# Dynamic quantizer synthesis based on invariant set analysis for SISO systems with discrete-valued input

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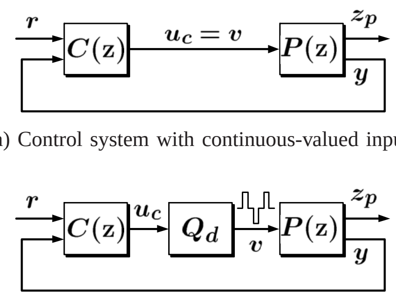
***Abstract*— This paper proposes analysis and synthesis methods of dynamic quantizers for linear feedback single input single output (SISO) systems with discrete-valued input in terms of invariant set analysis. First, this paper derives the quantizer analysis and synthesis conditions that clarify an optimal quantizer within the ellipsoidal invariant set analysis framework. In the case of minimum phase feedback systems, next, this paper presents that the structure of the proposed quantizer is also optimal in the sense that the quantizer gives an optimal output approximation property. Finally, this paper points out that the proposed design method can design a stable quantizer for non-minimum phase feedback systems through a numerical example.**

## I. INTRODUCTION

Recently, one of the most remarkable control studies is the discrete-valued control problem. In the systems with discrete-valued input such as the networked systems, continuous-valued signals are quantized and transmitted over communication channels. In this case, we need appropriate quantization methods to achieve some control performance requirements such as stabilization over communication channels. Since the discrete-valued control theory can be applied not only to the networked control and but also to the various devices such as D/A converters and ON/OFF actuators, this control topic has been actively studied [1]–[8] so far.

For the above challenging problem, references [5]–[8] have focused on optimality in terms of control and provided an optimal dynamic quantizer for the discrete-valued control. When a plant $P(\mathbf{z})$ and a controller $C(\mathbf{z})$ are given in the linear feedback system in Fig. 1 (b), those methods can provide a "dynamic" quantizer $Q_d$ such that the system in Fig. 1 (b) "optimally" approximates the usual system in Fig. 1 (a) in the sense of the input-output relation. The optimal dynamic quantizer enables us to design the controller $C(\mathbf{z})$ in Fig. 1 (b) based on the conventional linear control system theory [9]–[12]. However, their optimal quantizer is not always stable since the quantizer does not take its stability into account. In the case where the given system has unstable zeros, the optimal dynamic quantizer in [5], [6] becomes unstable and its quantized signal often goes unbounded as mentioned in [7]. Although the numerical design method in [8] can provide the stable optimal dynamic quantizer, its infinite time control performance is not always guaranteed and the order of the obtained quantizer is basically (in some cases much) higher than that of the given system.




(a) Control system with continuous-valued input

(b) Control system with discrete-valued input

Fig. 1. Two control systems

Therefore this paper reconsiders control problems for single input single output (SISO) systems with discrete-valued input in Fig. 1 (b) as a first step. Our approach is based on the invariant set analysis [13] and the linear matrix inequality (LMI) technique [9]. This framework allows us to analyze and synthesize dynamic quantizers guaranteed with infinite quantization period even if the given system has unstable zeros. First, this paper proposes a quantizer analysis condition. Second, this paper discusses quantizer synthesis conditions. As a result, the synthesis conditions can be characterized by the transmission zeros of the given system. Third, this paper reconsiders the optimal dynamic quantizer proposed in [5]–[8]. In the case where the given system is minimum phase, this paper clarifies that the invariant analysis framework also provides an optimal dynamic quantizer in terms of the output approximation property. Compared with [8], it is clarified that our design method can design a stable suboptimal quantizer such that the order of the quantizer is exactly the same as that of the given system and the infinite time control performance is always guaranteed. Finally, this paper points out that our method is helpful in quantizer synthesis through numerical examples.

**Notation:** The set of $n \times m$ (positive) real matrices is denoted by $\mathbb{R}^{n\times m}$ ($\mathbb{R}_+^{n\times m}$). The set of $n \times m$ (positive) integer matrices is denoted by $\mathbb{N}^{n\times m}$ ($\mathbb{N}_+^{n\times m}$). $0_{n\times m}$ and $I_m$ (or for simplicity of notation, $0$ and $I$) denote the $n\times m$ zero matrix and the $m\times m$ identity matrix, respectively. $\lfloor a \rfloor$ denotes the floor of the positive number $a$. For a matrix $M$, $M^{\mathrm{T}}$ and $\rho(M)$ denote its transpose and its spectrum radius, respectively. For a symmetric matrix $X$, $X > 0$ ($X \geq 0$) means that $X$ is positive (semi) definite. For a vector $x$ and a sequence of vectors $X := \{x_1, x_2, ...\}$, $\|x\|$ and $\|X\|$

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denote their $\infty$-norms, respectively. For matrices $G$ and $K$, the linear fractional transformation of $K$ by $G$ is defined to be the matrix that represents the mapping from $w$ to $z$ in Fig. 2, and is denoted by $G \circledast K$.

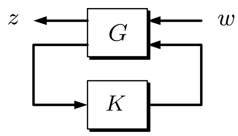


Fig. 2. Linear fractional transformation $G \circledast K$

## II. PRELIMINARIES

Consider the linear time invariant (LTI) discrete-time system given by

$$\xi(k+1) = \mathcal{A}\xi(k) + \mathcal{B}w(k) \tag{1}$$

where $\xi \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ denote the state vector and disturbance input, respectively. We define the invariant set.

*Definition 1:* Define the invariant set of the system (1) to be a set $\mathcal{X}$ which satisfies

$$\xi \in \mathcal{X}, \quad w^{\mathrm{T}}w \le 1 \quad \Rightarrow \quad \mathcal{A}\xi + \mathcal{B}w \in \mathcal{X}. \tag{2}$$

The analysis condition can be expressed in terms of matrix inequalities using the $S$-procedure [9] and the ellipsoid defined by a level set of quadratic Lyapunov function as summarized in the following proposition [13].

*Proposition 1: Consider the system (1). For a matrix* $0 < \mathcal{P} \in \mathbb{R}^{n\times n}$*, the ellipsoid* $\mathbb{E}(\mathcal{P}) := \left\{\xi \in \mathbb{R}^n : \xi^{\mathrm{T}}\mathcal{P}\xi \le 1\right\}$ *is an invariant set if and only if there exists a scalar* $\alpha \in [0, 1-\rho(\mathcal{A})^2]$ *satisfying*

$$\begin{bmatrix} \mathcal{A}^{\mathrm{T}}\mathcal{P}\mathcal{A} - (1-\alpha)\mathcal{P} & \mathcal{A}^{\mathrm{T}}\mathcal{P}\mathcal{B} \\ \mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{A} & \mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{B} - \alpha I_m \end{bmatrix} \le 0. \tag{3}$$

All the ellipsoidal invariant sets are parameterized by Proposition 1. Also, the ellipsoidal invariant set allows us to approximate the reachable set from outside since the former covers the latter. Reference [13] considers the criterion $f(\mathcal{P})$ for the approximation of $\mathbb{E}(\mathcal{P})$ to the reachable set because the matrix $\mathcal{P}$ determines the ellipsoid. $f(\mathcal{P})$ has the monotonical decreasingness in the sense that its value for the set of inside is less than that of outside. When $\alpha$ is fixed in (3), reference [13] clarifies that the infimum of $f(\mathcal{P})$ does not change even if $\mathcal{P}$ is restricted to $\mathcal{P}(\alpha)$ given by

$$\mathcal{P}(\alpha)^{-1} = \sum_{k=0}^{\infty} \frac{1}{\alpha(1-\alpha)^k}\mathcal{A}^k\mathcal{B}\mathcal{B}^{\mathrm{T}}(\mathcal{A}^{\mathrm{T}})^k, \qquad 0 < \alpha < 1-\rho(\mathcal{A})^2. \tag{4}$$

Therefore, all the ellipsoidal invariant sets approximating the reachable set in (3) can be parameterized by $\alpha \in (0, 1-\rho(\mathcal{A})^2)$. Also, the criterion $f(\mathcal{P})$ can be replaced by $f(\mathcal{P}(\alpha))$. Denote by $\xi(k, \xi(0), w)$ the state trajectory of the system (1) at the $k^{\mathrm{th}}$ time. For the set $\mathbb{E}(\mathcal{P})$ characterized by Proposition 1, the property

$$\lim_{k\to\infty} \inf_{\xi\in\mathbb{E}(\mathcal{P})} \|\xi(k, \xi(0), w) - \xi\| = 0 \tag{5}$$

also holds clearly [14].

## III. PROBLEM FORMULATION

Consider the control system with discrete-valued input depicted in Fig. 1 (b), which is consists of the discrete-time LTI plant $P(\mathbf{z})$, the discrete-time LTI controller $C(\mathbf{z})$, and the dynamic quantizer $Q_d$. The systems $P(\mathbf{z})$ and $C(\mathbf{z})$ are represented by

$$P(\mathbf{z}) : \begin{bmatrix} x_p(k+1) \\ z_p(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A_p & B_p \\ C_{p1} & 0 \\ C_{p2} & 0 \end{bmatrix} \begin{bmatrix} x_p(k) \\ v(k) \end{bmatrix}, \tag{6}$$

$$C(\mathbf{z}) : \begin{bmatrix} x_c(k+1) \\ u_c(k) \end{bmatrix} = \begin{bmatrix} A_c & B_{c1} & B_{c2} \\ C_c & D_{c1} & D_{c2} \end{bmatrix} \begin{bmatrix} x_c(k) \\ y(k) \\ r(k) \end{bmatrix} \tag{7}$$

where $x_p \in \mathbb{R}^{n_p}$, $x_c \in \mathbb{R}^{n_c}$, $v \in \mathbb{R}^m$, $u_c \in \mathbb{R}^m$, $z_p \in \mathbb{R}^q$, $y \in \mathbb{R}^p$, and $r \in \mathbb{R}^q$ denote the plant state vector, the controller state vector, the control input, the controller output, the controlled output, the measured output, and the exogenous input, respectively. The systems $P(\mathbf{z})$ and $C(\mathbf{z})$ are given such that the control system with continuous-valued input in Fig. 1 (a) is stable in the discrete domain.

For the systems, we consider the dynamic quantizer $v = Q_d u_c$ with the state vector $x_q \in \mathbb{R}^{n_q}$. The system $Q_d$ consists of the static quantizer $Q_{st} : \mathbb{R}^m \to d\mathbb{N}^m$, i.e.,

$$v(k) = Q_{st}(u(k)), \quad u(k) := u_q(k) + u_c(k) \tag{8}$$

and the dynamic compensator $Q(\mathbf{z})$

$$\begin{bmatrix} x_q(k+1) \\ u_q(k) \end{bmatrix} = \begin{bmatrix} A_q & B_q \\ C_q & 0 \end{bmatrix} \begin{bmatrix} x_q(k) \\ e_q(k) \end{bmatrix} \tag{9}$$

where $e_q(k) := v(k) - u_c(k)$. Note that $Q_{st}$ is of the nearest-neighbor type toward $-\infty$ with the quantization interval $d \in \mathbb{R}_+$ and the initial state is given by $x_q(0) = 0$ for the drift-free of $Q_d$ [5]–[8]. One such static quantizer is the midtread type quantizer in Fig. 3.

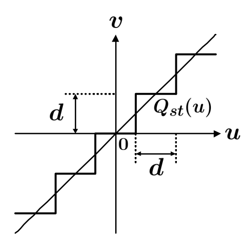


Fig. 3. Midtread type quantization

We define the following matrices:

$$A := \begin{bmatrix} A_p + B_pD_{c1}C_{p2} & B_pC_c \\ B_{c1}C_{p2} & A_c \end{bmatrix}, \quad B_1 := \begin{bmatrix} B_p \\ 0 \end{bmatrix},$$

$$B_2 := \begin{bmatrix} B_pD_{c2} \\ B_{c2} \end{bmatrix}, \quad C_1 := \begin{bmatrix} D_{c1}C_{p2} & C_c \end{bmatrix}, \quad D := D_{c2},$$

$$C_2 := \begin{bmatrix} C_{p1} & 0 \end{bmatrix}, \qquad \mathcal{A} := \begin{bmatrix} A & B_1C_q \\ 0 & A_q + B_qC_q \end{bmatrix},$$

$$\mathcal{B}_1 := \begin{bmatrix} B_1 \\ B_q \end{bmatrix}, \qquad \mathcal{B}_2 := \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \quad \mathcal{C}_1 := \begin{bmatrix} C_1 & C_q \end{bmatrix},$$

$$\mathcal{C}_2 := \begin{bmatrix} C_2 & 0 \end{bmatrix}, \qquad \mathcal{D} := D. \tag{10}$$

For the system in Fig. 1 (b), the systems $P(\mathbf{z})$ and $C(\mathbf{z})$ with the static quantizer $Q_{st}$ seen by the linear compensator $Q(\mathbf{z})$ can be recast as the linear fractional transformation (LFT) of a generalized plant $G(\mathbf{z})$:

$$\begin{bmatrix} x(k+1) \\ u(k) \\ z_p(k) \\ e_q(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 & B_1 \\ C_1 & 0 & D & I \\ C_2 & 0 & 0 & 0 \\ 0 & I & 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \\ r(k) \\ u_q(k) \end{bmatrix} \quad (11)$$

where $x := [\ x_p^{\mathrm{T}} \quad x_c^{\mathrm{T}}\ ]^{\mathrm{T}} \in \mathbb{R}^{n_g}(n_g := n_p + n_c)$, and the quantization error $Q_e$ in Fig. 4,

$$e(k) = Q_e(u(k)), \quad Q_e(u(k)) := v(k) - u(k). \quad (12)$$

In this case, the control system in Fig. 1 (b) can be described

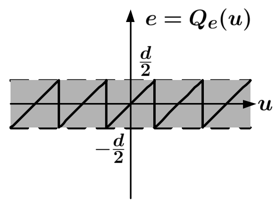


Fig. 4. Quantization error (Midtread type)

as a LFT (Fig. 5) of the quantization error $Q_e$ and a LTI system $H(\mathbf{z})$ represented by

$$e(k) = Q_e(u(k)),$$
$$H(\mathbf{z}) : \begin{bmatrix} \xi(k+1) \\ u(k) \\ z_p(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & 0 & \mathcal{D} \\ \mathcal{C}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi(k) \\ e(k) \\ r(k) \end{bmatrix} \quad (13)$$

where $\xi := [\ x^{\mathrm{T}} \quad x_q^{\mathrm{T}}\ ]^{\mathrm{T}} \in \mathbb{R}^n(n := n_g + n_q)$ and $H(\mathbf{z}) = G(\mathbf{z}) \circledast Q(\mathbf{z})$.

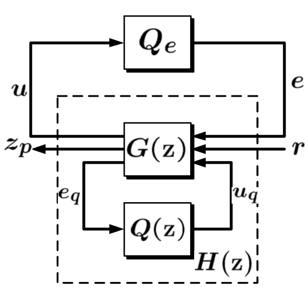


Fig. 5. Feedback system with quantization error

Suppose that $P(\mathbf{z})$ is the SISO system ($m = q = 1$). For the system in Fig. 1 (b) without the exogenous signal ($r(k) = 0\ \forall k$), $z_p(k, x_0)$ denotes the output of $z_p$ at the $k$-th time for the initial state $x_0 := x(0)$. In this case, this paper considers the following cost function:

$$L(Q_d) := \limsup_{k\to\infty} |z_p(k, x_0)|.$$

Our first objective is to solve the following dynamic quantizer synthesis problem **(L)**: *For the system (13) without the exogenous signal, suppose that the quantization interval $d \in \mathbb{R}_+$ and the performance level $\gamma \in \mathbb{R}_+$ are given. Characterize a stable dynamic quantizer $Q_d$ (i.e., find parameters $(n_q, A_q, B_q, C_q)$) achieving $L(Q_d) \le \gamma$ based on Proposition 1.* Note that the quantizer $Q_d$ is stable if and only if $A_q + B_qC_q$ is stable in the discrete domain [6]–[8].

$L(Q_d)$ indicates the difference between the systems in Fig. 1 (a) and (b) in terms of the same initial state. Because of the quantization error caused by the static quantizer, the controlled output $z_p$ in Fig. 1 (b) might not go to zero and might go unbounded no matter where it starts and no matter how long time passes unlike the usual system in Fig. 1 (a). If the minimum value of $\gamma$ is sufficient small, the dynamic quantizer minimizes the effect of the quantization error on the controlled output $z_p$ in a neighborhood of the origin.

Next, Let $T \in \mathbb{N}_+ \cup \{\infty\}$ be the period over which we consider the quantizer performance. For the system in Fig. 1 (b) with the exogenous signal sequence $R := \{r(0), r(1), ..., r(T-1)\} \in l_\infty$, $z_p(k, x_0, R)$ denotes the output of $z_p$ at the $k$-th time for the initial state $x_0$. Also, for the system in Fig. 1 (a) without the quantizer, $z_p^*(k, x_0, R)$ denotes its output at the $k$-th time for the initial state $x_0$. $Z_p(x_0, R)$ and $Z_p^*(x_0, R)$ denote the vector sequence of $z_p(k, x_0, R)$ and $z_p^*(k, x_0, R)$ for $k = 1, ..., T$, respectively. In this case, this paper considers the following cost function:

$$E_T(Q_d) := \sup_{(x_0,R)\in\mathbb{R}^n\times l_\infty} \|Z_p^*(x_0, R) - Z_p(x_0, R)\|$$

which is discussed in [5]–[8].

Our second objective is to solve the following dynamic quantizer synthesis problem **(E)**: *For the system (13) with the exogenous signal sequence $R := \{r(0), r(1), ..., r(T-1)\} \in l_\infty$, suppose that the quantization period $T \in \mathbb{N}_+ \cup \{\infty\}$ and the quantization interval $d \in \mathbb{R}_+$ are given. Characterize a stable dynamic quantizer $Q_d$ (i.e., find parameters $(n_q, A_q, B_q, C_q)$) minimizing $E_T(Q_d)$.*

If the quantizer minimizes $E_T(Q_d)$, the system in Fig. 1 (b) "optimally" approximates the usual system in Fig. 1 (a) in the sense of the input-output relation. The above problems analyze the dynamic quantizers within the invariant set analysis framework.

## IV. Main result

First, this paper analyzes the cost function $L(Q_d)$ characterized by Proposition 1 when the dynamic quantizer is given. Next, this paper presents solutions to the synthesis problem **(L)**. As a result, the solutions are characterized by the transmission zero property of the generalized plant $G(\mathbf{z})$. Finally, this paper clarifies a solution to the synthesis problem **(E)** based on the solutions to the problem **(L)**.

### *A. Quantizer Analysis*

Suppose that the given quantizer to be analyzed is stable. The quantization error $Q_e$ satisfies

$$|e(k)| \le \frac{d}{2} \quad \forall k \in \mathbb{N}_+. \quad (14)$$

In this case, $(2, 1)$ block of the transfer function $H(\mathbf{z})$ in (13) can be regard as the linear system with the $l_\infty$ bounded disturbance (14). Considering the reachable set to estimate the influences of the quantization error, this paper utilizes the ellipsoidal invariant set which covers the reachable set from outside. Define

$$\mathcal{A} := \mathcal{A}, \quad \mathcal{B} := \mathcal{B}_1 \frac{d}{2}, \quad \mathcal{C} := \mathcal{C}_2. \tag{15}$$

In this case, the ellipsoidal invariant set $\mathbb{E}(\mathcal{P})$ for the system (13) can be parameterized by Proposition 1. If there exists the set $\mathbb{E}(\mathcal{P})$, there exists a scalar $\gamma \in \mathbb{R}_+$ satisfying

$$\sup_{\xi \in \mathbb{E}(\mathcal{P})} |\mathcal{C}\xi| = \gamma \quad \Leftrightarrow \quad \begin{bmatrix} \mathcal{P} & \mathcal{C}^{\mathrm{T}} \\ \mathcal{C} & \gamma^2 \end{bmatrix} \geq 0 \tag{16}$$

as schematized in Fig. 6 [12]. Then, from the property (5) of the invariant set, the performance level $\gamma$ in (16) satisfies

$$L(Q_d) \leq \gamma. \tag{17}$$

In the statements of Proposition 1, the matrix $\mathcal{P}$ is assumed

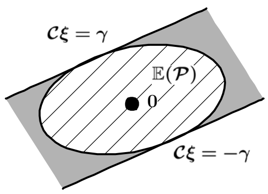


Fig. 6. Invariant set on the state space

to be given. For quantizer analysis, it is appropriate to treat $\mathcal{P}$ as a variable and search for $\mathcal{P}$ minimizing the performance level $\gamma$. For Proposition 1 defined by (15), we have the optimization problem **(Aop)**:

$$\min_{\mathcal{P}>0, 1-\rho(\mathcal{A})^2>\alpha>0, \gamma>0} \gamma \quad \text{s.t.} \quad (3) \quad \text{and} \quad (16).$$

If the scalar $\alpha$ is fixed, the conditions in (3) and (16) are linear matrix inequalities (LMIs) in terms of $\mathcal{P}$ and $\gamma$. Then, the problem can be solved numerically using standard LMI software in combination with the line search for $\alpha$.

Focusing on the left side of (16), we see that $\gamma$ is corresponding to the value of the criterion $f(\mathcal{P}(\alpha))$ for the approximation of $\mathbb{E}(\mathcal{P})$ to the reachable set. Then the infimum of $\gamma$ can be expressed by the following lemma.

*Lemma 1: For the feedback system (13), suppose that the quantization interval $d \in \mathbb{R}_+$ is given. For the optimization problem* **(Aop)** *defined in (15), the infimum of $\gamma$ is given by*

$$\inf \gamma = \inf_{\alpha} \frac{d}{2\sqrt{\alpha}} \sqrt{\sum_{k=0}^{\infty} \frac{1}{(1-\alpha)^k} |\mathcal{C}_2 \mathcal{A}^k \mathcal{B}_1|^2}, \quad 0 < \alpha < 1 - \rho(\mathcal{A})^2. \tag{18}$$

*Proof:* Define $\gamma(\alpha)$ which is obtained from the problem **(Aop)** for the fixed $\alpha \in (0, 1-\rho(\mathcal{A})^2)$. Considering (4) and (15), we obtain

$$(16) \Leftrightarrow \gamma(\alpha)^2 - \mathcal{C}\mathcal{P}(\alpha)^{-1}\mathcal{C}^{\mathrm{T}} \geq 0$$
$$\Leftrightarrow \gamma(\alpha)^2 \geq \frac{d^2}{4\alpha} \sum_{k=0}^{\infty} \frac{1}{(1-\alpha)^k} \mathcal{C}_2 \mathcal{A}^k \mathcal{B}_1 \mathcal{B}_1^{\mathrm{T}} (\mathcal{A}^{\mathrm{T}})^k \mathcal{C}_2^{\mathrm{T}}.$$

Hence, the infimum of $\gamma$ is given by (18). ■

From Lemma 1, we see that Proposition 1 allows us to characterize the infimum of the upper bound of the cost function $L(Q_d)$.

### B. QUANTIZER SYNTHESIS

From the optimization problem **(Aop)** defined in (15), the quantizer synthesis problem **(L)** reduces to the search for the quantizer parameters satisfying conditions (3) and (16). In this case, the matrix $\mathcal{A}$ ($\mathcal{A}$) is at least stable in the discrete-domain because of $(1,1)$ block of condition (3) for any $\alpha \in (0, 1-\rho(\mathcal{A})^2)$. Therefore, the matrix $A_q + B_q C_q$ characterized by Proposition 1 is also stable in the discrete domain because of the structure $\mathcal{A}$ in (10). The quantizer parameter search based on Proposition 1 turns out to be a matrix inequality problem as shown below.

*Theorem 1: For the feedback system (13), suppose that the quantization interval $d \in \mathbb{R}_+$ and the performance level $\gamma \in \mathbb{R}_+$ are given. For a scalar $\alpha \in (0,1)$, there exists a stable dynamic quantizer $Q_d$ achieving $L(Q_d) \leq \gamma$ if one of the following equivalent statements holds.*
*$(i)$ There exist a matrix $0 < \mathcal{P} \in \mathbb{R}^{n \times n}$ and a dynamic quantizer $Q_d$ satisfying (3) and (16).*
*$(ii)$ There exist matrices $0 < X \in \mathbb{R}^{n_g \times n_g}$, $0 < Y \in \mathbb{R}^{n_g \times n_g}$, $F \in \mathbb{R}^{1 \times n_g}$, $W \in \mathbb{R}^{n_g \times n_g}$, and $U \in \mathbb{R}^{n_g \times 1}$ satisfying*

$$\begin{bmatrix} (1-\alpha)\Xi_{\mathsf{P}} & 0 & \Xi_{\mathsf{A}}^{\mathrm{T}} \\ 0 & \frac{4\alpha}{d^2} & \Xi_{\mathsf{B}}^{\mathrm{T}} \\ \Xi_{\mathsf{A}} & \Xi_{\mathsf{B}} & \Xi_{\mathsf{P}} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Xi_{\mathsf{P}} & \Xi_{\mathsf{C}}^{\mathrm{T}} \\ \Xi_{\mathsf{C}} & \gamma^2 \end{bmatrix} \geq 0 \tag{19}$$

*where*

$$\Xi_{\mathsf{P}} := \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \qquad \Xi_{\mathsf{A}} := \begin{bmatrix} XA & W \\ A & AY + B_1 F \end{bmatrix},$$
$$\Xi_{\mathsf{B}} := \begin{bmatrix} U^{\mathrm{T}} & B_1^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \quad \Xi_{\mathsf{C}} := \begin{bmatrix} C_2 & C_2 Y \end{bmatrix}.$$

*In this case, one such quantizer is given by*

$$B_q = Z^{-1}(U - XB_1), \quad C_q = -FY^{-1}, \quad n_q = n_g,$$
$$A_q = Z^{-1}(XAY - UF - W)Y^{-1} \tag{20}$$

*where $Z = X - Y^{-1}$.*

If conditions (19) hold, therefore, the dynamic quantizer $Q_d$ is given by (20) such that $A_q + B_q C_q$ is stable and $L(Q_d) \leq \gamma$ is achieved. For the dynamic quantizer synthesis problem based on Theorem 1, we have the optimization problem **(Sop)**:

$$\min_{X>0, Y>0, F, W, U, 1>\alpha>0, \gamma>0} \gamma \quad \text{s.t.} \quad (19).$$

In synthesis, the parameters $(A_q, B_q, C_q)$ to be designed lead to $\alpha \in (0,1)$. When scalar $\alpha$ is fixed, the conditions in Theorem 1 are LMIs in terms of the other variables. Then, **(Sop)** can be solved numerically using standard LMI software in combination with the line search for $\alpha$ similar to **(Aop)**. Also, according to Lemma 1, the problem **(L)** is feasible if the problem **(Sop)** is feasible.

Under some circumstances, Proposition 1 gives a dynamic quantizer which is expressed by the generalized plant parameters. The following theorem denotes this fact.

*Theorem 2: Consider the following non-convex optimization problem* **(OP)** *defined in (15):*

$$\min_{\mathcal{P}>0, A_q, B_q, C_q, 0>\alpha>1-\rho(\mathcal{A})^2, \gamma>0} \gamma \quad \text{s.t.} \quad (3) \quad \text{and} \quad (16).$$

*Suppose that $G(\mathbf{z})$ is minimum phase in the case of $u_c = v$. An optimal solution of $(A_q, B_q, C_q)$ and its infimum of $\gamma \in \mathbb{R}_+$ to the problem* **(OP)** *are given by*

$$A_q = A, \quad B_q = B_1, \quad C_q = -(C_{p1}B_p)^{-1}C_2A \qquad (21)$$

*and*

$$\inf \gamma = (d|C_{p1}B_p|)/(2\sqrt{1-\rho(\mathcal{A})^2}). \qquad (22)$$

*Proof:* Since the cost function $L(Q_d)$ is characterized by (18), the parameters (21) leads to (22). ■

In the case of $m = p = 1$, the stable $A_q + B_qC_q$ in (21) implies that all the transmission zeros of the system $G(\mathbf{z})$ are stable [7]. That is, solutions of the problem **(L)** depend on the transmission zero property of the generalized plant. When $G(\mathbf{z})$ is minimum phase, Theorem 2 gives a solution to the problem **(L)**. When $G(\mathbf{z})$ is non-minimum phase, Theorem 1 gives a solution to the problem **(L)**. Figure 7 illustrates the relation between the problems **(OP)** and **(Sop)**.

$$\begin{array}{ll} & \text{Problem } (\mathbf{L}) \\ (\text{Lemma 1}) & \Uparrow^{*} \\ & \text{Problem } (\mathbf{OP}):\ (n_q, A_q, B_q, C_q) \\ (\text{Theorem 1}) & \Updownarrow^{\star} \\ & \text{Problem } (\mathbf{Sop}):\ (X, Y, F, W, U) \end{array}$$

$*$: **(L)** is feasible if **(OP)** is feasible.
$\star$: Equivalent for the fixed $\alpha \in (0, 1)$.

Fig. 7. Solutions to Problem **(L)**

### *C. Relation to the optimal dynamic quantizer*

The existing results [5]–[8] have proposed the optimal dynamic quantizers based on the cost function $E_T(Q_d)$. This subsection characterizes the cost function $E_T(Q_d)$ within the invariant set analysis framework. The usual (continuous-valued) feedback system in Fig. 1 (a) is given by

$$\begin{bmatrix} x^*(k+1) \\ z_p^*(k) \end{bmatrix} = \begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x^*(k) \\ r(k) \end{bmatrix}$$

where $x^* \in \mathbb{R}^{n_g}$, $z_p^* \in \mathbb{R}$ denote its state vector and controlled output respectively, and $x^*(0) = x(0)$. Define the signals as follows:

$$\xi_{cl} := x - x^*, \quad \xi := [\ \xi_{cl}^{\mathrm{T}} \ \ x_q^{\mathrm{T}}\ ]^{\mathrm{T}}, \quad z := z_p - z_p^*.$$

In this case, the difference between $z_p^*(k, x_0, R)$ and $z_p(k, x_0, R)$ is generated by the following error system $\mathcal{H}(\mathbf{z})$:

$$\begin{bmatrix} \xi(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 \\ \mathcal{C}_2 & 0 \end{bmatrix} \begin{bmatrix} \xi(k) \\ e(k) \end{bmatrix}, \quad \xi(0) = 0$$

where the signal $e \in \mathbb{R}$ is given by (12) and the matrices $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{C}_2$ are defined by (10). That is, the system $\mathcal{H}(\mathbf{z})$ can be regard as as the linear system with the $l_\infty$ bounded disturbance (14) similar to the system $H(\mathbf{z})$ in subsection IV-A. Therefore, by using the same procedure, Proposition 1 characterizes the cost function $E_T(Q_d)$ as follows:

$$E_\infty(Q_d) \le \gamma, \ \gamma := \min_{\mathcal{P}, \alpha, A_q, B_q, C_q} \sup_{\xi \in \mathbb{E}(\mathcal{P})} |\mathcal{C}_2\xi| \text{ s.t. } (3).$$

That is, the quantization period $T$ is $\infty$ within the invariant set analysis framework. Considering Theorems 1 and 2, a solution to the synthesis problem **(E)** is given by the following theorem.

*Theorem 3: For the feedback system (13), suppose that the quantization interval $d \in \mathbb{R}_+$ is given and $m = p = 1$. If the system $G(\mathbf{z})$ in (7) is minimum phase, a stable dynamic quantizer achieving $E_\infty(Q_d) \le \gamma$ and its performance are given by (21) and (22). If the system $G(\mathbf{z})$ is non-minimum phase, a stable dynamic quantizer achieving $E_\infty(Q_d) \le \gamma$ and its performance are characterized by (19) and (20).*

When the system $G(\mathbf{z})$ is minimum phase and $T = \infty$, references [5]–[8] have present an optimal dynamic quantizer $Q_d^{op}$ given by (21) and its performance given by

$$E_\infty(Q_d^{op}) = |C_{p1}B_p|\frac{d}{2}.$$

It is striking that the structure of their optimal dynamic quantizer is equivalent to our proposed one based on Proposition 1. That is, Theorem 3 points out that the proposed quantizer is also optimal in the sense that the quantizer gives an optimal output approximation property. Also, it is striking that both of the synthesis problems **(L)** and **(E)** are reduced to the optimization problems **(Sop)** and **(OP)** within invariant set analysis framework, although their goals are different.

Reference [8] has provided the numerical design method of the optimal dynamic quantizer. When the quantization period $T \in \mathbb{N}_+$ is given, the method has recast the stable and optimal quantizer synthesis problem as the following optimization problem:

$$\min_{A_q, B_q, C_q, \mathsf{P}} \left\| \sum_{k=0}^{T-1} |\mathcal{C}_2\mathcal{A}^k\mathcal{B}_1| \right\| \frac{d}{2} \left(= E_T(Q_d)\right)$$
$$\text{s.t. } (A_q + B_qC_q)^{\mathrm{T}}\mathsf{P}(A_q + B_qC_q) < \mathsf{P}, \quad \mathsf{P} > 0. \qquad (23)$$

Denote by $Q_{op}^\circ$ $(n_q^\circ, A_q^\circ, B_q^\circ, C_q^\circ)$ a solution to the relaxed problem in the sense that the condition (23) is removed from the original problem. Reference [8] has clarified that $Q_{op}^\circ$ is an optimal solution to the original problem if the matrix $A_q^\circ + B_q^\circ C_q^\circ$ is stable. Otherwise, the reference has presented a solution $\bar{Q}_{op}^\circ$ $(n_q^\circ T, \bar{A}_q^\circ, \bar{B}_q^\circ, \bar{C}_q^\circ)$ which is composed of $Q_{op}^\circ$ to be optimal for the original problem. Their numerical design method has provided a stable optimal quantizer $Q_{op}^\circ$ where its order is given by $n_q^\circ = \lfloor T/2 \rfloor + 1$. When the quantization period $T$ is set be large and/or $\bar{Q}_{op}^\circ$ becomes an optimal solution, we need the reduction technique of $Q_{op}^\circ$ and $\bar{Q}_{op}^\circ$ such as Ho-Kalman's method [15].

Compared with [5]–[8], our design method has the following advantages:*When the generalized plant $G(\mathbf{z})$ has unstable zeros, $(i)$ the stable quantizer order is $n_q = n_g$, $(ii)$ the infinite time control performance is always guaranteed, $(iii)$ the method provides a suboptimal dynamic quantizer in the sense that the upper bound of $E_\infty(Q_d)$ is minimized, $(iv)$ and the method naturally extends to multiobjective control problems as shown in [10]–[12].*

As shown in Table I, in the case of the problem **(E)** with $T = \infty$, we conclude that the invariant analysis framework provides an optimal dynamic quantizer for the SISO minimum phase system and a suboptimal dynamic quantizer for the SISO non-minimum phase system, respectively.

TABLE I

SOLUTIONS TO PROBLEM (E)

| | | quantization period $T$ | |
|---|---|---|---|
| | | $T \in \mathbb{N}_+$ | $T = \infty$ |
| zeros of $G^*(\mathbf{z})$ | stable | **(Fop)**<br>$n_q^\circ$ or $\bar{n}_q^\circ$ | **(Iop)**, **(OP)**<br>$n_{cl}$ |
| | unstable | **(Fop)**<br>$n_q^\circ$ or $\bar{n}_q^\circ$ | **(Sop)**<br>$n_{cl}$ |

## V. NUMERICAL EXAMPLE

The plant $P(\mathbf{z})$ is the discretized system of the unstable non-minimum phase continuous-time LTI system

$$\begin{bmatrix} \dot{x}_p(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & -1 & 1 \\ \hline -2 & 1 & 0 \end{array}\right] \begin{bmatrix} x_p(t) \\ v(t) \end{bmatrix}, \quad z_p(t) = y(t)$$

with the sampling time $h = 0.1$ [s] and zero-order-hold. Its eigenvalues are $\{1.064, 0.857\}$ and its unstable zero is $\{1.224\}$. The stabilizing controller $C(\mathbf{z})$ is given by

$$\begin{bmatrix} x_c(k+1) \\ u_c(k) \end{bmatrix} = \left[\begin{array}{c|c} 0.741 & 0.086 \\ \hline -1 & 1 \end{array}\right] \begin{bmatrix} x_c(k) \\ y(k) \end{bmatrix}.$$

We consider the no exogenous signal case. If the quantization period is set to be $T = 100$, the numerical method in [8] provides a stable quantizer $\bar{Q}_{op}^\circ$ with $n_q = 5100$. On the other hand, for the quantization interval $d = 10$, we obtain the quantizer with $n_q = 3$ achieving $\gamma = 2.16$ via **(Sop)**. The quantizer is given by

$$\begin{bmatrix} A_q & B_q \\ C_q & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 0.995 & 0.100 & -0.005 & -0.005 \\ -0.095 & -1.005 & -0.095 & -0.095 \\ -0.173 & 0.086 & 0.741 & 0 \\ \hline -20.338 & -11.590 & 1.000 & 0 \end{array}\right]$$

where $A_q = A$, $B_q = B_1$, $C_q \neq -(C_{p1}B_p)^{-1}C_2A$, and the eigenvalues of $A_q + B_qC_q$ are $\{0, 0.741, 0.797\}$. We see that our method can provide a stable quantizer with $n_q = n_g$. The matrix $C_q$ is modified from the optimal one such that the quantizer is stable.

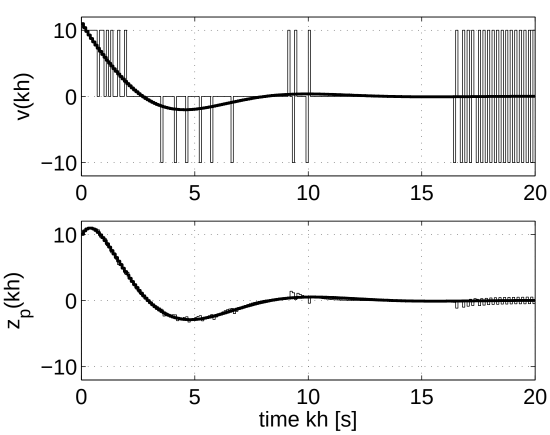


Fig. 8. Time responses of $v(kh)$ and $z_p(kh)$.

Figure 8 illustrates the time responses of $v(kh)$ and $z_p(kh)$ for the initial state $x(0) = [\ -5\ \ 0\ \ -1\ ]^{\mathrm{T}}$. In Fig. 8, the thin lines and the thick lines illustrate the time responses of the usual feedback system in Fig. 1 (a) and the feedback system with discrete-valued signal in Fig. 1 (b), respectively. We see that the controlled output of Fig. 1 (b) approximates that of Fig. 1 (a) even if the discrete-valued signals are applied to the input $v$. Then this numerical example shows the effectiveness of our method and examines the validity of Theorem 3 in dynamic quantizer synthesis.

## VI. CONCLUSION

Focusing on the control problems for SISO systems with discrete-valued input, we have proposed the dynamic quantizer analysis and synthesis conditions. Our approach is based on the invariant set analysis and the LMI technique. First, this paper has proposed the quantizer analysis conditions. Second, this paper has discussed the quantizer synthesis conditions which are characterized by the transmission zero property. Third, this paper has reconsidered the existing optimal dynamic quantizer. Finally, this paper has pointed out that the proposed method is helpful through numerical examples.

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