Analytic properties of matrix Riccati equation solutions

Ruth Curtain and Leiba Rodman

***Abstract*— For matrix Riccati equations of platoon-type systems and of systems arising from PDEs, assuming the coefficients are analytic functions in a suitable domain, the analyticity of the stabilizing solution is proved under various hypotheses. In addition, general results on the analytic behavior of stabilizing solutions are developed.**

I. NONSTANDARD RICCATI EQUATIONS

In [10], [7] and elsewhere continuity and real analytic properties of solutions of standard Riccati equations depending on a parameter z have been considered:

$$A(z)^*P(z) + P(z)A(z) + Q(z) = P(z)D(z)P(z) \tag{I.1}$$

In [1], in the context of spatially invariant systems, the question arose as to when the solution $P(z)$ will have a complex-analytic extension to some open connected set Ω. Here we report on some of the results from [6], where a more general version of this problem was studied. The key mathematical problem concerns the existence of analytic solutions to the following nonstandard Riccati equation

$$\begin{aligned} E(z)P(z) + P(z)A(z) + Q(z) \\ = P(z)D(z)P(z), \end{aligned} \tag{I.2}$$

where $E(z), A(z), D(z), Q(z)$ are $n\times n$ matrices that are analytic in Ω, a connected open subset of the complex plane. The properties of solutions (I.2) are determined by the Hamiltonian

$$H(z) = \begin{bmatrix} A(z) & -D(z) \\ -Q(z) & -E(z) \end{bmatrix}. \tag{I.3}$$

We call $P(z)$ a stabilizing solution of (I.2) if the eigenvalues of $A(z) - D(z)P(z)$ have negative real parts for all $z \in \Omega$. We cite three main results from [6].

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Theorem *1.1:* Suppose that following two assumptions hold

(a) $H(z)$ has no eigenvalues on the imaginary axis, for every $z \in \Omega$,

(b) for some $z_0 \in \Omega$, $H(z_0)$ is similar to either $-H(z_0)$ or $-H(z_0)^*$,

then either (A) or (B) holds:

(A) There are no stabilizing solutions for any $z \in \Omega$;

(B) There is an $n \times n$ matrix function $P(z)$ which is meromorphic on Ω and such that $P(z_0)$ is the unique stabilizing solution whenever z_0 is not a pole of $P(z)$. Moreover, there are no stabilizing solutions if z_0 is a pole of $P(z)$. □

Theorem *1.2:* If for every $z \in \Omega$ there is a unique stabilizing solution $P(z)$, then $P(z)$ is analytic in Ω. □

Theorem *1.3:* Assume in addition to the hypotheses of Theorem 1.1 that the entries of $H(z)$ are rational functions with poles outside Ω. If (B) of Theorem 1.1 holds, then the number of poles of $P(z)$ is either empty (i.e., $P(z)$ is analytic) or finite. □

In general, the region of analyticity of P will be larger than the region where $P(z)$ is a stabilizing solution as the following two examples illustrate.

Example *1.4:* Consider the analytic function

$$H(z) = \begin{bmatrix} 1+z & -1+2z^2 \\ -1 & -1+z \end{bmatrix},$$

noting that assumption (b) of Theorem 1.1 is satisfied in $z = 0$. For condition (a) let us examine where $H(z)$ has imaginary eigenvalues. This reduces to finding a real ω_0 and points $z \in \mathbb{C}$ such that

$$\jmath\omega_0 = z \pm \sqrt{2-2z^2}.$$

In fact this has infinitely many solutions parametrized by ω_0, namely,

$$z = (\pm\sqrt{6+2\omega_0^2} + \jmath\omega_0)/3.$$

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These points lie on the curve $\{x + \jmath y : x^2 - 2y^2 = 2/3\}$. So Theorem 1.1 predicts that, with the possible exception of poles of P, there will be stabilizing matrices in the open set $\Omega := \{x + \jmath y : x^2 - 2y^2 < 2/3\}$.

Solving the associated Riccati equation gives the two possible solutions

$$P(z) = \frac{1 \pm \sqrt{2 - 2z^2}}{1 - 2z^2},$$

which clearly have algebraic branch points at $z = \pm 1$ and poles at $z = \pm 1/\sqrt{2}$. The stabilizing solution is the one such that

$$A_P(z) := H_{11}(z) + H_{12}(z)P(z) = z \mp \sqrt{2 - 2z^2}$$

is stable, i.e.,

$$P_+(z) = \frac{1 + \sqrt{2 - 2z^2}}{1 - 2z^2}, \quad A_{P_+}(z) = z - \sqrt{2 - 2z^2}.$$

To examine the region where A_{P_+} is stable we calculate the real part of $A_{P_+}(z)$. Now with $z = x + \jmath y$ we have

$$\mathrm{Re}(A_{P_+}(z)) = x - \sqrt{1 + y^2 - x^2 + \sqrt{(1 + y^2 - x^2)^2 + 4x^2y^2}}.$$

Hence the region of stability of $A_{P_+}(z)$ is described by Ω as predicted by Theorem 1.1. Note that P_+ has poles at the points $z = \pm 1/\sqrt{2}$ and so at these points there is no stabilizing matrix. Outside Ω there will be no stabilizing solutions, but $P(z)$ will be analytic, except at the points $z = \pm 1$. □

Example *1.5:* Consider the analytic function

$$H(z) = \begin{bmatrix} -1 + z & -1 + 2z^2 \\ -1 & 1 + z \end{bmatrix},$$

noting that assumption (b) of Theorem 1.1 is satisfied in $z = 0$. As in Example 1.4, Theorem 1.1 predicts that there will be stabilizing matrices in the open set Ω of Example 1.4. Solving the associated Riccati equation gives the two possible solutions

$$P(z) = \frac{-1 \pm \sqrt{2 - 2z^2}}{1 - 2z^2},$$

which clearly have algebraic branch points at $z = \pm 1$ and poles at $z = \pm 1/\sqrt{2}$. The $H(z)$-stabilizing solution is given by

$$P_+(z) = \frac{-1 + \sqrt{2 - 2z^2}}{1 - 2z^2}, \quad A_{P_+}(z) = z - \sqrt{2 - 2z^2}.$$

Note that the apparent poles at $z = \pm 1/\sqrt{2}$ cancel out. As for Example 1.4 the region of stability of $A_{P_+}(z)$ is described by Ω and in this region $P_+(z)$ is stabilizing. Again, outside Ω there will be no stabilizing solutions, but $P(z)$ will be analytic, except at $z = \pm 1$. □

Concrete results for various classes of problems are obtained when the matrices $A(z)$, $E(z)$, $D(z)$, $Q(z)$ have the symmetry properties: $E(z) = A^{\sim}(z)$, $D^{\sim}(z) = D(z)$, $Q^{\sim}(z) = Q(z)$ with respect to a certain operation $^{\sim}$. These results have many applications depending the choice of Ω and the choice of $^{\sim}$. The two main sets of interest for the applications to spatially invariant systems are when Ω is an annulus around the unit circle $\mathfrak{A}(\tau) = \{z \in \mathbb{C} \,|\, e^{-\tau} < |z| < e^{\tau}\}, \quad \tau > 0$, and $A^{\sim}(z) := A(\overline{1/z})^*$ (or $A^{\sim}(z) := A(1/z)^T$) and when Ω is a strip around the imaginary axis $\mathfrak{S}(\alpha) := \{s \in \mathbb{C} : |\mathrm{Re}\,(s)| < \alpha\}$, for some $\alpha > 0$ and $A^{\sim}(s) := A(-\overline{s})^*$ (or $A^{\sim}(s) = A(-s)^T$).

In Sections II and III discuss the applications to control design for platoon-type systems and in Sections IV and V the applications to control design for partial differential systems defined on infinite domains.

II. Platoon-type systems

The platoon-type systems we consider are described by

$$\dot{x}_r(t) = \sum_{l \in \mathbb{Z}} A_l x_{r-l}(t) + \sum_{l \in \mathbb{Z}} B_l u_{r-l}(t), \qquad \text{(II.1)}$$

$$y_r(t) = \sum_{l \in \mathbb{Z}} C_l x_{r-l}(t) + \sum_{l \in \mathbb{Z}} D_l u_{r-l}(t), \qquad \text{(II.2)}$$

where $r \in \mathbb{Z}$, the set of integers, $A_l \in \mathbb{C}^{n \times n} B_l \in \mathbb{C}^{n \times m}, C_l \in \mathbb{C}^{p \times n}, D_l \in \mathbb{C}^{p \times m}$ and $x_r(t) \in \mathbb{C}^n$, $u_r(t) \in \mathbb{C}^m$ and $y_r(t) \in \mathbb{C}^p$ are the state, the input and the output vectors, respectively, at time $t \geq 0$ and spatial point $r \in \mathbb{Z}$. This class belongs to the class of spatially invariant systems introduced in [1]) and it is a special type of infinite-dimensional system. Using the terminology and formalism of [3] we can formulate (II.1), (II.2) as an infinite-dimensional linear system $\Sigma(A, B, C, D)$

$$\dot{x}(t) = (Ax)(t) + (Bu)(t), \qquad \text{(II.3)}$$
$$y(t) = (Cx)(t) + (Du)(t), \quad t \geq 0,$$

with the state space $X = \ell_2(\mathbb{C}^n)$, the input space $U = \ell_2(\mathbb{C}^m)$ and the output space $Y = \ell_2(\mathbb{C}^p)$. Note that X, U, Y are all infinite dimensional. So

$$x(t) = (x_r(t))_{r\in\mathbb{Z}},\ u(t) = (u_r(t))_{r\in\mathbb{Z}},$$

$$y(t) = (y_r(t))_{r\in\mathbb{Z}} \in \ell_2(\mathbb{C}^\bullet),$$

where $\bullet$ denotes an appropriate dimension and A, B, C, D are convolution operators. Denoting the signals and the convolution operators generically by $x(t)$ and T, respectively, we have

$$((Tx)(t))_r = \sum_{l\in\mathbb{Z}} T_l x_{r-l}(t) = \sum_{l\in\mathbb{Z}} T_{r-l} x_l(t).$$

Taking discrete Fourier transforms of the system equations (II.3): $\mathfrak{F} : \ell_2(\mathbb{C}^n) \to \mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$ we obtain

$$\begin{aligned} \dot{\check{x}}(t) &= \mathfrak{F}\dot{x}(t) = \check{A}\check{x}(t) + \check{B}\check{u}(t), \qquad \text{(II.4)} \\ \check{y}(t) &= \mathfrak{F}y(t) = \check{C}\check{x}(t) + \check{D}\check{u}(t), \end{aligned}$$

where $\check{A} = \mathfrak{F}A\mathfrak{F}^{-1}$, $\check{B} = \mathfrak{F}B\mathfrak{F}^{-1}$, $\check{C} = \mathfrak{F}C\mathfrak{F}^{-1}$ and $\check{D} = \mathfrak{F}D\mathfrak{F}^{-1}$ are multiplicative operators of the form $\check{A}(z) := \sum_{l\in\mathbb{Z}} A_l z^{-l}$, where $z \in \mathbb{T}$ the unit circle in the complex plane. If $\check{A}, \check{B}, \check{C}, \check{D} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{\bullet\times\bullet})$, then A, B, C, D are all bounded operators. In this case the linear system $\Sigma(A, B, C, D)$ on the state space $\ell_2(\mathbb{C}^n)$ is isometrically isomorphic to the linear system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ on the state space $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$ with input and output spaces $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^m)$ and $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^p)$ respectively. Their system theoretic properties are identical and are studied in [4].

For almost all $z \in \mathbb{T}$ the system (II.4) can be written as

$$\begin{aligned} \frac{\partial}{\partial t}\check{x}(z,t) &= \check{A}(z)\check{x}(z,t) + \check{B}(z)\check{u}(z,t) \qquad \text{(II.5)} \\ \check{y}(z,t) &= \check{C}(z)\check{x}(z,t) + \check{D}(z)\check{u}(z,t), \quad t \geq 0. \end{aligned}$$

The motivation for studying this special class of system stems from the interest shown in the literature for controlling infinite platoons of vehicles over the years (see [1], [2], [9], [11], [12], [13]). The models obtained for these configurations have the spatially invariant form (II.4) and $\check{A}, \check{B}, \check{C}, \check{D}$ have finitely many nonzero Fourier coefficients.

The following result is well-known.

Theorem *2.1:* Suppose that $\check{A}(z), \check{B}(z), \check{C}(z)$ are continuous in z on $\mathbb{T}$. If $(\check{A}(z), \check{B}(z), \check{C}(z))$ is stabilizable and detectable for each $z \in \mathbb{T}$, then the following family of Riccati equations has a unique stabilizing solution $\check{P}(z)$ for each $z \in \mathbb{T}$:

$$\begin{aligned} &\check{A}(z)^*\check{P}(z) + \check{P}(z)\check{A}(z) + \check{C}(z)^*\check{C}(z) \\ &\qquad = \check{P}(z)\check{B}(z)\check{B}(z)^*\check{P}(z) \qquad \text{(II.6)} \end{aligned}$$

and $\check{P}(z)$is positive semidefinite for each $z \in \mathbb{T}$. □

The stabilizing feedback in Theorem 2.1 $\check{K} = -\check{B}^*\check{P}$ has the form

$$\check{K}(z) = \sum_{l\in\mathbb{Z}} K_l z^l, \quad z \in \mathbb{T}$$

and the control action has the form

$$u(t)_r = \sum_{l\in\mathbb{Z}} K_{r-l} x_l(t) = \sum_{l\in\mathbb{Z}} K_l x_{r-l}(t).$$

For practical implementation it is desirable that the control action depend only on the nearest neighbours $x_r, x_{r\pm1}, ...x_{r\pm(r+s)}$, where s can be chosen as small as possible. This will be the case only if the Fourier coefficients k_r of K decay rapidly as $r \to \infty$. Consequently the authors of [14] sought conditions under which the solutions of Riccati equations will have a spatially decaying property. As they showed in [14, Theorem 1], it is sufficient that $\check{K}$ has an analytic extension to an annulus around the unit circle.

Lemma *2.2:* Suppose that $K(z)$ is a matrix function which is analytic in the annulus $\mathfrak{A}(\tau)$. Let $K(z) = \sum_{l\in\mathbb{Z}} K_l z^l$ be the Laurent series for $K(z)$. Then for every α, $0 \leq \alpha < \tau$, there exists a positive μ such that $\|K_l\| \leq \mu e^{-|l|\alpha}$. □

So the Fourier coeficients of $\check{K}$ will decay exponentially fast if $\check{B}$ and the solution to the Riccati equation (II.6) have an analytic extension to an annulus around the unit circle.

III. Riccati equation for the platoon-type systems

In this section we give conditions under which the solutions to the Riccati equation (II.6) have an analytic extension to an annulus around the unit circle. The analogue of the approach used in [1] is to seek an analytic extension $P(z)$ of the solution $\check{P}(z)$ to the Riccati equation (II.6) to an annulus $\mathfrak{A}(\alpha)$, i.e., $P(z) = \check{P}(z)$ for $z \in \mathbb{T}$. The obvious candidate is a solution $P(z)$ for $z \in \mathfrak{A}(\alpha)$ to the following nonstandard Riccati equation

$$\begin{aligned} &A^\sim(z)P(z) + P(z)A(z) + C^\sim(z)C(z) \\ &\quad = P(z)B(z)B^\sim(z)P(z), \qquad \text{(III.1)} \end{aligned}$$

where $A^{\sim}(z) := A(\overline{z^{-1}})^*$ and we suppose that $A(z), B(z), C(z)$ are $n \times n, n \times m$ and $p \times n$ matrix valued functions. In fact, we can consider a more general equation which allows for H-infinity type Riccati equations

$$A^{\sim}(z)P(z) + P(z)A(z) + Q(z) = P(z)D(z)P(z), \qquad \text{(III.2)}$$

where $D(z) = D^{\sim}(z)$ and $Q(z) = Q^{\sim}(z)$.

Using Theorem 1.1 we obtain sufficient conditions for the analyticity of $P(z)$ in some annulus around the unit circle $\mathbb{T}$.

Theorem *3.1:* Suppose that for some $\alpha > 0$

$$H(z) = \begin{bmatrix} A(z) & -D(z) \\ -Q(z) & -A^{\sim}(z) \end{bmatrix}$$

is analytic in the annulus $\mathfrak{A}(\alpha)$, and the following conditions hold:

(1) For every $z \in \mathbb{T}$, the matrix $H(z)$ has no eigenvalues on the imaginary axis;

(2) For every $z \in \mathbb{T}$, there exists a stabilizing solution;

(3) For at least one of the two points $z = \pm 1$, there exists a Hermitian stabilizing solution to (III.2).

Then for some β, $0 < \beta \leq \alpha$ and for every $z \in \mathfrak{A}(\beta)$, there exists a unique stabilizing solution $P(z)$ of $(III.2)$ with $P^{\sim}(z) = P(z)$ and $P(z)$ is analytic in $\mathfrak{A}(\beta)$ with $P^{\sim}(z) = P(z)$. $\square$

Theorems 3.1 and 2.1 provide the following useful corollary.

Corollary *3.2:* Suppose that for some $\alpha > 0$ $H(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$, $D(z) = B(z)B^{\sim}(z)$, $Q(z) = C^{\sim}(z)C(z)$ and $(A(z), B(z), C(z))$ is stabilizable and detectable for all $z \in \mathbb{T}$. Then there exists β, $0 < \beta \leq \alpha$, such that for every $z \in \mathfrak{A}(\beta)$, there exists a unique stabilizing solution $P(z)$ of $(III.1)$, and $P(z)$ is analytic in $\mathfrak{A}(\beta)$ with $P^{\sim}(z) = P(z)$. $\square$

The following theorem is another result asserting analyticity of stabilizing solutions, under suitable uniqueness hypothesis.

Theorem *3.3:* Assume that for some $\alpha > 0$ $H(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$. If for every $z \in \mathfrak{A}(\alpha)$ there is a unique $H(z)$-stabilizing matrix $P(z)$, and at least one of the two matrices $P(1)$ and $P(-1)$ is Hermitian, then $P^{\sim}(z) = P(z)$ for all $z \in \mathfrak{A}(\alpha)$ (thus $P(z)$ is a stabilizing solution of $(III.2)$), with $P^{\sim}(z) = P(z)$ and $P(z)$ is analytic in $\mathfrak{A}(\alpha)$. $\square$

So to obtain more information about the size of the annulus of analyticity we need to examine the existence of a stabilizing solution. An analogue of the following result was given in [1].

Theorem *3.4:* Suppose that for some $\alpha > 0$, the following properties hold:

1. $(III.2)$ has a stabilizing solution for all $z \in \mathbb{T}$.
2. $H(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$.
3. $H(z)$ has no eigenvalues on the imaginary axis for all $z \in \mathfrak{A}(\alpha)$.
4. The pair $(A(z), D(z))$ is stabilizable for all $z \in \mathfrak{A}(\alpha)$.
5. $D(z)$ is in the factored form $D(z) = B(z)B^{\sim}(z)$, where $B(z)$ is $n \times m$ and analytic in $\mathfrak{A}(\alpha)$, and for every $z \in \mathfrak{A}(\alpha)$, if for some vectors x, y there holds $y^T D(z)x = 0$, then $y^T B(z) = 0$ or $B^{\sim}(z)x = 0$.

Then for all $z \in \mathfrak{A}(\alpha)$ the equation $(III.2)$ has a unique stabilizing solution $P(z)$ with $P^{\sim}(z) = P(z)$. $\square$

Combining Theorem 3.4 with Theorem 3.3 gives our main result on the analyticity of solutions to (III.1):

Corollary *3.5:* Suppose that

1. for some $\alpha > 0$ the matrix function $H(z)$ with $D(z) = B(z)B^{\sim}(z)$, $Q(z) = C^{\sim}(z)C(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$;
2. $(A(z), B(z), C(z))$ is stabilizable and detectable for every $z \in \mathbb{T}$;
3. the matrix $H(z)$ has no eigenvalues on the imaginary axis for all $z \in \mathfrak{A}(\alpha)$;
4. $(A(z), B(z)B^{\sim}(z))$ is stabilizable for all $z \in \mathfrak{A}(\alpha)$;
5. for every $z \in \mathfrak{A}(\alpha)$, if for some vectors x, y there holds $y^T B(z)B^{\sim}(z)x = 0$, then $y^T B(z) = 0$ or $B^{\sim}(z)x = 0$.

Then for all $z \in \mathfrak{A}(\alpha)$ the equation $(III.1)$ has a unique stabilizing solution $P(z)$ that is analytic on $\mathfrak{A}(\alpha)$ with $P^{\sim}(z) = P(z)$. $\square$

We remark that the conditions in items 1-4 of the above Theorem are necessary (see Theorem 1.1), but the condition in item 5 of Theorem 3.4 is very restrictive. However, it does hold for rank one input operators of the form $B(z) = b_1(z)b_2^{\sim}(z)$, where $b_1(z) \in \mathbb{C}^n$, $b_2(z) \in \mathbb{C}^m$, provided that the scalar analytic function $b_2^{\sim}(z)b_2(z)$ has no zeros in the annulus $\mathfrak{A}(\alpha)$. This case occurs frequently in applications, but it is easy to construct examples for which the condition in item 5 of Theorem 3.4 is not necessary to achieve analyticity.

Example *3.6:* Consider the class of systems

$$A(z) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C(z) = \begin{bmatrix} c(z) & 0 \\ 0 & c(z) \end{bmatrix},$$

and B the 2×2 identity and assume that $c(z) \neq 0$, $z \in \mathbb{T}$, and $q(z) := c^{\sim}(z)c(z)$ is analytic in some annulus $\mathfrak{A}(\alpha)$. Then $H(z)$ will have imaginary eigenvalues only if the following equation has a real solution ω:

$$\omega^4 + 2q(z)\omega^2 + q(z) + q(z)^2 = 0.$$

So we choose $c(z)$ such that this equation has no real solutions in $\mathfrak{A}(\alpha)$. Then all the conditions in items 1-4 of Theorem 3.4 are satisfied, but the condition in item 5 is not satisfied.

The solution to the Riccati equation (III.1) is

$$P(z) = \begin{bmatrix} \sqrt{2qp_3} & p_3 \\ p_3 & \sqrt{2(1+q)p_3} \end{bmatrix},$$

where $p_3 = -q + \sqrt{q^2+q}, \quad q = q(z) = c^{\sim}(z)c(z)$. Under the above assumptions $P(z)$ will be the unique stabilizing solution in the annulus $\mathfrak{A}(\alpha)$.

It is possible to generate infinitely many examples of this class, but we give two. If $c(z) = z$, then $q(z) = 1$ and the unique stabilizing solution is

$$P = \begin{bmatrix} \sqrt{2}t & t^2 \\ t^2 & 2t \end{bmatrix}, \quad \text{where} \quad t = \sqrt{\sqrt{2}-1}.$$

If $c(z) = 2 + z$, then $q(z) = (2 + \frac{1}{z})(2 + z)$, the function $H(z)$ is analytic in $\mathfrak{A}(\log 2) = \{z : 1/2 < |z| < 2\}$, and $(A(z), B(z), C(z))$ is stabilizable and detectable in $\mathfrak{A}(\log 2)$. Computations show that $H(z)$ has no imaginary eigenvalues for every $z \in \mathfrak{A}(\log 2)$, and so $P(z)$ is the unique stabilizing solution for all $z \in \{z : 1/2 < |z| < 2\}$. $\square$

IV. Fourier transforms of systems arising from partial differential equations

A classical method of analyzing partial differential equations is to take Fourier transforms. This results in matrix multiplication operators on suitable Banach function spaces, for example, $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^n)$, where $\mathbb{J}$ denotes the imaginary axis. An $n \times n$ complex matrix $A(\cdot)$ induces a *matrix multiplication operator* M_A defined by

$$D(M_A) := \{f \in \mathbf{L}_2(\mathbb{J}; \mathbb{C}^n) : A(\lambda)f(\lambda) \in \mathbf{L}_2(\mathbb{J}; \mathbb{C}^n)\},$$
$$M_A f(\lambda) = A(\lambda)f(\lambda), \quad \forall f \in D(M_A), \quad \forall \lambda \in \mathbb{J}.$$

In [1] it is assumed that after taking Fourier transforms a controlled partial differential system takes the form like (II.4) with M_A generating a strongly continuous semigroup on the state space $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^n)$, input space $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^m)$ and output space $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^p)$. While this is the case for many partial differential systems, for example of parabolic type, in general it is not true that M_A will generate a semigroup on $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^n)$. Counterexamples are given in [5]. However, for the special case that M_A does generate a C_0-semigroup on $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^n)$ and $A(\lambda), B(\lambda), C(\lambda)$ are continuous for $\lambda \in \mathbb{J}$ the Fourier transformed system can also be written in the form like (II.5) with z replaced by $\lambda \in \mathbb{J}$. So this yields another class of spatially invariant systems as defined in [1] with the Riccati equation of the type (I.1).

The situation is analogous to that for platoon-type systems with the important difference that the platoon-type systems are parametrized over the unit circle, whereas the pde-type systems are parametrized over the imaginary axis which is not compact. Analogously to the platoon case, for an implementable control law the feedback gain should be localized. Typically the control law will have the form

$$u(x,t) = \int_{-\infty}^{\infty} \kappa(x-\xi) z(\xi,t)\, \mathrm{d}\xi,$$

where $z(x,t)$ is the solution of the controlled p.d.e., and κ is the distribution which is the inverse Fourier transform of $-B(s)^*P(s)$. Suppose that κ is a continuous function. Then this control law can be approximated by a localized control law provided that the kernel decays exponentially fast to zero as $|x| \to \infty$. A first step in proving this exponential decaying property is to establish the extension of $P(\lambda)$ to a function that is analytic in a vertical strip around the imaginary axis. This was done in [1, Theorem 6] for the special case that $A(s), B(s), C(s)$ are all rational analytic functions in a strip around the imaginary axis (and certain extra assumptions). We give more general results in the next section.

V. Analyticity of Riccati equation solutions on a vertical strip

Following the approach in Section III we seek solutions to the following nonstandard Riccati equation which are analytic in a strip $\mathfrak{S}(\alpha)$ around the imaginary axis:

$$A^{\sim}(s)P(s) + P(s)A(s) + C^{\sim}(s)C(s) = P(s)B(s)B^{\sim}(s)P(s). \qquad \text{(V.1)}$$

where $A(s)$, $B(s)$, $C(s)$ are matrix valued functions of sizes $n \times n$, $n \times p$, $q \times n$, respectively, of the variable s in $\mathfrak{S}(\alpha)$. In contrast with Section III, we now have $A^{\sim}(s) := A(-\overline{s})^*$, and as in Section III we consider also a more general form of (V.1)

$$A^{\sim}(s)P(s) + P(s)A(s) + Q(s) \qquad \text{(V.2)}$$

$$= P(s)D(s)P(s), \qquad \text{(V.3)}$$

where $D^{\sim}(s) = D(s)$, $Q^{\sim}(s) = Q(s)$ for $s \in \mathfrak{S}(\alpha)$.

The Hamiltonian matrix function $H(s)$ is defined by the same formula (I.3).

The following theorems are the analogues of Theorems 3.1, 3.3 and 3.4 in Section III.

Theorem *5.1:* Assume that for some $\alpha > 0$ $H(s)$ is analytic in the strip $\mathfrak{S}(\alpha)$. If for every $s \in \mathfrak{S}(\alpha)$ there is a unique stabilizing matrix $P(s)$, and the matrix $P(0)$ is Hermitian, then $P^{\sim}(s) = P(s)$ for all $s \in \mathfrak{S}(\alpha)$, and $P(s)$ is analytic in $\mathfrak{S}(\alpha)$.

Theorem *5.2:* Suppose that $H(s)$ is a rational matrix function and the following conditions hold:

(1) There is $\alpha > 0$ such that $H(s)$ is analytic in the strip $\mathfrak{S}(\alpha)$, and for every $s \in \mathfrak{S}(\alpha)$ and $H(s)$ has no eigenvalues on the imaginary axis;

(2) For every $z \in \mathbb{J}$, there exists a stabilizing matrix $P(s)$ and $P(0)$ is Hermitian;

Then there exists β, $0 < \beta \leq \alpha$, such that for every s in the strip $\mathfrak{S}(\beta)$, there exists a unique stabilizing solution $P(s)$ of $(V.2)$ satisfying $P^{\sim}(s) = P(s)$ and $P(s)$ is analytic in $\mathfrak{S}(\beta)$. □

Note that $H(s)$ is allowed to have a pole at infinity under the hypotheses of Theorem 5.2. We also point out that the hypothesis of $H(s)$ being rational yields at most finitely many poles of $P(s)$ (Theorem 1.3), and as a result the conclusions of the theorem hold for some strip $\mathfrak{S}(\beta)$. Assuming that $H(s)$ is merely analytic in $\mathfrak{S}(\alpha)$, the conclusions of the theorem would hold for some open set $\Omega \subseteq \mathfrak{S}(\beta)$ such that $z \in \Omega \Rightarrow -z \in \Omega$, but Ω need not a be a strip of the form $\mathfrak{S}(\beta)$. The difference with the analogous result in Theorem 3.3 lies in the fact that, unlike $\mathfrak{A}(\alpha)$, $\mathfrak{S}(\alpha)$ is not compact. The translation of Theorem 3.4 to the strip yields the following existence result that was proven in [1, Theorem 6].

Theorem *5.3:* Suppose that:

1. $(V.2)$ has a stabilizing solution for all $s \in \mathbb{J}$.
2. For some $\alpha > 0$ $H(s)$ is analytic in the strip $\mathfrak{S}(\alpha)$.
3. $H(s)$ has no eigenvalues on the imaginary axis for all $s \in \mathfrak{S}(\alpha)$.
4. $(A(s), D(s))$ is stabilizable for all $s \in \mathfrak{S}(\alpha)$.
5. $D(s)$ is in the factored form $D(s) = B(s)B^{\sim}(s)$, where $B(s)$ is $n \times m$ and analytic in $\mathfrak{S}(\alpha)$, and for every $s \in \mathfrak{S}(\alpha)$, if for some vectors x, y there holds $y^T D(s)x = 0$, then $y^T B(s) = 0$ or $B^{\sim}(s)x = 0$.

Then for all $s \in \mathfrak{S}(\alpha)$, equation $(V.2)$ has a unique stabilizing solution $P(s)$. □

Combining Theorems 5.3, 5.1 yields the proof of the following result which generalizes that in [1, Theorem 6] where it was assumed that $H(s)$ was rational.

Corollary *5.4:* Suppose that $A(s), B(s), C(s)$ are matrix functions of suitable sizes, and the following conditions hold:

1. $(A(s), B(s), C(s))$ is stabilizable and detectable for all $s \in \mathbb{J}$.
2. For some $\alpha > 0$ $A(s), B(s), C(s)$ are analytic in the strip $\mathfrak{S}(\alpha)$.
3. $H(s)$ has no eigenvalues on the imaginary axis for all $s \in \mathfrak{S}(\alpha)$.
4. $(A(s), B(s)B^{\sim}(s))$ is stabilizable for all $s \in \mathfrak{S}(\alpha)$;
5. for every $s \in \mathfrak{S}(\alpha)$, if for some vectors x, y there holds $y^T B(s)B^{\sim}(s)x = 0$, then $y^T B(s) = 0$ or $B^{\sim}(s)x = 0$.

Then for all $s \in \mathfrak{S}(\alpha)$ the equation $(V.1)$ has a unique stabilizing solution $P(s)$ that is analytic on $\mathfrak{S}(\alpha)$and $P^{\sim}(s) = P(s)$. □

This corollary explains the difference between Examples 1.4 and 1.5. In both examples $H(s)$ is analytic in $\mathbb{C}$ and $H(s)$ has no eigenvalues on the imaginary axis for $s \in \mathfrak{S}(\sqrt{2/3})$. In Example 1.4 $(A(s), B(s)) = (1 + s, 1 + \sqrt{2}s)$ is stabilizable in $\mathfrak{S}(1/\sqrt{2})$ and so by Corollary 5.4 $P(s)$ is the stabilizing solution in $\mathfrak{S}(1/\sqrt{2})$. In Example 1.5 $(A(s), B(s)) = (-1 + s, 1 + \sqrt{2}s)$ is stabilizable in $\mathbb{C}$ and so $P(s)$ is the stabilizing solution in the larger strip $\mathfrak{S}(\sqrt{2/3})$.

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