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A sparse stability test for sparse matrices

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***Abstract*— In the study of distributed control systems, it is of fundamental interest to understand how specifications on local interconnections influence properties of the global system. In this paper, we consider linear continuous time systems described by a sparse matrix. The property of interest is stability. In particular, for matrices with sparsity structure corresponding to a chordal graph, we show that conditions for Hurwitz stability can be written with the same sparsity structure.**

I. Positivity and Stability of Banded Matrices

The main idea behind the results of the paper dates back at least to the 1960s. It is illustrated in Figure 1 and stated rigorously in the following proposition:

Proposition 1: Suppose that the matrix $M \in \mathbf{R}^{n\times n}$, with coefficients M_{ij}, is symmetric and banded, i.e. there exists $d < n-1$ such that $M_{ij} = 0$ for $|i-j| > d$. If M is positive semi-definite, then there exist positive semi-definite matrices $M^k = (M^k_{ij})$, $k = 1+d, \dots, n-d$ such that $M = M^{1+d} + \cdots + M^{n-d}$ and $M^k_{ij} = 0$ for $\min\{|i-k|, |j-k|\} > d$.

Proposition 1 is an immediate consequence of the following lemma [2]. See Figure 2.

Lemma 1: Suppose that $M = (M_{ij}) \in \mathbf{R}^{n\times n}$ is symmetric and banded, i.e. there exists $d < n-1$ such that $M_{ij} = 0$ for $|i-j| > d$. Then there exists a lower triangular matrix $L = (L_{ij})$ such that $LL^T = M$ and $L_{ij} = 0$ for $|i-j| > d$.

Given the positivity test of Proposition 1, the step to stability verification is not long:

Proposition 2: Assume $A \in \mathbf{R}^{n\times n}$ is banded, i.e. there exists $d_1 \le 0 \le d_2$ such that $A_{ij} = 0$ for $i-j < d_1$ and for $i-j > d_2$. Then A is Hurwitz if and only if there exist rational matrix functions $M^k(\cdot,\cdot) = (M^k_{ij})$ with

$$(sI-A)^*(sI-A) = M^1(\bar{s},s) + \cdots + M^{n-d}(\bar{s},s) \quad (1)$$

where $M^k(\bar{s},s)$ is positive definite for Re $s \ge 0$, while $M^k_{ij} = 0$ for $\min\{|i-k|, |j-k|\} > d_2 - d_1$.
A proof will follow later.

Remark 1. At first sight, it may look like the equality condition (1) makes Proposition 2 very sensitive to perturbations in the system coefficients A_{ij}. However, the equality can be recoverded by just updating the local M^k-matrix. This works as long as the matrix remains positive definite for s in the right half plane.

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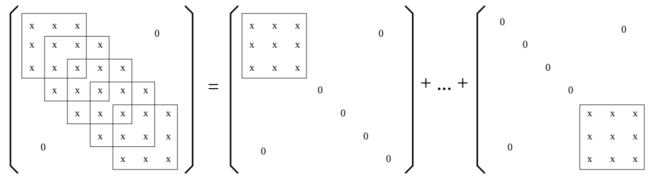

Fig. 1. Illustration of Proposition 1 and Proposition 2. A banded positive semi-definite matrix can be decomposed as a sum of positive semi-definite matrices, where each term is non-zero only in a small square. As a consequence, the banded matrix A is Hurwitz stable if and only if the matrix $(sI-A)^*(sI-A)$ can be decomposed this way for every s with Re $s \ge 0$.

Proposition 2 may be used for distributed system verification as follows: Suppose that a system has been designed to behave according to the dynamics described by the matrix A and matrix functions M^k have been computed to satisfy the conditions of Proposition 2. Suppose that each node has access to local model data A_{ij} and M^k for the neighboring nodes. Then the i:th row of condition (1) can be verified in node i before the system is put into operation. If a component changes and the local dynamics is updated, a local update of M^i could be sufficient to maintain equality in the i:th row. However, if the positivity of M^i is lost, it could be necessary to recompute all the performance certificates M^i, or conclude that stability has been lost.

II. Graph Notation and Preliminaries

Consider an undirected graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ with nodes $\mathbb{V} = \{1, 2, \dots, J\}$ and edges $\mathbb{E}$. A *cycle* in the graph is a sequence of pairwise distinct nodes $(n_1, \dots, n_s)$ such that $(n_1, n_2), (n_2, n_3), \dots, (n_{s-1}, n_s), (n_s, n_1) \in \mathbb{E}$.

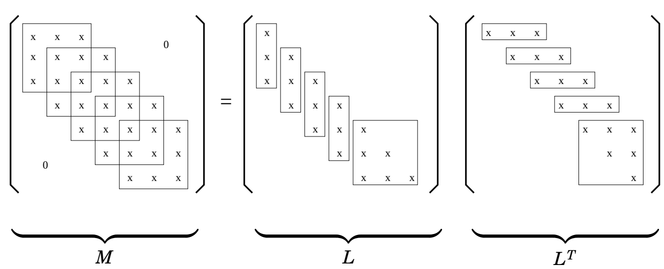

Fig. 2. Illustration of Lemma 1. A banded positive semi-definite matrix has a banded Cholesky-factorization. Multiplying the first column of L with the first row of L^T gives the first term on the right hand side of Figure 1. The other terms are obtained analogously.

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A *chord* of the graph cycle $(n_1,\dots,n_s)$ is an edge $(n_i,n_j)\in\mathbb{E}$, where $1\le i<j\le s$, $(i,j)\neq(1,s)$ and $|i-j|\ge 2$. The graph is called *chordal* if every cycle of length ≥ 4 has a chord.

A *clique* of the graph $(\mathbb{V},\mathbb{E})$ is a subset $\mathbb{C}$ of $\mathbb{V}$ having the property that $(i,j)\in\mathbb{E}$ for all $i,j\in\mathbb{C}$. An *ordering* of the graph is a bijection $\alpha:\mathbb{V}\to\{1,2,\dots,J\}$. The ordering α is a *perfect elimination ordering* if for every $i\in\mathbb{V}$, the set $\{j\mid (i,j)\in\mathbb{E},\ \alpha(i)<\alpha(j)\}$ is a clique. The following result is cited from [4]:

Proposition 3: A graph has a perfect elimination ordering if and only if it is chordal.

Chordal graphs have appeared in the literature in relation to the problem of reordering rows and columns of a sparse matrix in such a way that no nonzero elements are created in the course of Gaussian elimination [4].

Given a graph node i, define $[i]$ as the set of all j such that $(i,j)\in\mathbb{E}$. For each node i of the graph, we assign a vector $x_i\in\mathbf{C}^{n_i}$. The concatenation of all vectors is $x\in\mathbf{C}^n$. Similarly $x_{[i]}$ is the concatenation of only those vectors x_j where $(i,j)\in\mathbb{E}$. The dimension of $x_{[i]}$ is denoted $n_{[i]}$.

For a matrix $A\in\mathbf{R}^{n\times n}$ with blocks $A_{ij}\in\mathbf{R}^{n_i\times n_j}$, let $A_{[i]}\in\mathbf{R}^{n\times n_{[i]}}$ be the submatrix consisting of blocks A_{kj} with $j\in[i]$. Let $E_{[i]}\in\mathbf{R}^{n\times n_{[i]}}$ be the corresponding submatrix of the identity matrix $I\in\mathbf{R}^{n\times n}$. Similarly, $E_i\in\mathbf{R}^{n\times n_i}$ is the block column matrix where all blocks are zero except the i:th block, which is identity.

III. Main result

We are now ready to give generalizations of the results in section I beyond banded matrices. The following generalization of Lemma 1, known since the 1970s [3], [1], indicates that this should be possible.

Lemma 2: Given a chordal graph $\mathbb{G}=(\mathbb{V},\mathbb{E})$, suppose that $M=(M_{ij})\in\mathbf{R}^{n\times n}$ is symmetric and $M_{ij}=0$ for $(i,j)\notin\mathbb{E}$. Then there exists a lower triangular matrix $L=(L_{ij})$ such that $LL^T=M$ and $L_{ij}=0$ for $(i,j)\notin\mathbb{E}$.

In order to verify that $M^i(\cdot,\cdot)$ in Proposition 2 and its generalizations can be chosen as rational functions, we state the following main result.

Theorem 1: Given a chordal graph $\mathbb{G}=(\mathbb{V},\mathbb{E})$, let $M(\cdot,\cdot)$ be a rational matrix with blocks $M_{ij}(s)$ that are non-zero only for $(i,j)\in\mathbb{E}$. Suppose that $M(\bar{s},s)$ is Hermitean positive definite for Re $s\ge 0$. Then there exist rational matrix functions $M^1(\cdot,\cdot),\dots,M^J(\cdot,\cdot)$ such that for each i and Re $s\ge 0$ the matrix $M^i(\bar{s},s)$ is identically zero outside the submatrix ${E_{[i]}}^T M^i(\bar{s},s)E_{[i]}$ which is positive definite, and

$$M(\bar{s},s)=\sum_{i=1}^{J}M^i(\bar{s},s)\qquad(2)$$

Remark 2. Notice that (2) holds if and only if

$${E_{[i]}}^T M(\bar{s},s)E_{[i]}=\sum_{j\in[i]}{E_{[i]}}^T M^j(\bar{s},s)E_{[i]}*$$

for all i. Hence the condition has the desired sparsity and can be verified node by node.

Proof. Theorem 1 will be proved by induction over J. The statement is trivial for $J=1$. Suppose that the statement holds for $J=n$ and we want to prove it for $J=n+1$. Consider a chordal graph with $n+1$ nodes. After a permutation of the nodes, by Proposition 3, we may assume without restriction that the set

$$\langle i\rangle:=\{j\mid (i,j)\in\mathbb{E},\ i<j\}$$

is a clique for every i. In particular, the graph $\mathbb{G}_1$, obtained by removing the first node from $\mathbb{G}$, is also chordal. Partition the matrix $M(\bar{s},s)$ as

$$M(\bar{s},s)=\left[\begin{array}{c|c}M_{1,1}&N_{12}\\\hline N_{21}&N_{22}\end{array}\right]$$

Then $M(\bar{s},s)=M^1(\bar{s},s)+M^2(\bar{s},s)$ where

$$M^1(\bar{s},s)=\begin{bmatrix}M_{1,1}&N_{12}\\N_{21}&N_{21}M_{1,1}^{-1}N_{12}\end{bmatrix}+\epsilon(E_{[1]}E_{[1]}^T-E_1E_1^T)$$

$$M^2(\bar{s},s)=\begin{bmatrix}0&0\\0&N_{22}-N_{21}M_{1,1}^{-1}N_{12}\end{bmatrix}-\epsilon(E_{[1]}E_{[1]}^T-E_1E_1^T)$$

The matrix $M^1(\bar{s},s)$ vanishes outside the submatrix ${E_{[1]}}^T M^1(\bar{s},s)E_{[1]}$, which is positive definite for $\epsilon>0$. For sufficiently small ϵ, also the lower right corner of $M^2(\bar{s},s)$ is positive definite. The fact that $\langle 1\rangle$ is a clique implies that the lower right corner of $M^2(\bar{s},s)$ has the same sparsity structure as N_{22}, namely the structure corresponding to the chordal graph $\mathbb{G}_1$. Hence, the desired decomposition follows by induction. □

A proof of Proposition 2 follows immediately:

Proof of Proposition 2. The (if)-statement is the simple direction is trivial. The (only if)-statement follows from Theorem 1 with $M(\bar{s},s)=(sI-A)^*(sI-A)$. □

IV. Acknowledgement

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