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Lur'e Equations and Singular Optimal Control

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***Abstract*— In this work we give an overview about Lur'e matrix equations, linear-quadratic infinite time horizon optimal control problems and their connections to the eigenstructure of certain even matrix pencils. We characterize the set of solutions in terms of deflating subspaces of even matrix pencils. In particular, it is shown that these special solutions can be constructed deflating subspaces of even matrix pencils.**

I. Introduction

Consider the linear-quadratic optimal control problem

Minimize

$$\mathcal{J}(u,x_0)=\frac{1}{2}\int_0^\infty \begin{bmatrix}x(t)\\u(t)\end{bmatrix}^*\begin{bmatrix}Q & C\\C^* & R\end{bmatrix}\begin{bmatrix}x(t)\\u(t)\end{bmatrix}dt \quad (1a)$$

subject to $u\in L_2(\mathbb{R}^+,\mathbb{C}^m)$ and

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t),\\x(0)&=x_0,\\\lim_{t\to\infty}x(t)&=0,\end{aligned} \quad (1b)$$

where $A\in\mathbb{C}^{n,n}$, $B,C\in\mathbb{C}^{n,m}$, and Hermitian matrices $Q\in\mathbb{C}^{n,n}$, $R\in\mathbb{C}^{m,m}$. *Feasibility* of this minimization problem means that for all $x_0\in\mathbb{C}^n$, there exists some $u\in L_2(\mathbb{R}^+,\mathbb{C}^m)$ with $\mathcal{J}(u,x_0)<\infty$ and, additionally, the quantity

$$\mathcal{J}_{opt}(x_0)=\inf\{\mathcal{J}(u,x_0)\,:\,u\in L_2(\mathbb{R}^+,\mathbb{C}^m)\} \quad (2)$$

fulfills $\mathcal{J}_{opt}(x_0)>\infty$. In frequency domain, this condition is equivalent to the positive semi-definiteness of the so-called *spectral density function*, that is

$$\Phi(i\omega)=\begin{bmatrix}(i\omega I-A)^{-1}B\\I_m\end{bmatrix}^*\begin{bmatrix}Q & C\\C^* & R\end{bmatrix}\begin{bmatrix}(i\omega I-A)^{-1}B\\I_m\end{bmatrix}\geq 0 \quad (3)$$

for all $\omega\in\mathbb{R}$. Under certain assumptions on the structure of the uncontrollable modes of (1b), it is shown in [6] that the positivity of the spectral density function is equivalent to the solvability of the linear matrix inequality

$$\begin{bmatrix}A^*X+XA+Q & XB+C\\B^*X+C^* & R\end{bmatrix}\geq 0, \quad (4)$$

for some Hermitian matrix $X\in\mathbb{C}^{n,n}$ (for Hermitian matrices M,N, $M\geq N$ means that $M-N$ is positive definite). Of particular interest is the so-called *maximal solution* X_+ which solves (4) and, additionally, $X_+\geq X$ for all solutions X of (4). It is further known [6] that the maximal solution minimizes the rank of the matrix on the left hand side of (4). This type is called *Lur'e equations* and can equivalently be written as a system

$$\begin{aligned}A^*X+XA+Q&=K^*K,\\XB+C&=K^*L,\\R&=L^*L,\end{aligned} \quad (5a)$$

that has to be solved for the triple

$$(X,K,L)\in\mathbb{C}^{n,n}\times\mathbb{C}^{p,n}\times\mathbb{C}^{p,m} \quad (5b)$$

with Hermitian X and $p\in\mathbb{N}$ as small as possible. In case of solvability, it is shown in [6] that $p\leq m$. Equations of type (5) were first introduced by A.I. Lur'e [10] in 1951 (see [4] for an historical overview) and play a fundamental role in systems theory, e.g. since also properties like dissipativity of linear systems can be characterized via their solvability [1], [2], [3], [17]. This type of equations moreover appears in balancing-related model reduction [5], [8], [11], [12], [14]. In the case where R is invertible, it can be readily verfied that $p=m$ and the matrices K and L can be eliminated by obtaining the algebraic Riccati equation

$$A^*X+XA-(XB+C)R^{-1}(XB+C)^*+Q=0. \quad (6)$$

It is well-known [9], [18] that solvability criteria and the construction of solutions can be constructed via consideration of the eigenstructure of the Hamiltonian matrix

$$\mathcal{A}_H=\begin{bmatrix}A-BR^{-1}C & -BR^{-1}B^*\\C^*R^{-1}C-Q & -(A-BR^{-1}C)^*\end{bmatrix}. \quad (7)$$

In this work, we place particular emphasis on the case where R is not invertible. In this case, the optimal control problem (1) is called *singular*. The Hamiltonian matrix (7) clearly cannot be built. Instead we consider the *even matrix pencil*

$$s\mathcal{E}-\mathcal{A}=\begin{bmatrix}0 & -sI+A & B\\sI+A^* & Q & C\\B^* & C^* & R\end{bmatrix}. \quad (8)$$

The aim of this work is to collect equivalences for the solvability of the Lur'e equations, the eigenstructure of the pencil (8) and conntions to the spectral density function as well as the feasibility of the optimal control problem (1).

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II. Deflating Subspaces and Even Matrix Pencils

In this part we introduce the basic facts about matrix pencils and their normal forms.

Theorem 1: Let $sE - A$ be a matrix pencil with $E, A \in \mathbb{C}^{m,n}$. Then $sE - A$ is called *regular* if $m = n$ and $\det(\lambda E - A)$ does not vanish identically. If $sE - A$ is not regular, then it is said to be *singular*. A pencil $sE - A$ is called *even* if $E = -E^*$ and $A = A^*$.

Now we consider the concept of deflating subspaces of matrix pencils. That is, roughly speaking, a generalization of the concept of invariant subspaces for matrices.

Theorem 2: A subspace $\mathcal{V} \subset \mathbb{C}^N$ is called *(right) deflating subspace* for the pencil $sE - A$ with $E, A \in \mathbb{C}^{M,N}$ if for a matrix $V \in \mathbb{C}^{N,k}$ with full column rank and $\operatorname{im} V = \mathcal{V}$, there exists an $l \leq k$ and matrices $W \in \mathbb{C}^{M,l}$, $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{l,k}$ with

$$(sE - A)V = W(s\widetilde{E} - \widetilde{A}). \tag{9}$$

Deflating subspaces can be well characterized by means of the *Kronecker canonical form*. In the following we introduce two kinds of Kronecker forms. The first one is the classical form that applies to general matrix pencils. After that, we consider a special type [16] for even matrix pencils. For these forms, we require the following special matrices $J_k, M_k, N_k \in \mathbb{R}^{k,k}$, $K_k, L_k \in \mathbb{R}^{k-1,k}$ with

$$J_k = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}, \qquad K_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix},$$

$$L_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix},$$

$$M_k = \begin{bmatrix} & & 1 & 0 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{bmatrix}, \quad N_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Type	Size	$\mathcal{C}_j(s)$	Parameters
W1	$k_j \times k_j$	$(s-\lambda)I_{k_j} - N_{k_j}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}$
W2	$k_j \times k_j$	$sN_{k_j} - I_{k_j}$	$k_j \in \mathbb{N}$
W3	$(k_j - 1) \times k_j$	$sK_{k_j} - L_{k_j}$	$k_j \in \mathbb{N}$
W4	$k_j \times (k_j - 1)$	$sK_{k_j}^T - L_{k_j}^T$	$k_j \in \mathbb{N}$

TABLE I

Block types in Kronecker canonical form

Theorem 1 (Kronecker Form (KCF), [7]): For a matrix pencil $sE - A$ with $E, A \in \mathbb{C}^{n,m}$, there exist matrices $U_l \in \mathrm{Gl}_n(\mathbb{C})$, $U_r \in \mathrm{Gl}_m(\mathbb{C})$, such that

$$U_l(sE - A)U_r = \operatorname{diag}(\mathcal{C}_1(s), \ldots, \mathcal{C}_k(s)), \tag{10}$$

where each of the pencils $\mathcal{C}_j(s)$ is of one of the types presented in Table I.

The numbers λ appearing in the blocks of type W1 are called the *(generalized) eigenvalues* of $sE - A$. Blocks of type W2 are said to be corresponding to *infinite eigenvalues*.

Since each block of type W3 (W4) leads to an additional column (resp. row) rank deficiency of 1, the regularity of a pencil is equivalent the absence of blocks of type W3 and W4 in its KCF.

In the following, we review a special modification of the KCF from [15] for even matrix pencils, the *even Kronecker canonical form (EKCF)*. This form is achieved by a congruence transform of $sE - A$ and therefore preserves the evenness.

Theorem 2 (Even Kronecker Form (EKCF), [15]): For an even matrix pencil $sE - A$ with $E, A \in \mathbb{C}^{n,n}$, there exists a matrix $U \in \mathrm{Gl}_n(\mathbb{C})$ such that

$$U^*(sE - A)U = \operatorname{diag}(\mathcal{D}_1(s), \ldots, \mathcal{D}_k(s)), \tag{11}$$

where each of the pencils $\mathcal{D}_j(s)$ is of one of the types presented in Table II. The numbers ε_j in the blocks of type E2 and E3 are called the *block signatures*.

The appearance of block of type E1 shows that generalized eigenvalues $\lambda \notin i\mathbb{R}$ occur in pairs $(\lambda, -\overline{\lambda})$. The blocks of type E2 and E3 respectively correspond to the purely imaginary and infinite eigenvalues. The additional sign parameter is contained which is basically due to the fact that for a fixed $\lambda \in i\mathbb{R}$ the congruence transformation with U preserves the inertia of the Hermitian matrix $\lambda E - A$. Blocks of type E4 consist of a combination of blocks that are equivalent to those of type W3 and W4.

III. Solvability of Lur'e Equations

In this section we collect criteria for the solvability of Lur'e equations. Those are given in terms of the fesibiliy of the optimal control problem (1), the solvability of the LMI (4), the positive definiteness of the spectral density function (3) and the eigenstructure of the even matrix pencil (8). The following result requires the pair (A, B) to be *stabilizable* (resp. *controllable*). That is, for all $s \in \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ ($s \in \mathbb{C}$) holds $\operatorname{rank}[\, sI - A \,,\, B \,] = n$. In terms of dynamics, stabilizablitity is equivalent to the fact that for all $x_0 \in \mathbb{C}^n$ holds that there exists some $u \in L_2(\mathbb{R}^+, \mathbb{C}^m)$ such that (1b) is fulfilled.

Theorem 3: Let the Lur'e equations (5) with associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (8) and spectral density function Φ as in (3) be given. Assume that at least one of the claims

(i) the pair (A, B) is stabilizable and the pencil $s\mathcal{E} - \mathcal{A}$ as in (8) is regular;

(ii) the pair (A, B) is controllable

holds true. Then the following statements are equivalent:

1) For all $x_0 \in \mathbb{C}^n$, there exists some constant $c \in \mathbb{R}$ such that $\mathcal{J}(u, x_0) > c$ for all $u \in L_2(\mathbb{R}^+, \mathbb{C}^m)$.
2) For all $u \in L_2(\mathbb{R}^+, \mathbb{C}^m)$ holds $\mathcal{J}(u, 0) \geq 0$.
3) For all $\omega \in \mathbb{R}$ holds $\Phi(i\omega) \geq 0$.

Type	Size	$\mathcal{D}_j(s)$	Parameters
E1	$2k_j \times 2k_j$	$\begin{bmatrix} 0_{k_j,k_j} & (\lambda-s)I_{k_j}-N_{k_j} \\ (\overline{\lambda}+s)I_{k_j}-N_{k_j}^T & 0_{k_j,k_j} \end{bmatrix}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$
E2	$k_j \times k_j$	$\epsilon_j((-is-\mu)J_{k_j} + M_{k_j})$	$k_j \in \mathbb{N}, \mu \in \mathbb{R}$, $\epsilon_j \in \{-1,1\}$
E3	$k_j \times k_j$	$\epsilon_j(isM_{k_j} + J_{k_j})$	$k_j \in \mathbb{N}$, $\epsilon_j \in \{-1,1\}$
E4	$(2k_j-1)\times$ $(2k_j-1)$	$\begin{bmatrix} 0_{k_j-1,k_j-1} & -sK_{k_j} + L_{k_j} \\ sK_{k_j}^T + L_{k_j}^T & 0_{k_j,k_j} \end{bmatrix}$	$k_j \in \mathbb{N}$

TABLE II

BLOCK TYPES IN EVEN KRONECKER CANONICAL FORM

4) There exists a solution (X, K, L) of the Lur'e equations.
5) There exists some Hermitian $X \in \mathbb{C}^{n,n}$ that solves the LMI (4).
6) There exists some Hermitian $X_+ \in \mathbb{C}^{n,n}$ that solves the LMI (4) such that for all other Hermitian solutions $X \in \mathbb{C}^{n,n}$ of the LMI (4) holds $X_+ \geq X$.
7) For all $\omega \in \mathbb{R}$ such that $i\omega$ is not an eigenvalue of A holds $\Phi(i\omega) \geq 0$;
8) In the EKCF of $s\mathcal{E} - \mathcal{A}$, all blocks of type E2 have positive signature and even size, and all blocks of type E3 have negative sign and odd size.
9) In the EKCF of $s\mathcal{E} - \mathcal{A}$, all blocks of type E2 have even size, and all blocks of type E3 have negative sign and odd size.

In particular, solutions of the Lur'e equations fulfill $(X, K, L) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,p} \times \mathbb{C}^{m,p}$ with

$$p = \max_{\omega\in\mathbb{R}} \operatorname{rank} \Phi(i\omega) \tag{12}$$

Proof: The equivalences of the statements 3)-9) as well as $p = \max_{\omega\in\mathbb{R}} \operatorname{rank} \Phi(i\omega)$ is shown in [13]. The proof the outstanding equivalences, we show that 2)⇔3), 1)⇒2) and 4)⇒1).

The equivalence between 2) and 3) follows from the fact that, for $\widehat{u}$ being the Fourier transform of u, Parseval's theorem yields that

$$\mathcal{J}(u,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}^*(i\omega)\Phi(i\omega)\widehat{u}(i\omega)d\omega.$$

Now we show that 1) implies 2): Assume that 2) is not fulfilled, i.e., there exists some or all $\bar{u} \in L_2(\mathbb{R}^+, \mathbb{C}^m)$ with $\mathcal{J}(\bar{u}, 0) = c < 0$. Then, by definition of $\mathcal{J}(\cdot,\cdot)$, we have for all $n \in \mathbb{N}$ that

$$\mathcal{J}(n\bar{u}, 0) = n^2 c.$$

Therefore, the set $\{\mathcal{J}(u, 0) \ : \ u \in L_2(\mathbb{R}^+, \mathbb{C}^m)\}$ is not bounded from below. This is a contradiction to 1).

For the proof that 5) implies 2), we assume that X solves the LMI (4). Since (A, B) is stabilizable, there exists some $u \in L_2(\mathbb{R}^+, \mathbb{C}^m)$ with $\mathcal{J}(u, x_0) < \infty$. Then we obtain

$$\begin{aligned} x_0^* X x_0 &= -\int_0^\infty \frac{d}{dt} x^*(t)Xx(t)dt \\ &= -\int_0^\infty \dot{x}^*(t)Xx(t) + x^*(t)X\dot{x}(t)dt \\ &= -\int_0^\infty x^*(t)A^*Xx(t) + u^*(t)B^*Xx(t) \\ &\quad + x^*(t)XAx(t) + x^*(t)XBu(t)dt \\ &= \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} -A^*X - XA & -B \\ -B^* & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \\ &\leq \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & C \\ C^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt = \mathcal{J}(u, x_0). \end{aligned}$$

This shows that 1) holds true. ■

The maximal solution X_+ as described in 6) can be shown to minimize the rank of the LMI (4), i.e., it is part of a solution $(X, K, L) \in \mathbb{C}^{n,n} \times \mathbb{C}^{p,n} \times \mathbb{C}^{p,m}$ of the Lur'e equations (5). Note that, if the stabilizability assumption in Theorem 3 is replaced by *anti-stabilizability* (that is, stabilizability of $(-A, B)$), then there exists a minimal solution of the LMI (4). This is closely related to linear-quadratic optimal control on the negative time horizon.

IV. CONSTRUCTION OF SOLUTIONS VIA DEFLATING SUBSPACES

In this part, we relate the solutions of the Lur'e equations (5) to certain deflating subspaces of the even matrix pencil (8). By making use of the three equations in (5) for a solution $(X, K, L) \in \mathbb{C}^{n,n} \times \mathbb{C}^{p,n} \times \mathbb{C}^{p,m}$, we obtain the deflating subspace relation

$$\begin{aligned} &\begin{bmatrix} 0 & -sI + A & B \\ sI + A^* & Q & C \\ B^* & C^* & R \end{bmatrix} \begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \\ -X & K^* \\ 0 & L^* \end{bmatrix} \begin{bmatrix} -sI + A & B \\ K & L \end{bmatrix}. \end{aligned} \tag{13}$$

This means that any solution of the Lur'e equations (5) correspond to a certain deflating subspace of the associated even matrix pencil (8). The following result characterizes the deflating subspcaes of associated even matrix pencil that correspond to solutions of the Lur'e equations. $s\mathcal{E}-\mathcal{A}$.

Theorem 4 ([13]): Let the assumptions of Theorem 3 be fulfilled and, moreover, let the Lur'e equations (5) be solvable and p be defined as in (12). Then there exist $V_\mu, V_x \in \mathbb{C}^{n,n+m}$, $V_u \in \mathbb{C}^{m,n+m}$, $W_\mu, W_x \in \mathbb{C}^{n,n+p}$, $W_u \in \mathbb{C}^{m,n+p}$ and $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{n+p,n+m}$ such that

(i) For

$$V = \begin{bmatrix} V_\mu \\ V_x \\ V_u \end{bmatrix}, \quad W = \begin{bmatrix} W_\mu \\ W_x \\ W_u \end{bmatrix} \tag{14a}$$

holds

$$(s\mathcal{E}-\mathcal{A})V = W(s\widetilde{E}-\widetilde{A}); \tag{14b}$$

(ii) $\mathcal{V} = \operatorname{im}[\, V_\mu^T \,,\, V_x^T \,,\, V_u^T \,]^T$ is $\mathcal{E}$-neutral;

(iii) $\operatorname{rank} V_x = n$.

In particular, for some arbitrary right inverse V_x^- of V_x holds that the matrix

$$X = V_\mu V_x^-. \tag{15}$$

is part of a solution of the Lur'e equations (5), i.e., there exist $K \in \mathbb{C}^{p,n}$, $L \in \mathbb{C}^{p,m}$ such that (5) holds true.

In the following, we relate the deflating subspaces with the properties as in Theorem 4 to the even Kronecker form as presented in Theorem 2.

Theorem 5 ([13]): Let the assumptions of Theorem 3 be fulfilled and let $s\mathcal{E}-\mathcal{A}$ as in (8) be given. For blocks $\mathcal{D}_j(s)$, $j = 1, \ldots, k$ as presented in Table II, let (11) be the EKCF of $s\mathcal{E}-\mathcal{A}$. Further assume that statement 9) of Theorem 3 holds true. Consider the partitioning $U = [\, U_1 \,, \ldots, U_k \,]$ according to the block structure of the EKCF. Then a matrix $V \in \mathbb{C}^{2n+m,n+m}$ has $\mathcal{E}$-neutral range and satisfies (14) for some $W \in \mathbb{C}^{2n+m,n+p}$, $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{n+p,n+m}$ and a full rank matrix $V_x \in \mathbb{C}^{n,n+m}$ if

$$V = \begin{bmatrix} V_1 & \ldots & V_k \end{bmatrix} \quad \text{for } V_j = U_j Z_j, \tag{16}$$

where

$$Z_j = \begin{cases} \text{either } [\, I_{k_j} \,,\, 0_{k_j} \,]^T \\ \quad \text{or } [\, 0_{k_j} \,,\, I_{k_j} \,]^T, & \text{if } \mathcal{D}_j \text{ is of type E1,} \\ [\, I_{k_j/2} \,,\, 0_{k_j/2} \,]^T, & \text{if } \mathcal{D}_j \text{ is of type E2,} \\ [\, I_{(k_j-1)/2} \,,\, 0_{(k_j+1)/2} \,]^T, & \text{if } \mathcal{D}_j \text{ is of type E3,} \\ [\, I_{k_j} \,,\, 0_{k_j+1} \,]^T, & \text{if } \mathcal{D}_j \text{ is of type E4.} \end{cases}$$

If, moreover, for each block of type E1, the matrix Z_j is chosen as $Z_j = [\, I_{k_j} \,,\, 0_{k_j} \,]^T$, then X constructed by (15) is the maximal solution of the Lur'e equations.

V. Conclusion

In this work we have studied Lur'e matrix equations and their connction to the singular linear-quadratic optimal control problem. Under the assumption of either controllability or regularity of the associated even matrix pencil together with stabilizability, equivalent criteria for the solvability of Lur'e equations are given in terms of the feasibility of an optimal control problem, the solvability of a linear matrix equation, the positive semi-definiteness of the spectral density function and the eigenstructure of a certain associated even matrix pencil. This associated even matrix pencil was utilized to describe the solution set. It is shown that solutions of Lur'e equations correspond to certain deflating subspaces of the associated even matrix pencil. These particular deflating subspaces were further characterized in terms of the even Kronecker form. It is moreover shown that there exists a solution which is maximal in terms of definiteness. The corresponding deflating subspace was particularly analyzed.

References

[1] B.D.O. Anderson. Algebraic description of bounded real matrices. *Electronics Letters*, 2(12):464–465, 1966.

[2] B.D.O. Anderson and R.W. Newcomb. Impedance synthesis via state-space techniques. *Proc. IEE*, 115(7):928–936, 1968.

[3] B.D.O. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis*. Prentice Hall, Englewood Cliffs, NJ, 1973.

[4] N.E. Barabanov. Kalman-Yakubovich lemma in general finite-dimensional case. *Int. J. Robust Nonlinear Control*, 17:369–386, 2007.

[5] X. Chen and J.T. Wen. Positive realness preserving model reduction with $\mathcal{H}_\infty$ norm error bounds. *Systems Control Lett.*, 54(4):361–374, 2005.

[6] D.J. Clements, B.D.O. Anderson, A.J. Laub, and L.B. Matson. Spectral factorization with imaginary-axis zeros. *Linear Algebra Appl.*, 250:225–252, 1997.

[7] F.R. Gantmacher. *Theory of Matrices*. Chelsea Publishing Company, New York, 1959.

[8] S. Gugercin and A.C. Antoulas. A survey of model reduction by balanced truncation and some new results. *Internat. J. Control*, 77(8):748–766, 2004.

[9] P. Lancaster and L. Rodman. *Algebraic Riccati equations*. Clarendon Press, Oxford, 1995.

[10] A.I. Lur'e. *Certain Nonlinear Problems in the Theory of Automatic Control*. Gostekhizat, Moscow, Leningrad, 1951. Translated into English, H.M. Stationery, 1957.

[11] P.C. Opdenacker and E.A. Jonckheere. A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds. *IEEE Trans. Circuits Syst. I Regul. Pap.*, 35(2):184–189, 1988.

[12] J.R. Phillips, L. Daniel, and L.M. Silveira. Guaranteed passive balancing transformations for model order reduction. *IEEE Trans. Computer-Aided Design Integr. Circuits Syst.*, 22(8):1027–1041, 2003.

[13] T. Reis. Lur'e equations and even matrix pencils. Technical Report 09-672, DFG Research Center MATHEON, Berlin, 2009. submitted for publication.

[14] T. Reis and T. Stykel. Passive and bounded real balancing for model reduction of descriptor systems. *Internat. J. Control*, 83(1):74–88, 2010.

[15] R.C. Thompson. The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil. *Linear Algebra Appl.*, 14:135–177, 1976.

[16] R.C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.

[17] J.C. Willems. Dissipative dynamical systems. II: Linear systems with quadratic supply rates. *Arch. Ration. Mech. Anal.*, 45:352–393, 1972.

[18] K. Zhou, J. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Princeton, 1996.