Robust Control, Multidimensional Systems and Multivariable Function Theory: Commutative and Noncommutative Settings

Joseph A. Ball and Sanne ter Horst

Abstract— **In the classical 1-D case there is a seamless connection between state-space (time-domain) representations and transfer-function (frequency-domain) representations for linear systems. In particular, the first results on H^∞-control were developed in the frequency-domain leading to an active exchange of ideas between mathematicians with backgrounds in function theory and engineers. Eventually elegant computational algorithms for solving the standard H^∞-control problem were found in terms of state-space coordinates, first in terms of a pair of coupled Riccati equations, and then completely in terms of Linear Matrix Inequalities. Here we discuss two kinds of extensions of these ideas to the context of multidimensional systems and multivariable function theory, namely: (1) the commutative case, where the transfer-function is a function of several complex variables, and (2) the noncommutative case, where the transfer-function is a function of noncommuting operator (or matrix) variables. Perhaps surprisingly, we shall see that the noncommutative setting provides a much more complete parallel with the classical case than the commutative setting. Many of the ideas of the present report are taken from our survey article [17].**

I. INTRODUCTION

The early years of the MTNS meeting featured a lot of fruitful interaction between mathematicians and engineers inspired by the evolving theory and needs of what has become known as H^∞-control. Mathematicians attracted to this area were typically schooled in the intricacies of holomorphic operator-valued functions of a complex variable and associated connections with operator theory while engineers were focused on addressing practical control or signal-processing issues. Arguably much of the basis for this interaction was the essential equivalence between frequency-domain and state-space domain formulations.

In this Section we recall these basic ideas for the classical case in order to set the table on how they generalize to the setting of multidimensional systems and multivariable functions.

A. The standard problem of H^∞-control: frequency-domain formulation

Following the first book on H^∞-control [33], we see that the configuration for the so-called standard problem on H^∞-control is as in Figure 1. Here w, u, z, y consist of the *disturbance* (or *reference*) signal, the *control* signal, the *error* signal, and the *measurement* signal respectively, each a column vector of respective sizes $n_\mathcal{W}$, $n_\mathcal{U}$, $n_\mathcal{Z}$ and $n_\mathcal{Y}$ with rational-function entries, while v_1 and v_2 are tap

J.A. Ball is with the Department of Mathematics, Virginia Tech, Blacksburg, VA 240061 USA, `joball@math.vt.edu`

Sanne ter Horst is with the Department of Mathematics, Utrecht University, PO Box 80 010, 3508 TA Utrecht, The Netherlands, *S.terHorst@uu.nl*

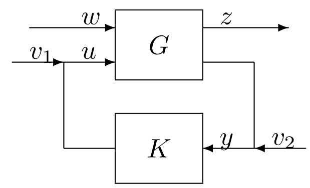

Fig. 1. Feedback with tap signals

signals of respective sizes $n_\mathcal{U}$ and $n_\mathcal{Y}$ used only to formulate the frequency-domain notion of internal stability. It is also understood that the *plant* $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{smallmatrix}\right]$ is a rational matrix function of size $(n_\mathcal{Z}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U})$ while the *controller* K is a rational matrix function of size $n_\mathcal{Y}\times n_\mathcal{U}$ and that the signal-flow diagram in Figure 1 is short hand for the system of equations

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix}\begin{bmatrix} w \\ u\end{bmatrix} = \begin{bmatrix} z \\ y - v_2\end{bmatrix}$$
$$Ky = u - v_1.$$

We say that the system $\Sigma(G,K)$ in Figure 1 is *well-posed* if one can solve for (z,u,y) uniquely in terms of (w,v_1,v_2). It is routine to verify that this happens exactly when $\det(I - G_{22}K) \neq 0$ and then the map from (w,v_1,v_2) to (z,u,y) is given by

$$\Theta(G,K) := \left[\begin{smallmatrix} I & -G_{12} & 0 \\ 0 & I & -K \\ 0 & -G_{22} & I\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ G_{21} & 0 & I\end{smallmatrix}\right]. \qquad (1)$$

We restrict ourselves to the discrete-time setting and declare that a signal is *stable* if it has finite energy, interpreted as all entries being analytic on the closed unit disk. A matrix function F is then said to be stable if it maps stable signals to stable signals, i.e., all entries of F should be bounded analytic functions on the closed unit disk $\mathbb{D}$. The configuration $\Sigma(G,K)$ in Figure 1 is said to be *internally stable* (in the frequency-domain sense) if $\Sigma(G,K)$ is well-posed and the map $\Theta(G,K)$ in (1) is stable (i.e., all nine block entries are stable). The configuration $\Sigma(G,K)$ is said to achieve *performance* if it is internally stable and in addition the (1,1)-entry

$$T_{zw} := \Theta(G,K)_{11} = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21},$$

referred to as the transfer-function from disturbance w to error z, has H^∞ norm at most one:

$$\|T_{zw}\|_{H^\infty} := \sup\{\|T_{zw}(\lambda)\| \colon \lambda\in\mathbb{D}\} \le 1. \qquad (2)$$

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The measurement feedback stabilization problem is: *given a plant G, design a controller K so that* $\Sigma(G,K)$ *is internally stable.* The H^∞-control problem is: *given a plant G, design a controller K so that the configuration* $\Sigma(G,K)$ *is both (a) internally stable, and (b) has performance.*

We mention an important special case of the stabilization and H^∞-control problems, namely, the case where G_{22} is zero. In this case well-posedness is automatic and the transfer-function $\Theta(G,K)$ in (1) collapses to

$$\Theta\left(\left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & 0 \end{smallmatrix}\right], K\right) = \begin{bmatrix} G_{11}+G_{12}KG_{21} & G_{22} & G_{12}K \\ KG_{21} & I & K \\ G_{21} & 0 & I \end{bmatrix}.$$

From this formula we read off that $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & 0 \end{smallmatrix}\right]$ is internally stabilizable via some controller K if and only if G itself is stable, and then a given controller internally stabilizes G if and only if K itself is stable. This particular case of the stabilization/H^∞-problem is often called the *Model-Matching Problem* [33].

One of the early results of the theory was that one can use a double coprime factorization of G_{22}, namely, the existence of stable transfer-functions $D, N, \widetilde{D}, \widetilde{N}, X, Y, \widetilde{X}, \widetilde{Y}$ of compatible sizes such that

$$G_{22} = D^{-1}N = \widetilde{N}\widetilde{D}^{-1}, \quad \left[\begin{smallmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{smallmatrix}\right]\left[\begin{smallmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I_{n_\mathcal{Y}} & 0 \\ 0 & I_{n_\mathcal{U}} \end{smallmatrix}\right], \tag{3}$$

to reduce a general stabilization/H^∞-problem to a problem in Model-Matching form. Specifically, given the existence of a double coprime factorization for G_{22}, then, if any stabilizing controllers for G exist, then a given K stabilizes G if and only if K stabilizes G_{22}, i.e., the lower right two-by-two block of $\Theta(G,K)$ is stable, and then any such stabilizing K is given by either of the formulas

$$K = (Y + \widetilde{D}\Lambda)(X + \widetilde{N}\Lambda)^{-1} = (\widetilde{X} + \Lambda N)^{-1}(\widetilde{Y} + \Lambda D)$$

where Λ is a free stable parameter of size $n_\mathcal{Y} \times n_\mathcal{U}$ subject to the constraint that $\det(X + \widetilde{N}\Lambda) = \det(\widetilde{X} + \Lambda N) \neq 0$.

Replacing the design parameter K by the free stable parameter Λ then converts the H^∞-problem to the Model-Matching form: *Given the stable functions* $\widetilde{G}_{11} := G_{11} + G_{12}YDG_{21}$, $\widetilde{G}_{21} := G_{12}\widetilde{D}$, $\widetilde{G}_{21} := DG_{21}$, *find a stable* Λ *so that*

$$F := \widetilde{G}_{11} + \widetilde{G}_{12}\Lambda\widetilde{G}_{21} \tag{4}$$

has $\|F\|_{H^\infty} := \sup\{\|F(\lambda)\| \colon \lambda \in \mathbb{D}\}$ *at most equal to* 1.

If we now do a second change of design parameter, namely, view the unknown as F rather than Λ, then the simplest case of the problem becomes a Nevanlinna-Pick interpolation problem as follows. Consider the SISO/scalar case where $n_\mathcal{W} = n_\mathcal{U} = n_\mathcal{Z} = n_\mathcal{Y} = 1$. We may absorb $\widetilde{G}_{21}$ into $\widetilde{G}_{12}$ to assume without loss of generality that $\widetilde{G}_{21} = 1$. We suppose that $\widetilde{G}_{12}$ has no zeros on the unit circle and let $\{\lambda_1, \dots, \lambda_N\}$ denote the zeros in the unit disk, all of which we assume of multiplicity one for simplicity. Let $\{w_1, \dots, w_N\}$ be the values $w_i = \widetilde{G}_{11}(\lambda_i)$ for $i = 1, \dots, N$ of G_{11} at these points $\lambda_1, \dots, \lambda_N$. Then it is not hard to see that: the function F has the form (4) if and only if F is analytic on the closed unit disk and satisfies the interpolation conditions

$$F(\lambda_i) = w_i \text{ for } i = 1, \dots, N. \tag{5}$$

Then the H^∞-problem assumes the form of a classical Nevanlinna-Pick interpolation problem: *Given the interpolation nodes* $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ *and interpolation values* $\mathbf{w} = (w_1, \dots, w_N)$, *find a holomorphic function mapping the unit disk* $\mathbb{D}$ *into the closed unit disk* $\overline{\mathbb{D}}$ *so that* $F(\lambda_i) = w_i$ *for* $i = 1, \dots, N$. The classical result is that solutions exist if and only if the associated Pick matrix is positive semidefinite:

$$\mathbb{P}(\text{''}\lambda, \mathbf{w}) = \left[\frac{1 - w_i\overline{w_j}}{1 - \lambda_i\overline{\lambda_j}}\right] \succeq 0. \tag{6}$$

This was the first evidence that the H^∞-control problem can actually be solved.

Extensions of this frequency-domain approach to the MIMO case (where the assumption that $n_\mathcal{W} = n_\mathcal{U} = n_\mathcal{Z} = n_\mathcal{Y} = 1$ is now dropped) ensued by developing a theory of Nevanlinna-Pick interpolation for matrix-valued functions as well as connections with J-inner-outer/spectral factorization and connections with Commutant Lifting theory (see [11], [15], [16], [20], [31], [37], [38], [44], [45]), with some of these works leading to state-space formulas for solutions of the H^∞-problem. These developments were largely overtaken by other developments on the state-space-coordinate front, which we now describe.

B. The standard H^∞-problem: state-space formulation and solution

In the state-space formulation of the stabilization/H^∞-control problem, we suppose that the plant $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$ is given in state-space form

$$G(\lambda) = \left[\begin{smallmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{smallmatrix}\right] + \lambda \left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right](I - \lambda A)^{-1}\left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right]$$

associated with the linear system (for the discrete-time case which we are considering here)

$$\Sigma_G \colon \left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) & = & C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) & = & C_2 x(t) + D_{21} w(t) + D_{22} u(t). \end{array} \right. \tag{7}$$

We then seek to impose a feedback connection with a controller K also given in state-space form

$$\Sigma_K \colon \left\{ \begin{array}{rcl} x_K(t+1) & = & A_K x_K(t) + B_K y(t) \\ u(t) & = & C_K x_K(t) + D_K y(t) \end{array} \right. \tag{8}$$

so that the closed-loop system $\Sigma(G,K)$ (a) is *well-posed*, (b) is *internally stable*, and possibly also (c) has *performance*. We consider each of these in turn.

1) Well-posedness: The state-space version of the *well-posedness* condition is that one can solve the combined system (7) and (8) uniquely for $x(t+1), x_K(t+1), z(t)$ in terms of $x(t), x_K(t), u(t)$; this condition in turn holds exactly when $I - D_K D_{22}$ is invertible. A simplifying assumption which guarantees that this happen and which simplifies all the subsequent formulas is that

$$D_{22} = 0. \tag{9}$$

Unlike the parallel situation in the frequency-domain setting where the assumption that $G_{22} = 0$ guarantees well-posedness and gives rise to a Model-Matching problem, the assumption (9) is considered innocuous due to a procedure known as *loop-shifting* (see e.g. [28, Exercise 8.11, page 277]). With (9) in force, the state-space system $\Sigma(G, K)$ ((7) together with (8)) is automatically well-posed and the closed-loop state-space system (with internal signals u, y ignored) has the form

$$\begin{aligned} \begin{bmatrix} x(t+1) \\ x_K(t+1) \end{bmatrix} &= A_{cl} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + B_{cl} w(t) \\ z(t) &= C_{cl} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + D_{cl} w(t) \end{aligned} \tag{10}$$

where the closed-loop system matrix $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$ is given explicitly as

$$\left[\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ \hline C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{array}\right] \tag{11}$$

One can reorganize this closed-loop system matrix as

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & D_{11} \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \underline{D}_{12} \end{bmatrix} J \begin{bmatrix} \underline{C} & \underline{D}_{21} \end{bmatrix} \tag{12}$$

which is affine in the system matrix $J_K := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ for the controller K with coefficient matrices completely determined from the given plant G in (7):

$$\overline{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$
$$\underline{B} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \; \underline{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, \; \underline{D}_{12} = \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \; \underline{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}.$$

2) Internal stability: The state-space formulation of internal stability is simply that the closed-loop system (10) has the property that the associated autonomous system $x_{cl}(t+1) = A_{cl} x_{cl}(t)$ be *asymptotically stable* ($x_{cl}(t) \to 0$ as $t \to \infty$ for any choice of $x_{cl}(0)$) or, equivalently in the finite-dimensional case, *exponentially stable* ($\|x_{cl}(t)\| \le K\rho^t \|x_{cl}(0)\|$ for some $K < \infty$ and $\rho < 1$).

In general it is known that a matrix A (we now drop the sub-cl notation) is stable in this sense if and only if any one of the following conditions holds:

(S1) A is exponentially stable, i.e., $\|A^t x\| \le K\rho^t \|x\|$ for $t = 0, 1, 2, \dots$ for appropriate constants $K < \infty$ and $\rho < 1$.

(S2) A has spectral radius strictly less than 1, i.e., $I - \lambda A$ is invertible for all λ in the closed unit disk $\overline{\mathbb{D}}$.

(S3) A is similar to a strict contraction: there exists an invertible matrix Θ so that $\|\Theta A \Theta^{-1}\| < 1$, or equivalently, there is a positive definite matrix X so that $A^* X A - X < 0$.

3) Performance: We say that the state-space closed-loop system $\Sigma(G, K)$ in (10) has *performance* if the energy of the error signal $z(t)$ is bounded by the energy of the disturbance signal $w(t)$ whenever the initial state is set equal to zero:

$$\sum_{t=0}^{\infty} \|z(t)\|^2 \le \sum_{t=0}^{\infty} \|w(t)\|^2 \text{ whenever } x_{cl}(0) = 0, \tag{13}$$

or, in equivalent frequency-domain terms, the closed-loop transfer-function $T_{zw}(\lambda) = D_{cl} + \lambda C_{cl}(I - \lambda A_{cl})^{-1} B_{cl}$ has H^∞-norm at most 1. Later evolutions of the theory allow the initial state to be nonzero and include an additional bias term $M\|x_{cl}(0)\|^2$ on the right-hand side of (13).

4) Solution of the feedback stabilization problem: Given an input-state pair (A, B) where A in an operator on a finite-dimensional state-space $\mathcal{X}$ and B is an operator from a finite-dimensional input space $\mathcal{U}$ to $\mathcal{X}$, we recall the standard definition that (A, B) is *stabilizable* if there is feedback operator $F\colon \mathcal{X} \to \mathcal{U}$ so that $A + BF$ is stable in any of the senses (S1), (S2), (S3) mentioned above. The following list of equivalent conditions is well known:

(FS1) (A, B) is Hautus-stabilizable, i.e., the matrix pencil $\begin{bmatrix} I - \lambda A & B \end{bmatrix}$ has full rank for λ in the closed unit disk $\overline{\mathbb{D}}$.

(FS2) (A, B) is operator-stabilizable, i.e., there is an F with $A + BF$ stable.

(FS3) (Linear Matrix Inequality (LMI)-stabilizability): There is a positive definite matrix $Y > 0$ so that $AYA^* - Y + BB^* < 0$; equivalently, there is a positive definite matrix $Y > 0$ so that $(B^*)_\perp^* (AYA^* - Y)(B^*)_\perp < 0$, where $(B^*)_\perp$ is any matrix with columns forming a basis for the kernel of B^*.

There is a dual theory for output pairs (C, A), where C is an operator from the state-space $\mathcal{X}$ to the output space $\mathcal{Y}$ (say). We then have the following equivalent formulations of the notion of *detectability*:

(D1) (C, A) is *Hautus-detectable*, i.e., the matrix pencil $\begin{bmatrix} I - \lambda A \\ C \end{bmatrix}$ has full rank for $\lambda \in \overline{\mathbb{D}}$.

(D2) (C, A) is *operator detectable*, i.e., there exists an output injection $L\colon \mathcal{Y} \to \mathcal{X}$ so that $A + LC$ is stable.

(D3) (LMI-detectability): There is a positive definite matrix $X > 0$ so that $A^* X A - X + C^* C < 0$, or, equivalently, there is a positive definite matrix $X > 0$ so that $(C_\perp)^* (A^* X A - X) C_\perp < 0$ where C is any matrix whose columns form a basis for the kernel of C.

The state-space measurement-feedback stabilization problem is: *given a system G as in* (7) *(with $D_{22} = 0$ for simplicity), design a controller state-space system K as in* (8) *so that the closed-loop matrix A_{cl} (appearing as the (1,1)-block entry in either* (11) *or* (12)*) is stable.* The solution combines Luenberger observability theory with the various equivalences for stabilizability and detectability given above (see [28, Propoition 5.2]): *The state-space measurement-feedback stabilization problem has a solution $J_K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ if and only if (A, B_2) is stabilizable and (C_2, A) are detectable. In this case a solution is given by $J_K = \begin{bmatrix} A + B_2 F + LC_2 & -L \\ F & 0 \end{bmatrix}$ where F, L are any choices of matrices for which $A + B_2 F$ and $A + LC_2$ are stable.*

5) Solution of the H^∞-problem: For a number of years the coupled Riccati-equation solution [26], [36] was considered the definitive solution of the state-space H^∞-problem. Here we describe the LMI solution due to [34], [41]. Let us say that the controller K in (8) solves the *strict H^∞-problem* if the closed loop system $\Sigma(G, K)$ is internally stable and the performance criterion (13) holds in the strict form $\sum_{t=0}^{\infty}\|z(t)\|^2 \leq \rho\sum_{t=0}^{\infty}\|w(t)\|^2$ for some $\rho < 1$ (or $\|T\|_{H^\infty} < 1$). Then *solutions exist to the strict H^∞-control problem if and only if there exists solutions X, Y of the LMIs*

$$\begin{gathered}
\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} AYA^*-Y & AYC_1^* & B_1 \\ C_1YA^* & C_1YC_1^*-I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0, \\
\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^*XA-X & A^*XB_1 & C_1^* \\ B_1^*XA & B_1^*XB_1-I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0, \\
Y > 0, \quad X > 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0,
\end{gathered} \tag{14}$$

where $N_c = (\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix})_\perp$ is a matrix with columns forming a basis for $\operatorname{Ker}\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}$ and $N_o = (\begin{bmatrix} C_2 & D_{21} \end{bmatrix})_\perp$ is a matrix with columns forming a basis for $\operatorname{Ker}\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$.

C. Equivalence between frequency-domain and state-space problems

Suppose that we are given a plant G with state-space realization as in (7) such that K as in (8) achieves internal stability in the state-space sense (i.e., the closed-loop state-update operator A_{cl} is stable). Then it is straightforward to see that internal stability also holds in the frequency-domain sense since it is routine to verify that $\Theta(G, K)$ as in (1) has a state-space realization with internal state operator equal to the same A_{cl}. Conversely, given that the pair (G, K) achieves internal stability in the frequency-domain sense, i.e., the nine block entries of $\Theta(G, K)$ are stable transfer-functions, it turns out that (G, K) also achieves internal stability in the state-space sense, as long as we choose realizations (7) for G and (8) for K so that (A, B_2) and (A_K, B_K) are stabilizable and (C_2, A) and (C_K, A_K) are detectable. This result follows easily from the following general proposition: *Suppose that $G(\lambda)$ is a stable transfer-function with realization $G(\lambda) = \lambda C(I - \lambda A)^{-1}B$ such that (A, B) is stabilizable and (C, A) is detectable. Then it follows that A is necessarily stable.*

II. Commutative multidimensional linear systems

There have appeared extensions of the ideas of feedback stabilization and H^∞-control to multivariable functions in a frequency-domain polydisk formulation [46], [47], [48] as well as in state-space form (cf., [27] and the references there). We discuss each of these in turn next.

A. The frequency-domain formulation

For the multivariable discrete-time setting (e.g., in the n-D circuit theory literature), it is common to define a multivariable rational function $s(z)$ in d variables $z = (z_1, \dots, z_d)$ to be *stable* if it is uniformly bounded on the polydisk

$$\mathbb{D}^d = \{z = (z_1, \dots, z_d) \colon |z_k| < 1 \text{ for } k = 1, \dots, d\}.$$

It has been known for some time that it can happen that a rational function $s(z)$ with coprime representation $s(z) = n(z)/d(z)$ (i.e., n and d are polynomials in d variables with no common factors) can be stable in this sense and yet the denominator polynomial $d(z)$ may have zeros on the boundary of $\mathbb{D}^d$; in this case $n(z)$ also vanishes at such points, despite the fact that n and d have no common factors—a multivariable phenomenon. To get rid of this irritation, Lin [46], [47] introduced the ring $\mathbb{C}(z)_{ss}$ of *structured stable* rational functions $\mathbb{C}(z)_{ss}$ defined as the space of rational functions on the closed polydisk $\overline{\mathbb{D}}^d$ of the form $s(z) = \frac{p(z)}{q(z)}$ with $p(z)$ and $q(z)$ polynomials so that $q(z)$ has no zeros in $\overline{\mathbb{D}}^d$; note that the quotient field of this integral domain $Q(\mathbb{C}(z)_{ss}) = \left\{\frac{v(z)}{u(z)} \colon v, u \in \mathbb{C}(z)_{ss}\right\}$ is just the collection $\mathbb{C}(z)$ of rational functions back again (the quotient field of the ring of polynomials).

It makes sense to formulate the feedback-stabilization just as in Section I-A: *Given a matrix-valued function G of the form $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ with all entries in $\mathbb{C}(z)$, find a controller K, also a matrix of rational functions, so that the block 3×3 rational matrix function $\Theta(G, K)$ as in (1) is stable (i.e., has all entries in $\mathbb{C}(z)_{ss}$).*

We then take the H^∞-problem for this multivariable problem to be: *design a stabilizing controller K so that $\Theta(G, K)$ is stable and the closed-loop transfer-function $T_{zw} = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}$ has infinity norm over the polydisk $\mathbb{D}^d$ at most 1:*

$$\|G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}\|_{H^\infty} \leq 1.$$

One can now follow a similar procedure as discussed in Section I-A. Under the assumption that $G(z)$ is stabilizable, the function $G_{22}(z)$ admits a double coprime factorization. This implication was only proved recently by Quadrat [52], answering a conjecture posed by Lin [46], [47], [48] in the affirmative. Given a double coprime factorization of $G_{22}(z)$, one obtains a Youla-Kučera parametrization [54] for the set of all stabilizing controllers K in terms of a free stable parameter Λ. Replacing K by Λ then leads to the model matching form for the closed-loop transfer-function T_{zw}:

$$T_{zw}(z) := \widetilde{G}_{11}(z) + \widetilde{G}_{12}(z)\Lambda(z)\widetilde{G}_{21}(z) \tag{15}$$

where the functions $\widetilde{G}_{11}$, $\widetilde{G}_{12}$ and $\widetilde{G}_{21}$ are defined analogous to the case in (4) in terms of G_{11}, G_{12}, G_{21} and the coprime factors of G_{22}.

Following standard terminology in the mathematical literature, we say that a holomorphic function S on the polydisk $\mathbb{D}^d$ with values in the space of bounded operators between input space $\mathcal{W}$ and output space $\mathcal{Z}$ is in the d-variable *Schur class* $\mathcal{S}_d(\mathcal{W}, \mathcal{Z})$ if $\|S(z)\| \leq 1$ for all $z \in \mathbb{D}^d$.

Rename $T_1 := G_{11}$, $T_2 := G_{12}$ and $T_2 := G_{21}$. Then, with the change of design parameter $K \mapsto \Lambda$ followed by $\Lambda \mapsto S := T_1 + T_2\Lambda T_3$, we see that the H^∞-problem assumes the model-matching optimization form: *Find a stable function $\Lambda(z)$ (i.e., with entries in $\mathbb{C}(z)_{ss}$) so that the function S defined by*

$$S(z) = T_1(z) + T_2(z)\Lambda(z)T_3(z)$$

is in the Schur class $\mathcal{S}_d(\mathcal{W}, \mathcal{Z})$.

To reduced this model matching problem to a classical-type Nevanlinna-Pick interpolation problem on the polydisk, following [19], [39] we assume that T_1 and T_3 are scalar-valued, $T_3 \equiv 1$, and that the values of $T_2(z)$ are row vectors of length J: $T_2(z) = \begin{bmatrix} T_{2,1}(z) & \cdots & T_{2,J}(z) \end{bmatrix}$. In this case we seek functions $\Lambda_1(z), \dots, \Lambda_J(z)$ in $\mathbb{C}_d(z)_{ss}$ so that

$$S(z) = T_1(z) + T_{2,1}(z)\Lambda_1(z) + \cdots + T_{2,J}(z)\Lambda_J(z). \quad (16)$$

is in the scalar Schur class $\mathcal{S}_d := \mathcal{S}_d(\mathbb{C}, \mathbb{C})$.

For mathematical convenience we shall now widen the class of admissible solutions and allow $\Lambda_1, \dots, \Lambda_J$ to be in the Banach algebra $H^\infty(\mathbb{D}^d)$ of bounded analytic functions on $\mathbb{D}^d$. Under the assumption that the intersection of the zero varieties of $T_{2,1}, \dots, T_{2,J}$ within the closed polydisk $\overline{\mathbb{D}}^d$ consists of finitely many (say N) points

$$z_1 = (, z_{1,1}, \dots, z_{1,d}), \cdots, z_N = (z_{N,1}, \dots, z_{N,d})$$

all of which are in the open polydisk $\mathbb{D}^d$, and if we let $w_1, \dots, w_N$ be the values of T_1 at these points

$$w_1 = T_1(z_1), \dots, w_N = T_1(z_N),$$

then it is not hard to see that a function $S \in \mathcal{S}_d$ has the form (16) if and only if it satisfies the interpolation conditions

$$S(z_i) = w_i \text{ for } i = 1, \dots, N. \quad (17)$$

In this case the model-matching problem thus becomes the following finite-point Nevanlinna-Pick interpolation problem over $\mathbb{D}^d$: *find $S \in \mathcal{S}_d$ subject to $|S(z)| \le 1$ for all $z \in \mathbb{D}^d$ which satisfies the interpolation conditions* (17).

A second case (see [19]) where the polydisk Model-Matching Problem can be reduced to an Nevanlinna-Pick interpolation problem is the case where the values of $T_2(z)$ and $T_3(z)$ are square operators (i.e., acting between Hilbert spaces of the same dimension) with invertible values on the distinguished boundary of the polydisk; under these assumptions it is shown in [19, Theorem 3.5] how the model-matching problem is equivalent to a *bitangential Nevanlinna-Pick interpolation problem along a subvariety*, i.e., bitangential interpolation conditions are specified along all points of a codimension-1 subvariety of $\mathbb{D}^d$ (namely, the union of the zero sets of $\det T_2(z)$ and $\det T_3(z)$ intersected with $\mathbb{D}^d$).

However for both cases, and for $d \ge 2$, there is no theory with results parallel to those of the classical 1-variable case. Nevertheless, another modification makes a parallel theory possible. To formulate this adjustment, for given coefficient Hilbert spaces $\mathcal{W}$ and $\mathcal{Z}$ we define the *d-variable Schur-Agler class $\mathcal{SA}_d(\mathcal{W}, \mathcal{Z})$* to consist of those $\mathcal{L}(\mathcal{W}, \mathcal{Z})$-valued functions S analytic on the polydisk $\mathbb{C}^d$ for which the operator $S(\delta_1, \dots, \delta_d)$ has norm at most one for any collection $\delta_1, \dots, \delta_d$ of d commuting strict contraction operators on a separable Hilbert space $\mathcal{K}$; here $S(\delta_1, \dots, \delta_d)$ is defined as an operator from $\mathcal{W} \otimes \mathcal{K}$ to $\mathcal{Z} \otimes \mathcal{K}$ via the power series for S:

$$S(\delta_1, \dots, \delta_d) = \sum_{n \in \mathbb{Z}_+^d} S_n \otimes \delta^n \text{ if } S(z) = \sum_{n \in \mathbb{Z}_+^d} S_n z^n$$

whenever the series converges, where we use the standard multivariable notation

$$\delta^n = \delta_1^{n_1} \cdots \delta_d^{n_d} \quad \text{and} \quad z^n = z_1^{n_1} \cdots z_d^{n_d}$$
$$\text{for} \quad n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d.$$

For the cases $d = 1, 2$, it turns out, as a consequence of the von Neumann inequality or the Sz.-Nagy dilation theorem for $d = 1$ and of the Andô dilation theorem [8] for $d = 2$ (see [57], [21] for a full discussion), that the Schur-Agler class $\mathcal{SA}_d(\mathcal{W}, \mathcal{Z})$ and the Schur class $\mathcal{S}_d(\mathcal{W}, \mathcal{Z})$ coincide, while, due to an explicit example of Varopoulos, the inclusion $\mathcal{SA}_d(\mathcal{W}, \mathcal{Z}) \subset \mathcal{S}_d(\mathcal{W}, \mathcal{Z})$ is strict for $d \ge 3$.

There is a result due originally to Agler [1] and developed and refined in a number of directions since (see [4] for an overview) which parallels the one-variable case; for the case of a simple set of interpolation conditions (17) the result is as follows: *there exists a function S in the Schur-Agler class $\mathcal{SA}_d$ which satisfies the set of interpolation conditions $S(z_i) = w_i$ for $i = 1, \dots, N$ if and only if there exist d positive semidefinite matrices $\mathbb{P}^{(1)}, \dots, \mathbb{P}^{(d)}$ of size $N \times N$ so that*

$$1 - w_i \overline{w_j} = \sum_{k=1}^{d} (1 - z_{i,k} \overline{z_{j,k}}) \mathbb{P}_{i,j}^{(k)}.$$

For the case $d = 1$, the Pick matrix $\mathbb{P} = \left[\frac{1 - w_i \overline{w}_j}{1 - z_i \overline{z}_j} \right]_{i,j=1}^N$ is the unique solution of this equation, and we recover the classical criterion $\mathbb{P} \ge 0$ for the existence of solutions to the Nevanlinna-Pick problem.

Direct application of the Agler result to the bitangential Nevanlinna-Pick interpolation problem along a subvariety, however, gives a solution criterion involving an infinite Linear Matrix Inequality (where the unknown matrices have infinitely many rows and columns indexed by the points of the interpolation-node subvariety)—see [19, Theorem 4.1]. Alternatively, one can use the polydisk Commutant Lifting Theorem from [18] to get a solution criterion involving a Linear Operator Inequality [19, Theorem 5.2]. Without further massaging, either approach is computationally unattractive.

B. The multidimensional-system state-space formulation

One can also study such multivariable feedback-stabilization and H^∞-problems from the point of view of multidimensional state-space systems [27]. While there has been much work over the past couple of decades on the study of multidimensional systems from a coordinate-free point of view independent of the form of any particular input/state/output (i/s/o) representation (see e.g. the influential work of Oberst [51]), we focus here on Givone-Roesser i/s/o state-space systems as this representation has the most immediate tie-in with function theory on the polydisk. A Givone-Roesser system matrix has the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1d} & B_1 \\ \vdots & & \vdots & \vdots \\ A_{d1} & \cdots & A_{dd} & B_d \\ C_1 & \cdots & C_d & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_d \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_d \\ \mathcal{Y} \end{bmatrix}. \quad (18)$$

One then associates a multidimensional system in state-space form with evolution along the integer lattice

$$\mathbb{Z}_+^d = \{t = (t_1, \dots, t_d) \colon t_k \in \mathbb{Z}_+\},$$

with $\mathbb{Z}_+$ indicating the nonnegative integers, defined as

$$\Sigma_G : \begin{cases} \begin{bmatrix} x_1(t+e_1) \\ \vdots \\ x_d(t+e_d) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{bmatrix} + Bu(t) & (t \in \mathbb{Z}_+^d), \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{19}$$

with initial conditions a specification of the state values $x_k(\sum_{j \neq k} t_j e_j)$ for $t = (t_1, \dots, t_d) \in \mathbb{Z}_+^d$ subject to $t_k = 0$ where $k = 1, \dots, d$. Here e_k stands for the k-th unit vector in $\mathbb{C}^d$ and $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{bmatrix}$.

For the definition of internal stability, as in the 1-D case in Section I-B, one again considers the autonomous part of the system:

$$\begin{bmatrix} x_1(t+e_1) \\ \vdots \\ x_d(t+e_d) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{bmatrix} \qquad (t \in \mathbb{Z}_+^d).$$

Following [42], the Givone-Roesser system (19) is said to be *asymptotically stable* in case, for zero input $u(t) = 0$ for $t \in \mathbb{Z}_+^d$ and initial conditions with the property

$$\sup_{t \in \mathbb{Z}_+^d \colon t_k = 0} \|x_k(t)\| < \infty \text{ for } k = 1, \dots, d,$$

the state sequence x satisfies

$$\sup_{t \in \mathbb{Z}_+^d} \|x(t)\| < \infty \quad \text{and} \quad \lim_{t \to \infty} \|x(t)\| = 0,$$

where $t \to \infty$ is to be interpreted as $\min\{t_1, \dots t_d\} \to \infty$ when $t = (t_1, \dots, t_d) \in \mathbb{Z}_+^d$; one could also formulate an analogue of exponential stability the analogue of (S1) in Section I-B.2. We will not dwell on these except to say that at least it is argued in [42] that the asymptotic stability condition is equivalent to the *Hautus stability* condition (the analogue of (S2) in Section I-B.2): The operator

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1d} \\ \vdots & & \vdots \\ A_{d1} & \cdots & A_{dd} \end{bmatrix} \tag{20}$$

is *GR-Hautus-stable* if $I - Z(z)A$ is invertible for all z in the closed polydisk $\overline{\mathbb{D}}^d$, where we use the notation $Z(z) = \operatorname{diag}(z_1 I_{\mathcal{X}_1}, \dots, z_d I_{\mathcal{X}_d})$ with $\operatorname{diag}(T_1, \dots, T_d)$ indicating the block diagonal operator with diagonal blocks $T_1, \dots, T_d$.

One can then pose the feedback-stabilization and H^∞-problem analogously to the 1-D case as done in Section I-B as follows. Given a system G by (19) with the state-space refinement from (18), that is, $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$, we are assured that the closed-loop state-space system is well-posed if we make the innocuous assumption (via the loop-shifting procedure) that $D_{22} = 0$. We then seek a controller K in multidimensional state-space form

$$\Sigma_K : \begin{cases} \begin{bmatrix} x_{K1}(t+e_1) \\ \vdots \\ x_{Kd}(t+e_d) \end{bmatrix} = A_K \begin{bmatrix} x_{K1}(t) \\ \vdots \\ x_{Kd}(t) \end{bmatrix} + B_K y(t) \\ u(t) = C_K x_K(t) + D_K y(t) \qquad (t \in \mathbb{Z}_+^d) \end{cases} \tag{21}$$

(here also $\mathcal{X}_K$ has a finer decomposition $\mathcal{X}_K = \mathcal{X}_{K1} \oplus \cdots \oplus \mathcal{X}_{Kd}$) so that (a) (internal stability) the closed-loop system matrix A_{cl} which is also given by (11) or (12) is Hautus-stable (in the sense that $I - \begin{bmatrix} Z(z) & 0 \\ 0 & Z_K(z) \end{bmatrix} A_{cl}$ is invertible for all $z \in \overline{\mathbb{D}}^d$), and possibly also (b) (H^∞-performance) the closed-loop transfer-function

$$T_{zw}(z) = G_{11}(z) + G_{12}(z)(I - K(z)G_{22}(z))^{-1} K(z) G_{21}(z)$$

has norm bounded by one on the unit polydisk $\mathbb{D}^d$, where $G(z)$ and $K(z)$ are the associated Givone-Roesser transfer-functions

$$\begin{aligned} G(z) &= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \\ K(z) &= D_K + C_K (I - Z_K(z) A_K)^{-1} Z_K(z) B_K. \end{aligned}$$

We shall not attempt here to develop the analogue of condition (13).

In addition to the analogues of (S1) and (S2) mentioned above for GR-stability of a block state matrix A, we also mention the analogue of (S3): we say that the matrix A is *GR-scaled-stable* if there is an invertible block-diagonal matrix $\Theta = \operatorname{diag}[\Theta_1, \dots, \Theta_d]$ so that $\|\Theta A \Theta^{-1}\| < 1$, or equivalently, if there is a positive definite block-diagonal matrix $X = \operatorname{diag}[X_1, \dots, X_d]$ so that $A^* X A - X < 0$. Then in this multidimensional-system framework, rather than the equivalence of (S1)–(S3), we have only the implications (S3) $\Rightarrow$ (S2) $\Leftrightarrow$ (S1).

A similar phenomenon holds with respect to the notions of stabilizability and detectability. Given an input pair (A, B), we have the following analogues of (FS1)–(FS3) in Section I-B.4:

(FS1′) (A, B) is GR-Hautus-stabilizable if the matrix pencil $\begin{bmatrix} I - Z(z)A & B \end{bmatrix}$ has full rank for all z in the closed polydisk $\overline{\mathbb{D}}^d$.

(FS2′) (A, B) is GR-operator-stabilizable if there exists a matrix F so that $A + BF$ is GR-Hautus stable.

(FS3′) (A, B) is GR-LMI-stabilizable if there exists a structured positive definite matrix $Y = \operatorname{diag}[Y_1, \dots, Y_d$ so that $AYA^* - Y + BB^* < 0$, or equivalently, if there exists such a Y so that $(B^*)_\perp^* (AYA^* - Y)(B^*)_\perp < 0$.

We leave it to the reader to formulate the dual definitions and statements for GR-detectability.

Then the implications (FS3′) $\Rightarrow$ (FS2′) $\Rightarrow$ (FS1′) hold, with the reverse implications in general failing. The analogue of the result on measurement-feedback stabilization in Section I-B.4 then is: *The measurement-feedback stabilization problem has a solution if and only if (A, B_2) is GR-operator-stabilizable and (C_2, A) is GR-operator-detectable.* The difficulty here is that operator-stabilizability/detectability, unlike the situation in the 1-D case, has no practical test for existence or practical algorithm for the construction of the feedback F or the output injection L.

This situation can be remedied somewhat by considering a more conservative problem: replace the requirement that A_{cl} be GR-stable by the stronger condition that A_{cl} be GR-scaled-stable (the GR-analogue of condition (S3) mentioned above). Then we recover a definitive LMI solution criterion for this problem: *The measurement-feedback scaled-stabilization problem for G as in* (19) *is solvable if and only* *if* (A, B_2) *is GR-LMI-stabilizable and* (C_2, A) *is GR-LMI-detectable. Moreover, controllers* K *as in* (21) *solving the measurement-feedback GR-scaled-stabilization problem are given by system matrices* $\left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right] = \left[\begin{smallmatrix} A+B_2F+LC_2 & -L \\ F & 0 \end{smallmatrix}\right]$ *where* F, L *are any choices of matrices for which* $A + B_2F$ *and* $A + LC_2$ *are GR-scaled-stable.* Note that we do not discuss a scaled analogue of the Hautus-stability of Hautus-stabilizability (FS1′); we return to this theme in the context of noncommutative multidimensional systems to come.

The story for the d-D multidimensional system H^∞-problem is similar; for the problem as formulated there are no practical solution criteria or computational algorithms. What is done instead is to formulate a *scaled* version of the H^∞-problem which does have good solution criteria. Given a GR-plant G via a state-space realization with system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, we say that G achieves *scaled-performance* if there is a structured positive definite matrix $\Theta = \operatorname{diag}(\Theta_1, \dots, \Theta_d)$ so that

$$\left\| \begin{bmatrix} \Theta & 0 \\ 0 & I_{\mathcal{Z}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Theta^{-1} & 0 \\ 0 & I_{\mathcal{W}} \end{bmatrix} \right\| < 1.$$

Given a GR-plant G in state-space form by (19), the scaled H^∞-problem is to design a controller K as in (21) so that (a) the closed-loop state-update matrix A_{cl} is scaled-stable, and (b) the closed-loop transfer-function T_{zw}, with system matrix $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix}\right]$, has scaled performance. (Note that condition (b) actually already implies condition (a).) Then this problem has a clean LMI solution: *The scaled* H^∞*-problem for a given GR-plant* G *by* (19) *has a solution if and only if the LMIs in* (14) *have structured solutions* $X = \operatorname{diag}[X_1, \dots, X_d]$ *and* $Y = \operatorname{diag}[Y_1, \dots, Y_d]$.

C. Applications

Besides multidimensional-system applications (e.g., in image processing, discretizations of distributed-parameter control systems involving partial differential equations, etc. [27]), it is of interest that this multidimensional-system theory has applications to more specialized robust control problems in 1-D systems.

1) Linear Parameter-Varying Control: In this application (see [56] and [28, Section 11.1]), we suppose that we are given a 1-D plant as in (7) but where the coefficient matrices A, B, C, D all depend on some real scalar parameters $\alpha_1(t), \dots, \alpha_r(t)$ which vary with respect to the discrete time variable t (see [28, Section 11.1]); the variation with t is not known a priori but we assume that the functional dependence $A = A(\alpha)$, $B = B(\alpha)$, $C = C(\alpha)$, $D = D(\alpha)$ on $\alpha = (\alpha_1, \dots, \alpha_r)$ is known. We assume that the current values $\alpha(t) = (\alpha_1(t), \dots, \alpha_r(t))$ are measurable on-line so that the controller K with system matrix $J_K = \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right]$ can also be a function of α: $J_K(\alpha) = \left[\begin{smallmatrix} A_K(\alpha) & B_K(\alpha) \\ C_K(\alpha) & D_K(\alpha) \end{smallmatrix}\right]$. Then the LPV robust stabilization/H^∞-problem is: *given* $A(\alpha), B(\alpha), C(\alpha), D(\alpha)$ *as above, design a controller system matrix* $J_K(\alpha)$ *so that (a) the closed-loop state matrix* $A_{cl}(\alpha)$ *is stable for all choices of* α *in the closed polydisk* $\overline{\mathbb{D}}^r$*, and possibly also (b) the closed-loop transfer-function* $T_{zw}(\alpha)(\lambda) = D_{cl}(\alpha) + \lambda C_{cl}(\alpha)(I - \lambda A_{cl}(\alpha))^{-1} B_{cl}(\alpha)$ *has* $\|T_{zw}(\alpha)\|_{H^\infty} \le 1$ *(or* $\|T_{zw}(\alpha)\|_{H^\infty} \le \rho$ *for some* $\rho < 1$ *for the strict version) for all* α *in the closed polydisk* $\overline{\mathbb{D}}^r$.

To analyze this problem, it is usually further assumed that $A(\alpha), B(\alpha), C(\alpha), D(\alpha)$ are rational functions of the parameters α with GR-state-space realization: we assume that $\left[\begin{smallmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{smallmatrix}\right]$ is given by

$$\left[\begin{smallmatrix} A_{11} & B_1 \\ C_1 & D \end{smallmatrix}\right] + \left[\begin{smallmatrix} A_{10} \\ C_0 \end{smallmatrix}\right] (I - Z_p(\alpha) A_{00})^{-1} Z_p(\alpha) \left[\begin{smallmatrix} A_{01} & B_0 \end{smallmatrix}\right].$$

where we have set $Z_p(\alpha) = \operatorname{diag}(\alpha_1 I_{\mathcal{X}_{p1}}, \dots, \alpha_r I_{\mathcal{X}_{pr}})$, making use of the GR-decomposition $\mathcal{X}_p = \mathcal{X}_{p1} \oplus \cdots \oplus \mathcal{X}_{pr}$ for the parameter state-space $\mathcal{X}_p$. Then the state-space system

$$\begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x' \\ z \end{bmatrix}$$

can be written as a linear system with feedback connection

$$\begin{bmatrix} A_{00} & A_{01} & B_0 \\ A_{10} & A_{11} & B_1 \\ C_0 & C_1 & D \end{bmatrix} \begin{bmatrix} q \\ x \\ w \end{bmatrix} = \begin{bmatrix} p \\ x' \\ z \end{bmatrix}, \qquad q = Z_p(\alpha)$$

In the control application, w becomes $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right]$, z becomes $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ and the operators B_0, B_1, C_0, C_1 expand to

$$B_0 = \left[\begin{smallmatrix} B_{01} & B_{02} \end{smallmatrix}\right], \quad B_1 = \left[\begin{smallmatrix} B_{11} & B_{12} \end{smallmatrix}\right]$$
$$C_0 = \left[\begin{smallmatrix} C_{10} \\ C_{20} \end{smallmatrix}\right], \quad C_1 = \left[\begin{smallmatrix} C_{11} \\ C_{21} \end{smallmatrix}\right], \quad D = \left[\begin{smallmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{smallmatrix}\right].$$

To keep the notation simple we shall use only the condensed notation below; the precise meaning can be determined from the context.

To solve the LPV robust control problem, it is natural to assume that the controller state-matrix $J_K(\alpha)$ to be designed is also given in the rational form

$$J_K(\alpha) = \left[\begin{smallmatrix} A_{K11} & B_{K1} \\ C_{K1} & D_K \end{smallmatrix}\right] + \left[\begin{smallmatrix} A_{K10} \\ C_{K0} \end{smallmatrix}\right] (I - Z_{Kp}(\alpha) A_{K00})^{-1} Z_{Kp}(\alpha) \left[\begin{smallmatrix} A_{K01} & B_{K0} \end{smallmatrix}\right].$$

This suggests that we let

$$z = (\alpha_1, \dots, \alpha_r, \lambda) =: (z_1, \dots, z_d) \quad (\text{with } d = r + 1)$$

and introduce the GR-state-space system

$$\begin{bmatrix} x_1(t+e_1) \\ \vdots \\ x_d(t+e_d) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{bmatrix} + Bw(t)$$
$$y(t) = C \begin{bmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{bmatrix} + Dw(t)$$

where we have now set

$$A = \left[\begin{smallmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{smallmatrix}\right], \quad B = \left[\begin{smallmatrix} B_0 \\ B_1 \end{smallmatrix}\right], \quad C = \left[\begin{smallmatrix} C_0 & C_1 \end{smallmatrix}\right]. \qquad (22)$$

By making use of the Main Loop Theorem [62, page 284], one can now show that the robust LPV control problem reduces to the GR-stabilization/H^∞-problem (state-space non-scaled version) associated with the system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ (a condensed version of $\left[\begin{smallmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{smallmatrix}\right]$ derived as in (22)).

2) Robust Control against Time-Invariant Uncertainty: A variation on the robust LPV problem discussed above is the case where the controller is not allowed to be a function of the unknown parameters $\alpha_1, \dots, \alpha_r$ but only of the frequency variable λ. Thus we are given a state-space system of the form

$$\begin{bmatrix} A_{00} & A_{01} & B_{01} & B_{02} \\ A_{10} & A_{11} & B_{11} & B_{12} \\ C_{10} & C_{12} & D_{11} & D_{12} \\ C_{20} & C_{21} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} q \\ x \\ w \\ u \end{bmatrix} = \begin{bmatrix} p \\ x' \\ z \\ y \end{bmatrix}.$$

If we impose the feedback loop $x = \lambda x'$ we arrive at a system of the form

$$\begin{bmatrix} A_{00}(\lambda) & B_{01}(\lambda) & B_{02}(\lambda) \\ C_{10}(\lambda) & D_{11}(\lambda) & D_{12}(\lambda) \\ C_{20}(\lambda) & D_{21}(\lambda) & 0 \end{bmatrix} \begin{bmatrix} q \\ w \\ u \end{bmatrix} = \begin{bmatrix} p \\ z \\ y \end{bmatrix}$$

with system matrix depending rationally on λ given by

$$\begin{bmatrix} A_{00}(\lambda) & B_{01}(\lambda) & B_{02}(\lambda) \\ C_{10}(\lambda) & D_{11}(\lambda) & D_{12}(\lambda) \\ C_{20}(\lambda) & D_{21}(\lambda) & 0 \end{bmatrix} = \begin{bmatrix} A_{00} & B_{01} & B_{02} \\ C_{10} & D_{11} & D_{12} \\ C_{20} & D_{21} & 0 \end{bmatrix} + \lambda \begin{bmatrix} A_{01} \\ C_{11} \\ C_{21} \end{bmatrix} (I - \lambda A_{11})^{-1} \begin{bmatrix} A_{10} & B_{11} & B_{12} \end{bmatrix}.$$

We assume that the controller has the state-space form

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix} = \begin{bmatrix} x'_K \\ u \end{bmatrix}.$$

If we impose the controller feedback loop $x_K = \lambda x'_K$ as well as the frequency loop $x_K = \lambda x'_K$, we arrive at the closed-loop model-matching form

$$\begin{bmatrix} A_{cl00}(\lambda) & B_{cl01}(\lambda) \\ C_{cl10}(\lambda) & D_{cl11}(\lambda) \end{bmatrix} = \begin{bmatrix} A_{00} & B_{01} \\ C_{10} & D_{11} \end{bmatrix}(\lambda) + \begin{bmatrix} B_{02} \\ D_{12} \end{bmatrix}(\lambda)\widehat{K}(\lambda)\begin{bmatrix} C_{20} & D_{21} \end{bmatrix}(\lambda) \colon \begin{bmatrix} q \\ w \end{bmatrix} \mapsto \begin{bmatrix} p \\ z \end{bmatrix}.$$

We can then consider the robust control problem: *Given the system matrix* $\begin{bmatrix} A_{00} & B_{01} & B_{02} \\ C_{10} & D_{11} & D_{12} \\ C_{21} & D_{21} & 0 \end{bmatrix}(\lambda)$ *depending rationally on the frequency variable* λ*, find a rational matrix-function controller* $\widehat{K}(\lambda)$ *so that the disturbed closed-loop transfer matrix* $T_{zw}(\alpha, \lambda) = D_{cl11}(\lambda) + C_{cl10}(\lambda) + (I - Z_p(\alpha)A_{cl00}(\lambda))^{-1}B_{cl01}(\lambda)$ *has norm at most one for all* $(\alpha, \lambda) \in \overline{\mathbb{D}}^{r+1}$. This is the problem to which one can reduce the *robust synthesis against* $\boldsymbol{\Delta_{TI}}$*-problem* as discussed in [28, Section 9.3.2] and is also the problem studied at length from a more function-theory point of view in the mathematical papers [23], [24], [25], [5], [6]. This mathematical work deals only with special cases of the general problem but has inspired followup work in complex geometry and operator theory.

No tractable exact solution algorithm is known for this problem; Section 9.3.3 of [28] does present a synthesis heuristic (D-K iteration) which sometimes is effective but is not guaranteed to work in all cases. Alternatively, one can consider the scaled version of the problem as a route to a sufficiency analysis; this is the point of view of [9].

D. Equivalence between frequency-domain and state-space formulations

As we have seen in the multidimensional system/multivariable function context, there exists a lot of work on the frequency-domain version of the stabilization/H^∞-problem [5], [6], [19], [23], [24], [25], [46], [47], [48], [52], [54] and on the state-space version of the same problem [9], [49], [27]; with the exception of [17] there has not been much discussion on how these problems are related. Unlike in the single-variable case, stabilizing controllers do not always exist; in the frequency-domain setting, one requires that G_{22} have a double coprime factorization while, in the state-space setting, one requires that (A, B_2) and (C_2, A) are GR-operator stabilizable/detectable.

As for the connection between the two settings, one direction is straightforward: If we are given GR-realizations for the plant G as in (7) and for the controller K, as in (8) which solve the state-space version of the stabilization/H^∞-problem, then the associated transfer-functions (G, K) solve the frequency-domain version of the problem. This follows just as in the 1-D case; one sees that the 3×3-block transfer-function $\Theta(G, K)$ has GR-state-space realization having A_{cl} as the state operator. If A_{cl} is such that $I - Z_{cl}(z)A_{cl}$ is invertible on the closed polydisk, it follows that all nine block entries of $\Theta(G, K)$ have matrix entries in the stable class $\mathbb{C}(z)_{ss}$.

For the converse direction, suppose that (G, K) are rational matrix functions solving the frequency-domain stabilization/H^∞-problem. If one can find realizations (7) for G and (8) for K so that (A, B_2) and (A_K, B_K) are GR-Hautus stabilizable and (C_2, A) and (C_K, A_K) are GR-Hautus detectable, it then follows that these realizations for G and K give rise to a solution of the state-space version of the stabilization/H^∞-problem.[1] However the existence of such realizations in the multivariable-function setting is problematical, even in the presence of the additional hypothesis that G_{22} has a double coprime factorization. Thus the precise conditions under which a frequency-domain solution can be identified with a state-space solution are not clear.

However, if we are in the Model-Matching setup where $G_{22} = 0$), then a little more can be said. We first note the following general fact.

Proposition 2.1: *If F is a rational matrix function over $\mathbb{C}(z)_{ss}$, then F has a GR-realization $F(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ with state matrix A scaled-stable.*

Proof: If F is a rational matrix function over $\mathbb{C}(z)_{ss}$, then F is holomorphic on a neighborhood of the closed polydisk $\overline{\mathbb{D}}^d$ and hence has a power series representation $F(z) = \sum_{n \in \mathbb{Z}^d_+} F_n z^n$ with $\sum_{n \in \mathbb{Z}^d_+} \rho^{|n|} \|F_n\| =: M < \infty$ for some $\rho > 1$ ($|n| = n_1 + \cdots + n_d$ if $n = (n_1, \dots, n_d)$). If $\delta = (\delta_1, \dots, \delta_d)$ is a commutative tuple of contraction operators on $\mathcal{K}$, we conclude that $F(\rho\delta_1, \dots, \rho\delta_d)$ is a well defined operator from $\mathcal{W} \otimes \mathcal{K}$ to $\mathcal{Z} \otimes \mathcal{K}$ with $\|F(\rho\delta_1, \dots, \rho\delta_d)\| \le M$. We conclude that $\widetilde{F}(z) := \frac{1}{M}F(\rho z)$ is in the Schur-Agler class. Thus, by the realization theorem in [2] (see also [3], [22]), we see that $\widetilde{F}(z)$ has a GR-realization

[1] In [17] it is asserted that this result holds with the notions of of *modally stabilizable* and *modally detectable* in place of *GR-Hautus stabilizable* and *GR-Hautus detectable*. However the proof of Lemma 4.13 there appears to have a gap and we now believe that the correct assumption for the argument to go through is GR-Hautus stabilizable and GR-Hautus detectable.

$\widetilde{F}(z) = \widetilde{D} + \widetilde{C}(I - Z(z)\widetilde{A})^{-1}Z(z)\widetilde{B}$ with $\left\| \left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{smallmatrix} \right] \right\| \leq 1$. Lifting to noncommuting variables and using the Kalman decomposition and state-space similarity theorem of [12] (see Section III-A below), we see that it can be arranged that $\widetilde{A}$ acts on a finite-dimensional state space $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$ if $\widetilde{F}$ is rational (with finite-dimensional input and output spaces $\mathcal{W}$ and $\mathcal{Z}$). It then follows that F itself has a GR-realization $F(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ with

$$D = \frac{1}{M}\widetilde{D}, \quad ,C = \frac{1}{M}\widetilde{C}, \quad A = \frac{1}{\rho}\widetilde{A}, \quad B = \frac{1}{\rho}\widetilde{B}.$$

In particular $A = \frac{1}{\rho}\widetilde{A}$ is strictly contractive, and thus even scaled-stable. ■

If G is the Model-Matching setup ($G_{22} = 0$), then we know that stabilizability of G is equivalent to G being already stable, i.e., all matrix entries are in $\mathbb{C}(z)_{ss}$, and a given controller K stabilizes if and only if K is also stable. By Proposition 2.1 it then follows that we may choose stable (even scaled-stable) realizations for G and for K. In particular, these realizations have the property that (C_2, A) and (C_K, A_K) are Hautus-detectable and (A, B_2) and (A_K, B_K) are Hautus-stabilizable, from which it follows that the associated A_{cl} is Hautus-stable. It then follows that the realizations for G and K chosen in this way yield a solution of the state-space version of the stabilization problem.

Also of interest is the connection between the scaled-H^∞ problem (in state-space coordinates) and the frequency-domain H^∞-problem but with Schur-Agler-class performance criterion. For simplicity we restrict the discussion to the Model-Matching setup $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & 0 \end{smallmatrix} \right]$. We suppose that we are given $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & 0 \end{smallmatrix} \right]$ with realization as in (7) with A even scaled-stable and we suppose that K with realization $J = \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix} \right]$ yields a solution of the scaled-H^∞-problem. This implies that the realization $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix} \right]$ for T_{zw} can be brought to a new contractive realization $\left[\begin{smallmatrix} A'_{cl} & B'_{cl} \\ C'_{cl} & D'_{cl} \end{smallmatrix} \right]$ via a structured state-space similarity from which we conclude that T_{zw} is in the Schur-Agler class. Conversely, suppose that we are given stable realizations for G and K where $S := T_{zw} = G_{11} + G_{12}KG_{21}$ is in the Schur-Agler class. Then interconnection of realizations gives us a state-space realization $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix} \right]$ for the Schur-Agler class function S. On the other hand, by the realization theorem from [2] already mentioned in the proof of Proposition 2.1 above, we know that S has contractive realizations. However, in the absence of a strict Bounded Real Lemma or State-Space Similarity Theorem for this commutative-variable setting (however see [12], [14] and the proof of Theorem 3.3 below for noncommutative versions of these results), there is no apparent guarantee that the realization $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix} \right]$ generated from given stable realizations for G and K will be one of the realizations for S which is structured-similar to a contractive realization, as required for (G, K) to qualify as a solution of the scaled-H^∞ problem. Thus again the passage from the frequency-domain side to the state-space-coordinate side is problematical.

III. NONCOMMUTATIVE MULTIDIMENSIONAL LINEAR SYSTEMS

We now turn to another version of multidimensional systems which has origins in the theory of automata and formal languages [29], [30], [58], [59] but has only recently been revived in connection with problems in control theory [12], [13], [40]. We reverse the order of the previous sections, and consider the state-space setting first, then the "frequency"-domain setting.

A. *Noncommutative multidimensional linear systems: evolution along a tree*

We let $\mathcal{F}_d$ denote the free semigroup generated by the d letters $\{1, \ldots, d\}$. Thus elements of $\mathcal{F}_d$ consist of words $w = i_N i_{N-1} \cdots i_1$ where each letter i_k is one of the letters from the alphabet consisting of the d letters denoted simply as $1, 2, \ldots, d$. The multiplication in $\mathcal{F}_d$ is via concatenation: if $w = i_N \cdots i_1$ and $v = j_M \cdots j_1$, then we define a new word wv by $wv = i_N \ldots i_1 j_M \ldots j_1$. We include the empty word, denoted by $\emptyset$ as an element of $\mathcal{F}_d$, which serves as the unit element for the semigroup $\mathcal{F}_d$. Note that one can view $\mathcal{F}_d$ as a homogeneous tree of order d with root $\emptyset$: given a word w its immediate successors (d branches emanating out from w) are $1w, 2w, \ldots, dw$. The free semigroup (or homogeneous tree) $\mathcal{F}_d$ will serve as the "time domain" for our noncommutative multidimensional linear system. A very general setup was introduced in [12], [13], but we focus here on the so-called noncommutative Givone-Roesser systems defined as follows. The system matrix U for a noncommutative Givone-Roesser system has exactly the same form as in (18) in Section II-B. We then introduce system equations

$$\begin{aligned} \begin{bmatrix} x_1(1w) \\ \vdots \\ x_d(dw) \end{bmatrix} &= A \begin{bmatrix} x_1(w) \\ \vdots \\ x_d(w) \end{bmatrix} + Bu(w) \\ y(w) &= C \begin{bmatrix} x_1(w) \\ \vdots \\ x_d(w) \end{bmatrix} + Du(w). \end{aligned} \tag{23}$$

To compute the whole system trajectory $(u(w), x(w), y(w))$ for w an arbitrary word in $\mathcal{F}_d$ in a well-defined way, the appropriate choice of initial condition for the state vector $x(w)$ is a specification of the k-th component $x_k(v)$ over all words v which do not begin with the letter k, i.e., $\{x_k(v) \colon v \in \partial_k \mathcal{F}_d\}$ where we let

$$\partial_k \mathcal{F}_d = \{v \in \mathcal{F}_d \colon v \neq kv' \text{ for any } v' \in \mathcal{F}_d\}$$

denote the *k-boundary of* $\mathcal{F}_d$. If we then specify $x_k|_{\partial_k \mathcal{F}_d}$ for $k = 1, \ldots, d$ along with any choice of input signal $\{u(w) \colon w \in \mathcal{F}_d\}$, then the system equations (23) recursively determine the rest of the system trajectory $\{(u(w), x(w), y(w)) \colon w \in \mathcal{F}_d\}$.

To define stability we introduce the associated autonomous system

$$\begin{bmatrix} x_1(1w) \\ \vdots \\ x_d(dw) \end{bmatrix} = A \begin{bmatrix} x_1(w) \\ \vdots \\ x_d(w) \end{bmatrix}.$$

and suppose we specify an initial condition $\{x_k|_{\partial_k \mathcal{F}_d} \colon k = 1, \dots, d\}$ with total energy

$$M_\emptyset = \sum_{k=1}^{d} \sum_{v \in \partial_k \mathcal{F}_d} \|x_k(v)\|^2.$$

A natural notion of *exponential stability* analogous to the discussion in Sections I-B.2 and II-B is that

$$M_w \leq K\rho^{|w|} M_\emptyset \text{ for some } \rho < 1 \text{ and } K < \infty \quad \text{(NC-S1)}$$

where $|w|$ indicates the length of the word w (the number of letters in w) and where we set

$$M_w = \sum_{k=1}^{d} \sum_{v \in \partial_k \mathcal{F}_d} \|x_k(vw)\|^2 .$$

However, this notion of stability to this point has not been analyzed. What is used is the following noncommutative Hautus notion: Let A be an operator with decomposition as in (20) and $\mathcal{K}$ a fixed separable infinite-dimensional Hilbert space. We set $\mathbf{A} = A \otimes I_{\mathcal{K}}$ on $\boldsymbol{\mathcal{X}} := \mathcal{X} \otimes \mathcal{K}$ and define $Z(\delta)$ for a (not necessarily commutative) d-tuples $\delta = (\delta_1, \dots, \delta_d)$ of contraction operators on $\mathcal{K}$ by $Z(\delta) = \operatorname{diag}(I_{\mathcal{X}_1} \otimes \delta_1, \dots, I_{\mathcal{X}_d} \otimes \delta_d)$ with respect to the decomposition $\boldsymbol{\mathcal{X}} \cong (\mathcal{X} \otimes \mathcal{K}) \oplus \cdots \oplus (\mathcal{X}_d \otimes \mathcal{K})$. We then say that the operator A is *nc-Hautus stable* if $I_{\boldsymbol{\mathcal{X}}} - Z(\delta)\mathbf{A}$ is invertible on $\boldsymbol{\mathcal{X}}$ for all d-tuples $\delta = (\delta_1, \dots, \delta_d)$ of contraction operators on $\mathcal{K}$. Formally this notion arises from the notion of Hautus-stable for the matrix A in Section II-B but with the commutative set of variables $z = (z_1, \dots, z_d)$ in the closed polydisk replaced by the noncommutative set of variables $\delta = (\delta_1, \dots, \delta_d)$ in what one can think of as the noncommutative closed polydisk

$$\overline{\mathbb{D}}^d_{nc} = \{\delta = (\delta_1, \dots, \delta_d) \in \mathcal{L}(\mathcal{K})^d \colon \|\delta_k\| \leq 1 \text{ for each } k = 1, \dots, d\}.$$

With this notion of stability, we get equivalence with an LMI condition, just as in the 1-D case.

Proposition 3.1: Given a matrix A as in (20), *the following conditions are equivalent.*

(NC-S2) A *is* nc-Hautus stable.

(NC-S3) A *is similar to a strict contraction operator via a structured similarity transformation, i.e., there is an invertible matrix Θ of the block-diagonal form $\Theta = \operatorname{diag}(\Theta_1, \dots, \Theta_d)$ so that $\|\Theta A \Theta^{-1}\| < 1$, or equivalently, there is a positive-definite structured matrix $X = \operatorname{diag}(X_1, \dots, X_d) > 0$ satisfying the LMI condition: $A^* X A - X < 0$.*

This remarkable result in this form is due to Paganini [55], adapting key ideas from [60] and [50]; a very nice exposition can be found in [28, Chapter 8 and Appendix B].

There are corresponding noncommutative notions of *stabilizable* and *detectable*. Let us make the following definitions. Given an input pair (A, B) we say that:

(FS1$''$) (A, B) is *nc-Hautus-stabilizable* if the infinite-dimensional operator pencil $\left[\, I - \mathbf{A}Z(\delta) \;\; \mathbf{B} \,\right]$, $\mathbf{B} = B \otimes \mathcal{K}$, is bounded right-invertible for all $\delta = (\delta_1, \dots, \delta_d)$ in the closed noncommutative polydisk $\overline{\mathbb{D}}^d_{nc}$.

(FS2$''$) (A, B) is *nc-operator stabilizable* if there exists a feedback matrix $F \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ so that $A + BF$ is nc-Hautus stable.

(FS3$''$) (A, B) is *LMI-stabilizable* if there exists a structured positive definite matrix $Y = \operatorname{diag}[Y_1, \dots, Y_d]$ so that $AYA^* - Y + BB^* < 0$.

(Note that condition (FS3$''$) is the same as condition (FS3$'$) in Section II-B.) Similarly, given an output pair (C, A) we say that:

(D1$''$) (C, A) is *nc-Hautus detectable* if the infinite-dimensional operator pencil $\left[\begin{smallmatrix} I - \mathbf{A}Z(\delta) \\ \mathbf{C} \end{smallmatrix}\right]$, with $\mathbf{C} = C \otimes \mathcal{K}$, is boundedly left invertible for all δ in the closed noncommutative polydisk $\overline{\mathbb{D}}^d_{nc}$.

(D2$''$) (C, A) is *nc-operator stabilizable* if there exists an output injection matrix $L \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ so that $A + LC$ is nc-Hautus stable.

(D3$''$) (C, A) is *LMI-detectable* if there exists a structured positive definite matrix $X = \operatorname{diag}[X_1, \dots, X_d]$ so that $A^* X A - X + C^* C < 0$.

Then we have the following remarkable result (see [55]) which completely parallels the classical 1-D case, unlike the corresponding results for the commutative d-D case.

Proposition 3.2: Suppose that (A, B) is an input pair. Then conditions (FS1$''$), (FS2$''$) *and* (FS3$''$) *are all equivalent. Similarly, if (C, A) is an output pair, then conditions* (D1$''$), (D2$''$) *and* (D3$''$) *are equivalent.*

This gives a satisfactory understanding of nc-Hautus stabilization via state-feedback or output-injection. We next discuss the measurement-feedback scheme. For this purpose we suppose that we are given a noncommutative GR-system G via (23) with state-space having GR-decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$. The *nc-measurement-feedback stabilization problem* then is to design a controller K given by system equations

$$\begin{bmatrix} x_{K1}(1w) \\ \vdots \\ x_{Kd}(dw) \end{bmatrix} = A_K \begin{bmatrix} x_{K1}(w) \\ \vdots \\ x_{Kd}(w) \end{bmatrix} + B_K y(w)$$
$$u(w) = C_K \begin{bmatrix} x_{K1}(w) \\ \vdots \\ x_{Kd}(w) \end{bmatrix} + D_K y(w). \qquad (24)$$

(with state-space again having GR-decomposition $\mathcal{X}_K = \mathcal{X}_{K1} \oplus \cdots \oplus \mathcal{X}_{Kd}$) so as to guarantee that the closed-loop matrix A_{cl} as appearing in either (11) or (12) is nc-Hautus stable. The answer is a perfect analogue of the 1-D case: *The nc-measurement-feedback stabilization problem for the plant G (7) has a solution if and only if (A, B_2) is LMI-stabilizable and (C_2, A) is LMI-detectable. In this case the controller system matrix $J = \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right]$ solves the problem whenever J has the form $J = \left[\begin{smallmatrix} A + B_2 F + LC_2 & -L \\ F & 0 \end{smallmatrix}\right]$ where F and L are chosen so that $A + B_2 F$ and $A + LC_2$ are nc-Hautus stable.*

To formulate the nc-H^∞-problem (state-space version), rather than finding an analogue of the energy-balance relation

(13) for the noncommutative GR-system, we first proceed to a noncommutative frequency-domain.

As in [12], we introduce a formal noncommutative Z-transform on signals defined on the tree $\mathcal{F}_d$ as follows. In this section we let $z = (z_1, \dots z_d)$ be a d-tuple of freely noncommuting indeterminates. For $w = i_N \cdots i_1$ a word in $\mathcal{F}_d$, we let z^w denote the noncommutative monomial $z^w = z_{i_N} \cdots z_{i_1}$. Given two noncommutative monomials z^w and z^v, it is natural to define the product as $z^w \cdot z^v = z^{wv}$. For $\mathcal{X}$ any linear space, we let $\mathcal{X}\langle\langle z \rangle\rangle$ denote the space of all formal power series $f(z) = \sum_{w \in \mathcal{F}_d} f_w z^w$ with coefficients f_w from $\mathcal{X}$. Note that we can add and multiply formal power series

$$\left(\sum_{w \in \mathcal{F}_d} f_w z^w\right) + \left(\sum_{v \in \mathcal{F}_d} g_v z^v\right) = \sum_{w \in \mathcal{F}_d} (f_w + g_w) z^w,$$

$$\left(\sum_{w \in \mathcal{F}_d} f_w z^w\right) \cdot \left(\sum_{v \in \mathcal{F}_d} g_v z^v\right) = \sum_{w,v \in \mathcal{F}_d} f_w g_v z^{vw}$$

$$= \sum_{w \in \mathcal{F}_d} \left(\sum_{\alpha,\beta\colon \alpha\beta = w} f_\alpha g_\beta\right) z^w.$$

whenever the sum $f_w + g_w$ and products $f_\alpha g_\beta$ of the associated coefficients makes sense; e.g., the product makes sense if $f(z) \in \mathcal{L}(\mathcal{Z}, \mathcal{W})\langle\langle z \rangle\rangle$ has operator coefficients and $g(z) \in \mathcal{W}\langle\langle z \rangle\rangle$ has conformable vector coefficients.

For $\{x(w) \colon w \in \mathcal{F}_d\}$ any vector-valued signal defined along the tree $\mathcal{F}_d$, we define the Z-transform $\widehat{x}(z)$ of x to be the formal power series given by

$$\widehat{x}(z) = \sum_{w \in \mathcal{F}_d} x(w) z^w.$$

If we apply the Z-transform to the system equations (23) we arrive at the identity of formal power series

$$\begin{bmatrix} \sum_{w \in \mathcal{F}_d} x(1w) z^w \\ \vdots \\ \sum_{w \in \mathcal{F}_d} x(dw) z^w \end{bmatrix} = A \begin{bmatrix} \widehat{x}(z) \\ \vdots \\ \widehat{x}(z) \end{bmatrix} + B\widehat{u}(z)$$
$$y(z) = C\widehat{x}(z) + D\widehat{u}(z). \tag{25}$$

If we assume that the system is run with zero initial condition ($x_k|_{\partial_k \mathcal{F}_d} = 0$ for each $k = 1, \dots, d$), then we see that

$$z_k \sum_{w \in \mathcal{F}_d} x(kw) z^w = \sum_{w \in \mathcal{F}_d} x(kw) z_k z^w = \widehat{x}_k(z).$$

Hence the first equation in (25) after multiplying by $Z(z) = \operatorname{diag}(z_1 I_{\mathcal{X}_1}, \dots, z_d I_{\mathcal{X}_d})$ converts to

$$\widehat{x}(z) = Z(z)A\widehat{x}(z) + Z(z)B\widehat{u}(z).$$

Since a formal power series is invertible whenever the coefficient of $z^\emptyset$ (the constant term) is invertible, we can invert $I - Z(z)A$ and solve for $\widehat{x}(z)$:

$$\widehat{x}(z) = (I - Z(z)A)^{-1} B\widehat{u}(z).$$

Substitution of this in the second equation of (25) then gives us $\widehat{y}(z) = \widehat{G}(z)\widehat{u}(z)$ where we set

$$\widehat{G}(z) = D + C(I - Z(z)A)Z(z)B.$$

We call $\widehat{G}(z)$ the transfer-function of the noncommutative GR-system (23). Moreover, the power-series coefficients can be identified explicitly in terms of the system matrices in (23) according to the following formula: $\widehat{G}(z) = \sum_{w \in \mathcal{F}_d} \widehat{G}_w z^w$ where $\widehat{G}_\emptyset = D$ and $\widehat{G}_{i_N \cdots i_1} = C_{i_N} A_{i_N i_{N-1}} \cdots A_{i_2 i_1} B_{i_1}$. We mention that the converse realization question: *Given a collection of operators T_w between two finite-dimensional spaces $\mathcal{U}$ and $\mathcal{Y}$ indexed by words $\{T_w \colon w \in \mathcal{F}_d\}$, when is it the case that there is a finite-dimensional state-space $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$ and a colligation matrix so that we recover T_w as in the moment form $T_\emptyset = D$, $T_{i_N \cdots i_1} = C_{i_N} A_{i_N i_{N-1}} \cdots A_{i_2 i_1} B_{i_1}$?* The paper [12] presents a solution of this problem in terms of noncommutative Hankel matrices; this result is closely related to results of Fliess [29] who obtained analogous results for the case of so-called *recognizable series*, i.e., realizing a given collection of matrices f_w ($w \in \mathcal{F}_d$) as having the form $f_w = CA^w B$ (where now $A = (A_1, \dots, A_d)$ and $A^w = A_{i_N} \cdots A_{i_1}$ if $w = i_N \cdots i_1$) arising from a formal power series $\sum_{w \in \mathcal{F}_d} f_w z^w$ from a recognizable series $\sum_{w \in \mathcal{F}_d} f_w z^w = C(I - z_1 A_1 - \cdots - z_d A_d)^{-1} B$

One can use formal noncommutative power series in a collection $z = (z_1, \dots, z_d)$ of noncommuting indeterminates to define a functional calculus for (possibly noncommutative) tuples of operators $\delta = (\delta_1, \dots, \delta_d)$ on a fixed Hilbert space $\mathcal{K}$ as follows. Namely, if $f(z) = \sum_{w \in \mathcal{F}_d} f_w z^w$ is a formal power series with coefficients $f_w \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ and if $\delta = (\delta_1, \dots, \delta_d)$ is a d-tuple of operators on the Hilbert space $\mathcal{K}$, we define $f(\delta) \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$ by

$$f(\delta) = \sum_{w \in \mathcal{F}_d} f_w \otimes \delta^w := \lim_{N \to \infty} \sum_{n=0}^{N} \left(\sum_{w \in \mathcal{F}_d \colon |w| = N} f_w \otimes \delta^w\right)$$

whenever the limit exists in a suitable operator topology, where here we use the convention

$$\delta^w = \delta_{i_N} \cdots \delta_{i_1} \text{ whenever } w = i_N \cdots i_1 \in \mathcal{F}_d.$$

Following [13], we then say that $\widehat{G}$ is in the noncommutative Schur-Agler class $\mathcal{SA}_{nc,d}(\mathcal{U}, \mathcal{Y})$ if it happens that $S(\delta)$ is defined and defines a contraction operator from $\mathcal{U} \otimes \mathcal{K}$ into $\mathcal{Y} \otimes \mathcal{K}$ for all δ in what we shall call the *open noncommutative polydisk* $\mathbb{D}^d_{nc}$, i.e., for all δ of the form $\delta = (\delta_1, \dots, \delta_d)$ where each δ_k is a strict contraction operator on the space $\mathcal{K}$ for each $k = 1, \dots, N$. We can now formulate the noncommutative H^∞-problem (state-space version). As for the nc-measurement-feedback stabilization problem, we suppose that we are given a noncommutative GR-system (23) with state-space of the form $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$ and we seek to design a controller, via the noncommutative GR-system (24), with state-space $\mathcal{X}_K = \mathcal{X}_{K1} \oplus \cdots \oplus \mathcal{X}_{Kd}$. The nc-$H^\infty$-problem then is: *design the controller, via a system matrix $J = \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right]$, so that that not only (a) A_{cl} is nc-Hautus stable, but also (b) the closed-loop transfer-function $T_{zw}(z) = D_{cl} + C_{cl}(I - Z_{cl}(z)A_{cl})^{-1} Z_{cl}(z) B_{cl}$ is in the noncommutative Schur-Agler class $\mathcal{S}_{nc,d}(\mathcal{W}, \mathcal{Z})$. In the strict version of the problem, one asks that T_{zw} be in the* strict Schur-Agler class, *i.e., $\|T(\delta)\| \le \rho$ (for some $\rho < 1$) for* *all δ in the closed noncommutative polydisk $\overline{\mathbb{D}}_{nc}^d$ consisting of operator tuples $\delta = (\delta_1, \dots, \delta_d)$ with $\|\delta_k\| \leq 1$ for each* $k = 1, \dots, d$. Remarkably, the solution is the same as for the scaled H^∞-problem formulated for the commutative setting, as summarized in the following theorem.

Theorem 3.3: (See Theorem 11.5 in [28].) *Suppose that we are given the GR-system matrix* $U^{GR} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}$ *for the system* (23) *with GR-decomposition* $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$ *of the state space. Then the associated strict nc-H^∞-problem has a solution* $J = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ *if and only if the LMIs* (14) *have structured solutions* $X = \operatorname{diag}[X_1, \dots, X_d]$ *and* $Y = \operatorname{diag}[Y_1, \dots, Y_d]$.

Sketch of the proof: Given a system matrix $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ acting between finite-dimensional spaces $\mathcal{X} \oplus \mathcal{W}$ and $\mathcal{X} \oplus \mathcal{Z}$, let $\mathbf{U} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$ denote the operator $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes I_{\mathcal{K}}$ acting between $(\mathcal{X} \otimes \mathcal{K}) \oplus (\mathcal{W} \otimes \mathcal{K})$ and $(\mathcal{X} \otimes \mathcal{K}) \oplus (\mathcal{Z} \otimes \mathcal{K})$. One can then use the Main Loop Theorem [62, page 284] to reduce the robust performance criterion (i.e., membership in the strict Schur-Agler class) to a NC-Hautus stability criterion: *a noncommutative formal power series $G(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ is in the strict noncommutative Schur-Agler class if and only if the operator pencil $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} Z(\delta) & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$ is invertible for all $\delta = (\delta_1, \dots, \delta_d) \in \overline{\mathbb{D}}_{nc}^d$ and Δ an arbitrary contraction operator on $\mathcal{K}$.* One can then use the result of [55], [50], [60] mentioned above (adapted to the case of uncertainty structures more general than the scalar structure associated with GR-systems) to get a *strict Bounded Real Lemma* for the strict noncommutative Schur-Agler class: *the operator pencil $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} Z(\delta) & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$ is invertible for all $\delta = (\delta_1, \dots, \delta_d) \in \overline{\mathbb{D}}_{nc}^d$* and Δ (i.e., $G(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ is in the strict Schur-Agler class) *if and only if there exist a structured positive definite matrix $X = \operatorname{diag}[X_1, \dots, X_d]$ so that* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0$.[2] In this way the nc-$H^\infty$-problem reduces to the scaled H^∞-problem discussed in Section II-B above. One can then follow the analysis of [9] for the scaled H^∞-problem to arrive at the solution criterion (14) (with structured X and Y) as also being the solution criterion for the strict nc-H^∞-problem.

B. Applications of the nc-H^∞-problem to 1-D systems

We mention two applications of the nc-H^∞-problem to 1-D systems which parallel those mentioned in Section II-C for the commutative case.

1) Noncommutative Linear Parameter Varying control: We suppose that we are given a 1-D plant as in (7) inducing an input-output map from $\ell^2_{\mathcal{W} \oplus \mathcal{U}}$ to $\ell^2_{\mathcal{Z} \oplus \mathcal{Y}}$. This input-output operator can be viewed as the feedback-connection of the system matrix U tensored with I_{ℓ^2} loaded with the shift operator S on $\mathcal{X} \otimes \ell^2$:

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes I_{\ell^2} \right) \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x' \\ z \end{bmatrix}$$
$$x = Sx'$$

To condense notation, let us set $\mathbf{U} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes I_{\ell^2}$. Analogous to what was done in Section II-C.1, we suppose that $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$ is subject to disturbances $\delta = (\delta_1, \dots, \delta_r)$ which we take to be general linear operators on $\mathcal{K} = \ell^2$ of norm at most 1. The functional dependence of the system matrix $\mathbf{U}$ on the disturbance parameter δ is taken to have the linear-fractional form

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{A}_{10} \\ \boldsymbol{C}_0 \end{bmatrix} (I - Z_p(\delta)\boldsymbol{A}_{00})^{-1} Z_p(\delta) \begin{bmatrix} \boldsymbol{A}_{01} & \boldsymbol{B}_0 \end{bmatrix}$$

for an appropriate coefficient matrix of the form

$$\begin{bmatrix} \boldsymbol{A}_{00} & \boldsymbol{A}_{01} & \boldsymbol{B}_0 \\ \boldsymbol{A}_{10} & \boldsymbol{A}_{11} & \boldsymbol{B}_1 \\ \boldsymbol{C}_0 & \boldsymbol{C}_1 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & B_0 \\ A_{10} & A_{11} & B_1 \\ C_0 & C_1 & D \end{bmatrix} \otimes I_{\ell^2}$$

with $Z_p(\delta) = \operatorname{diag}(I_{\mathcal{X}_{p1}} \otimes \delta_1, \dots, I_{\mathcal{X}_{pr}} \otimes \delta_d)$. Then we pose the noncommutative stabilization/H^∞-problem as that of designing a controller from $\mathcal{Y} \otimes \ell^2$ to $\mathcal{U} \otimes \ell^2$ with a similar functional dependence on δ

$$\left(\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \otimes I_{\ell^2} \right)(\delta) \begin{bmatrix} x_K \\ y \end{bmatrix} = \begin{bmatrix} x'_K \\ u \end{bmatrix}$$
$$x_K = S_K x'_K$$

(with S_K equal to the shift operator on $\ell^2_{\mathcal{X}_K}$) where $\left(\begin{bmatrix} A_K & B_K \\ C_K & D \end{bmatrix} \otimes I_{\ell^2} \right)(\delta)$ given by

$$\begin{bmatrix} \mathbf{A}_{K11} & \mathbf{B}_{K1} \\ \mathbf{C}_{K1} & \mathbf{D}_K \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{K10} \\ \mathbf{C}_{K0} \end{bmatrix} (I - Z_{Kp}(\delta)\mathbf{A}_{K00})^{-1} Z_{Kp}(\delta) \begin{bmatrix} \mathbf{A}_{K01} & \mathbf{B}_{K0} \end{bmatrix}$$

with coefficient matrix of the form

$$\begin{bmatrix} \mathbf{A}_{K00} & \mathbf{A}_{K01} & \mathbf{B}_{K0} \\ \mathbf{A}_{K10} & \mathbf{A}_{K11} & \mathbf{B}_{K1} \\ \mathbf{C}_{K0} & \mathbf{C}_{K1} & \mathbf{D}_K \end{bmatrix} = \begin{bmatrix} A_{K00} & A_{K01} & B_{K0} \\ A_{K10} & A_{K11} & B_{K1} \\ C_{K0} & C_{K1} & D_K \end{bmatrix} \otimes I_{\ell^2}$$

so that (a) the closed-loop state operator $\mathbf{A}_{cl}(\delta)$ is nc-Hautus stable for all $\delta \in \overline{\mathbb{D}}_{nc}^r$ and (b) the closed-loop input-output operator $T_{zw}(\delta)$ has operator norm at most ρ (for some $\rho < 1$) for all choices of $\delta \in \overline{\mathbb{D}}_{nc}^r$. By an analysis parallel to the discussion in Section II-C.1, one can see that this problem reduces to the nc-stabilization/H^∞-problem solved by Theorem 3.3, with U^{GR} taken to be

$$U^{GR} = \begin{bmatrix} A_{K00} & A_{K01} & B_{K0} \\ A_{K10} & A_{K11} & B_{K1} \\ C_{K0} & C_{K1} & D_K \end{bmatrix}.$$

It is fair to say that this noncommutative-LPV problem is not very realistic: it is hard to imagine physical systems where the controller has on-line measurement access to disturbances which are (infinite-dimensional) operators on ℓ^2. Nevertheless this point of view provides a control interpretation for the a priori artificial scaled H^∞-control problem.

[2]The paper [14] arrives at this strict Bounded Real Lemma via a different route using the realization theory for the noncommutative Schur-Agler class functions developed in [13] combined with the state-space similarity theorem for noncommutative GR-systems worked out in [12].

2) Robust control against time-varying uncertainty: We next consider the same scenario as discussed in the previous section for noncommutative-LPV control, but where we do not allow the controller to have access to the disturbances $\delta = (\delta_1, \ldots, \delta_r)$. Thus the form of the controller K is of the classical 1-D form:

$$\left(\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \otimes I_{\ell^2} \right) \begin{bmatrix} x_K \\ y \end{bmatrix} = \begin{bmatrix} x'_K \\ u \end{bmatrix}$$
$$x_K = S_K x'_K.$$

We wish to interconnect this controller with the disturbed plant $G(\delta)$ which has the form

$$\left(\begin{bmatrix} A_{00} & A_{01} & B_{01} & B_{02} \\ A_{10} & A_{11} & B_{11} & B_{12} \\ C_{10} & C_{11} & D_{11} & D_{12} \\ C_{20} & C_{21} & D_{21} & 0 \end{bmatrix} \otimes I_{\ell^2} \right) \begin{bmatrix} q \\ x \\ w \\ u \end{bmatrix} = \begin{bmatrix} p \\ x' \\ z \\ y \end{bmatrix},$$
$$q = Z_p(\delta),$$
$$x = Sx',$$

so that (a) $\mathbf{A}_{cl}(\delta)$ is nc-Hautus stable for all $\delta \in \overline{\mathbb{D}}^r_{nc}$, and (b) the closed-loop input-output operator $T_{zw}(\delta) \colon \ell^2_{\mathcal{W}} \to \ell^2_{\mathcal{U}}$ has operator norm at most ρ (for some $\rho < 1$) for all $\delta \in \overline{\mathbb{D}}^r_{nc}$. Another application of the Main Loop Theorem can be used to convert this problem to the nc-stabilization/H^∞-problem associated with the *augmented* plant G_{aug} with system matrix

$$U_{\text{aug}} = \begin{bmatrix} A_{00} & A_{01} & B_{01} & B_{02} \\ A_{10} & A_{11} & B_{11} & B_{12} \\ C_{10} & C_{11} & D_{11} & D_{12} \\ C_{21} & C_{21} & D_{21} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_p \\ \mathcal{X} \\ \mathcal{W} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_p \\ \mathcal{X} \\ \mathcal{Z} \\ \mathcal{Y} \end{bmatrix}$$

with state-space $\left[\begin{smallmatrix} \mathcal{X}_p \\ \mathcal{X} \end{smallmatrix} \right]$, but with the caveat that the controller state-space corresponding to the parameter state-space $\mathcal{X}_p$ be 0, i.e., $n_{Kp1} = \cdots = n_{Kpr} = 0$. There is a more refined version of Theorem 3.3 which allows for such additional restrictions (see [9] and [28, Theorem 11.5]): *if* $X = \operatorname{diag}[X_{p1}, \ldots, X_{pr}, X_0]$ *and* $Y = \operatorname{diag}[Y_{p1}, \ldots, Y_{pr}, Y_0]$ *are the structured positive definite solutions of the LMIs* (14)*, one demands the additional nonconvex constraints:*

$$\operatorname{rank} \left[\begin{smallmatrix} X_{pk} & I \\ I & Y_{pk} \end{smallmatrix} \right] = \dim \mathcal{X}_{pk} \text{ for } k = 1, \ldots, r.$$

A somewhat different solution criterion for this problem (also in the end nonconvex) is derived in [28] (see the discrete-time versions of conditions (9.7), (9.8), (9.9) there).

C. Noncommutative multidimensional linear systems: the frequency-domain

We have already introduced the noncommutative Schur-Agler class $\mathcal{SA}_{nc}(\mathcal{W}, \mathcal{Z})$ in Section III-A. An element $S(z)$ of $\mathcal{SA}_{nc,d} := \mathcal{SA}_{nc,d}(\mathbb{C}, \mathbb{C})$ is formally a power series $s(z) = \sum_{w \in \mathcal{F}_d} s_w z^w$ in noncommuting indeterminates $z = (z_1, \ldots, z_d)$ with scalar coefficients $s_w \in \mathbb{C}$ which induces a contractive $\mathcal{K}$-valued function (also denoted as s) on the noncommutative polydisk $\mathbb{D}^d_{nc}$ (d-tuples of contraction operators $\delta = (\delta_1, \ldots, \delta_d)$ on a fixed Hilbert space $\mathcal{K}$), while a strict noncommutative Schur-Agler-class function defines a strictly contractive $\mathcal{L}(\mathcal{K})$-valued function on the closed noncommutative polydisk $\overline{\mathbb{D}}^d_{nc}$. We may therefore identify strict Schur-Agler-class formal power series with the associated $\mathcal{L}(\mathcal{K})$-valued function defined on $\overline{\mathbb{D}}^d_{nc}$. It is then natural to consider the class of all scalar multiples of such functions, denoted here as $H^{\infty,o}_{nc,d}$, as the candidate for the class of stable elements for a feedback-control theory in the spirit of the fractional-representation approach of [61], [53]. The fractional representation approach for this ring is not completely straightforward since $H^{\infty,o}_{nc,d}$ is inherently not commutative. Nevertheless, if one restricts to "rational" $H^{\infty,o}_{nc,d}$, denoted by $\mathcal{R}H^{\infty,o}_{nc,d}$, namely, those functions having finite-dimensional noncommutative GR-state-space realizations, and considers the algebra generated by quotients of such functions with denominator function having invertible value at the zero operator-tuple $0 \in \overline{\mathbb{D}}^d_{nc}$, the resulting quotient algebra consists of functions with well defined value at least on a neighborhood of $0 \in \mathbb{D}^d_{nc}$ given by a noncommutative power series representation. One can then formulate a noncommutative frequency-domain version of the stabilization/H^∞-problem as in Section I-A. In particular, if G as in (7) and K as in (8) solve the state-space nc-stabilization/H^∞-problem as discussed in Section III-A and if we define $\widehat{G}(\delta) = \mathbf{D} + \mathbf{C}(I - Z(\delta)\mathbf{A})^{-1} Z(\delta)\mathbf{B}$ and $\widehat{K}(\delta) = \mathbf{D}_K + \mathbf{C}_K(I - Z_K(\delta)\mathbf{A})^{-1} Z(\delta)\mathbf{B}_K$, then $(\widehat{G}, \widehat{K})$ solve the frequency-domain version of the problem (again, since the 3×3-block noncommutative transfer-function $\Theta(G, K)$ has a noncommutative GR-realization with state matrix equal to A_{cl}). Conversely, if $(\widehat{G}(\delta), \widehat{K}(\delta))$ solves the frequency-domain version of the nc-stabilization/H^∞-problem and if $\widehat{G}$ and $\widehat{K}$ have realizations $\left[\begin{smallmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{smallmatrix} \right]$ and $\left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix} \right]$ with (A, B_2) and (A_K, B_K) nc-Hautus stabilizable and with (C_2, A) and $C_K, A_K)$ nc-Hautus detectable, it then follows that (G, K) (with these realizations) solves the state-space version of the problem [17, Theorem 6.10]. Left unresolved in [17] is whether such nc-Hautus stabilizable/detectable realizations exist; also left unresolved is the status of the Lin conjecture for this situation: if $\widehat{G}$ is stabilizable (in the noncommutative frequency-domain sense), does it follow that $\widehat{G}_{22}$ has a nc-double coprime factorization?

One can work with the Model-Matching version of the frequency-domain nc-H^∞-problem to arrive at a noncommutative Sarason interpolation problem [17, Section 6.2]. As stable elements of $\mathcal{R}H^{\infty,0}_{nc,d}$ have stable noncommutative GR-realizations, one can use Theorem 3.3 to formulate a solution of the noncommutative Sarason interpolation problem [17, Theorem 6.11]. Furthermore, as there is a strict Bounded Real Lemma for this setting, the passage from frequency-domain solution to state-space solution in the Model-Matching context is smooth: *if* $(\widehat{G}, \widehat{K})$ *is a solution of the noncommutative Sarason interpolation problem and* $G = \left[\begin{smallmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{smallmatrix} \right]$ *and* $J = \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix} \right]$ *are stable nc-GR realizations for* G *and* K*, then* (G, J) *solves the nc-*H^∞ *problem.*

Still unexplored, however, is how the Model-

Matching/Sarason version of the H^∞-problem relates to some noncommutative analogue of Nevanlinna-Pick interpolation such as e.g. the version studied in [10].

This area of noncommutative function theory and associated noncommutative state-space realizations is an active and evolving area of active research; we mention in particular [7], [43] as significant recent contributions.

IV. Conclusions

Mathematicians and engineers found much common ground in the early years of the H^∞ theory in the theory of Nevanlinna-Pick interpolation and its extensions to matrix- and operator-valued functions. Since then H^∞-theory and interpolation theory have moved on in the direction (among others) of multidimensional linear systems and multivariable functions, respectively, with new kinds of questions and problems. Independently of each other the two communities settled on separate compromise problems: the scaled H^∞-problem in the engineering community and interpolation in the Schur-Agler class (as opposed to the Schur class) in the mathematical community. One of the conclusions of our discussion here is that the interpretation of the scaled H^∞-problem as robust control against time-varying structured uncertainty and the extension of the Schur-Agler class to the noncommutative Schur-Agler class brings the state-space theory and the function theory back together, much as in the classical case.

We have mentioned here only Givone-Roesser type multidimensional linear systems (both commutative and noncommutative). There are also commutative and noncommutative versions of Fornasini-Marchesini multidimensional linear systems [32] and a general formalism for handling all of these as particular examples, motivated by the application in robust control against uncertainty structures (as in Sections II-C and III-B) more general than the scalar-block uncertainties to which we limited ourselves here (see [12], [13]). We expect that the points brought out here apply also in this more general framework.

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