BDE:

$$\begin{bmatrix} 1 & \frac{s+4}{s+1} \\ 0 & \frac{s+4}{s+1} \\ 0 & 0 \end{bmatrix} X(s) + Y(s) \begin{bmatrix} \frac{s-5}{s+2} & \frac{1}{s+2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{s+6}{s+3} & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \quad (25)$$

In this case, the state-space realizations are:

$$P(s) = \left[\begin{array}{c|cc} -1 & 0 & 3 \\ \hline 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \quad M(s) = \left[\begin{array}{c|cc} -2 & -7 & 1 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

$$T(s) = \left[\begin{array}{c|cc} -3 & 3 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right].$$

Because the assumption of Corollary 1 is satisfied in this example, we can always find L_p such that (9) is guaranteed to have a unique solution, and it is convenient to check the solvability of (25) using the result of Corollary 1. As $A_p^\times = -4$ is Hurwitz, we may take $L_p = 0$. Then, substituting the problem data into (9) yields

$$\begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} - \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

It is readily seen that the unique solutions of this equation are $Z_1 = 3/8$ are $Z_2 = 0$ and they do satisfy (3b). Therefore, BDE (25) is solvable and all its solutions can be characterized by substituting these Z_1 and Z_2 into (4). This results in the following parameterizations:

$$X(s) = \begin{bmatrix} \frac{77}{8(s+2)} & -\frac{11}{8(s+2)} \\ 0 & 0 \end{bmatrix} + Q(s) \begin{bmatrix} \frac{s-5}{s+2} & \frac{1}{s+2} \\ 0 & 1 \end{bmatrix},$$

$$Y(s) = \begin{bmatrix} \frac{8s+27}{8(s+3)} & \frac{3}{8(s+3)} \\ 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \frac{s+4}{s+1} \\ 1 & \frac{s+4}{s+1} \\ 0 & 0 \end{bmatrix} Q(s),$$

which are given in terms of a single stable but otherwise arbitrary parameter Q.

Example 2

Consider now an example, for which the assumption of Corollary 1 is not satisfied:

$$\begin{bmatrix} 1 & \frac{s-5}{s+1} \\ 0 & \frac{s-5}{s+1} \\ 0 & 0 \end{bmatrix} X(s) + Y(s) \begin{bmatrix} \frac{s-5}{s+2} & \frac{1}{s+2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{s+6}{s+3} & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \quad (26)$$

Here $M(s)$ and $T(s)$ are the same as in (25) and $P(s)$ has the following state-space realization:

$$P(s) = \left[\begin{array}{c|cc} -1 & 0 & -6 \\ \hline 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

It is readily seen that the zero of $P(s)$ at $s = 5$ coincides with a zero of $M(s)$.

To check the solvability of (26), substitute the problem data into (3). We obtain the following equations for Z_1 and Z_2:

$$\begin{bmatrix} -3 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} - \begin{bmatrix} Z_1 \\ Z_2 \\ 0 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}. \quad (27)$$

It is readily verified that these equations are solvable, implying that so is the considered BDE. All solutions of (27) are given by $Z_1 = 3/8$ and $Z_2 = q$, where $q \in \mathbb{R}$ is arbitrary. The substitution of these solutions into the formulae of Theorem 1 yields that all solutions of (26) can be characterized by

$$X(s) = \begin{bmatrix} \frac{77}{8(s+2)} & -\frac{11}{8(s+2)} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{7}{s+2} & -\frac{1}{s+2} \end{bmatrix} q + Q(s) \begin{bmatrix} \frac{s-5}{s+2} & \frac{1}{s+2} \\ 0 & 1 \end{bmatrix}, \quad (28a)$$

$$Y(s) = \begin{bmatrix} \frac{8s+27}{8(s+3)} & \frac{3}{8(s+3)} \\ 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{-7}{s+1} & \frac{1}{s+1} \\ \frac{-7}{s+1} & \frac{1}{s+1} \\ 0 & 0 \end{bmatrix} q - \begin{bmatrix} 1 & \frac{s-5}{s+1} \\ 0 & \frac{s-5}{s+1} \\ 0 & 0 \end{bmatrix} Q(s), \quad (28b)$$

where $Q \in \mathcal{RH}_\infty$ but otherwise arbitrary.

Thus, we end up with the parameterization given in terms of two independent parameters: $Q \in \mathcal{RH}_\infty$ and $q \in \mathbb{R}$. The latter is present due to the freedom in choice of the solution of (3). This agrees with the discussion in Section II, which shows that this kind of static parameters may appear only if the assumption of Corollary 1 is not satisfied.

An interesting observation is that q in (28) can be allowed to be an $\mathcal{RH}_\infty$ transfer function rather then a real number. Indeed, straightforward substitutions yield that X and Y defined by (28) solve BDE (26) for any $q \in \mathcal{RH}_\infty$. The completeness of parametrization (28) implies, however, that introducing dynamics into q does not allow any additional degree of freedom. This can be seen via presenting $q \in \mathcal{RH}_\infty$ as

$$q(s) = q(5) + \frac{s-5}{s+5} \cdot \tilde{q}(s), \quad \tilde{q} \in \mathcal{RH}_\infty$$

(this is always possible). The substitution of this expression into (28) yields that $\tilde{q}(s)$ can be absorbed into $Q(s)$ via shifting Q as

$$Q(s) \mapsto Q(s) - \tilde{q}(s) \begin{bmatrix} 0 & 0 \\ \frac{7}{s+5} & -\frac{1}{s+5} \end{bmatrix}.$$

This implies that the only effective degree of freedom in $q(s)$ is its value at $s = 5$.

It should be emphasized that the real parameter q in (28) is not redundant. Namely, q can not be fully absorbed into $Q(s)$ without altering constraints on the latter. To show this, define

$$\tilde{Q}(s) := Q(s) + \begin{bmatrix} 0 & 0 \\ \frac{7}{s-5} & -\frac{1}{s-5} \end{bmatrix} q.$$

Then, (28) can be rewritten as

$$X(s) = \begin{bmatrix} \frac{77}{8(s+2)} & -\frac{11}{8(s+2)} \\ 0 & 0 \end{bmatrix} + \tilde{Q}(s) \begin{bmatrix} \frac{s-5}{s+2} & \frac{1}{s+2} \\ 0 & 1 \end{bmatrix},$$
$$Y(s) = \begin{bmatrix} \frac{8s+27}{8(s+3)} & \frac{3}{8(s+3)} \\ 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \frac{s-5}{s+1} \\ 0 & \frac{s-5}{s+1} \\ 0 & 0 \end{bmatrix} \tilde{Q}(s).$$

Although these expressions can also be thought of as a parameterization of all admissible solutions of (26) in terms of a single parameter $\tilde{Q}(s)$, the latter belongs to the space $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \frac{s+5}{s-5} \end{smallmatrix}\right] \mathcal{RH}_\infty$, satisfying

$$\mathcal{RH}_\infty \subset \left[\begin{smallmatrix} 1 & 0 \\ 0 & \frac{s+5}{s-5} \end{smallmatrix}\right] \mathcal{RH}_\infty \subset \mathcal{RL}_\infty.$$

In fact, it can be shown that in this example the set of all solutions cannot be characterized by a general affine parameterization of the form[2]

$$X(s) = \bar{X}(s) + Q(s)U(s), \; Y(s) = \bar{Y}(s) + V(s)Q(s), \quad (29)$$

for some given $\bar{X}, \bar{Y}, U, V \in \mathcal{RH}_\infty$ and a single $\mathcal{RH}_\infty$ parameter Q. To show this, note that the complete parameterization of solutions of (26), given in (28), implies that the value of $X_{21}(5)$ is not constrained by the equation, while the value of $X_{11}(5)$ is forced to be equal to $\frac{11}{8}$. This situation can never be captured by the parameterization in (29), where an interpolation constraint on $X_{11}(s)$ at any point s, say $s = 5$, is always accompanied by a constraint on $X_{21}(s)$ at the same point. Indeed, the $(1,1)$ and $(2,1)$ sub-blocks of X, given by (29), can be rewritten as

$$X_{11}(s) = \bar{X}_{11}(s) + Q_{11}(s)U_{11}(s) + Q_{12}(s)U_{21}(s),$$
$$X_{21}(s) = \bar{X}_{21}(s) + Q_{21}(s)U_{11}(s) + Q_{22}(s)U_{21}(s).$$

Assuming $X_{11}(5) = \frac{11}{8}$, the fact that $Q_{11}(5)$ and $Q_{12}(5)$ may be arbitrary necessarily implies that $\bar{X}_{11}(5) = \frac{11}{8}$ and $U_{11}(5) = U_{21}(5) = 0$. This, in turn, means $\bar{X}_{21}(5) = X_{21}(5)$ and therefore is constrained.

V. Summary

In this paper the rational BDE (2) with square and invertible $M(s)$ has been considered. Using state-space techniques, we have shown that such BDE can be reduced to the constrained matrix Sylvester equation (3). As a result, convenient and numerically feasible solvability conditions of (2) have been derived and a complete state-space parameterization of all solutions for this equation has been provided in terms of two parameters $Q \in \mathcal{RH}_\infty$ and Z_2, which is a set of all solutions to (3). In addition, we have shown that if zeros of $P(s)$ and $M(s)$ do not coincide, the solution of the constrained Sylvester equation is unique. In this case, a simplified parameterization of all BDE solutions was established in terms of a single $\mathcal{RH}_\infty$ parameter.

[2] The difference between this parameterization and (4) is that in (29) the transfer matrices $U(s)$ and $V(s)$ are not constrained to be $M(s)$ and $P(s)$ respectively.

Acknowledgment

The first author is thankful to Prof. Paul Fuhrmann from Ben-Gurion University of the Negev for helpful discussions during the work on this paper.

Appendix

Consider a (minimal) state-space realization

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad (30)$$

with left invertible D. The following result can be formulated:

Lemma 5: $z \in \mathbb{C}$ is a transmission zero of $G(s)$ iff it is an unobservable mode of $(D^\perp C, A^\times)$, where $A^\times := A - BD^+C$.

Proof: As the realization of G is minimal, it is sufficient to show that all unobservable modes of $(D^\perp C, A^\times)$ are *invariant zeros* of realization (30). It is known [23] that z is an invariant zero of this realization iff the Rosenbrock matrix

$$R_G(\lambda) = \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

looses its normal rank at $\lambda = z$. Pre-multiply $R_G(\lambda)$ by the nonsingular matrix

$$M_a := \left[\begin{array}{c:cc} I & -B & 0 \\ \hdashline 0 & 0 & I \\ 0 & I & 0 \end{array}\right] \left[\begin{array}{c:c} I & 0 \\ \hdashline 0 & D^+ \\ 0 & D^\perp \end{array}\right],$$

where the dashed lines refer to the partitioning of R_G. This yields

$$M_a R_G(\lambda) = \begin{bmatrix} A^\times - \lambda I & 0 \\ D^\perp C & 0 \\ D^+ C & I \end{bmatrix}.$$

Since $R_G(\lambda)$ and $M_a R_G(\lambda)$ are of the same rank for every $\lambda \in \mathbb{C}$, $R_G(\lambda)$ looses its normal rank iff so does

$$\begin{bmatrix} A^\times - \lambda I \\ D^\perp C \end{bmatrix},$$

namely, at the eigenvalues of $A^\times$ that are unobservable from $D^\perp C$. ■

Lemma 6: For any λ which is not a pole of $G(s)$, $G(\lambda)Y = 0$ iff there exists X such that

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0. \quad (31)$$

Proof: It is a trivial modification of the proof of [23, Lemma 3.3]. ■

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