A max-plus approach to the approximation of transient bounds for systems with nonlinear $\mathcal{L}_2$-gain

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***Abstract*— The notion of nonlinear $\mathcal{L}_2$-gain is a natural generalization of the extensively studied conventional (linear) $\mathcal{L}_2$-gain property that finds application in stability analysis and $\mathcal{H}_\infty$-control for nonlinear systems. As in the conventional formulation, notions of transient and gain bounds play an integral role in the statement of the property as an input / output inequality. These bounds summarize an imposed decoupling of system behaviour into transient and asymptotic parts, each of which are important in understanding and quantifying system performance. In this work, a variational approach to the characterization of transient bounds in the presence of a fixed nonlinear gain is considered. Based on an associated dynamic programming principle for this variational characterization, a max-plus eigenvector method for approximating tight transient bounds in the presence of a nonlinear $\mathcal{L}_2$-gain bound is considered. Convergence of the associated power method is considered in some detail, whilst it is shown that significant issues remain to be addressed in the approximation of the dynamic programming evolution operators associated with the attendant finite horizon optimization problems.**

I. Introduction

The notion of finite $\mathcal{L}_2$-gain plays a central role in nonlinear system analysis and control. It has been extensively investigated in the past two decades and becomes an important tool for nonlinear system analysis and synthesis [9], [11], [12]. Connections with other powerful notions such as dissipative systems theory [10], input-to-state stability (ISS) [6], [8] have been established. Where employed as a system design performance objective, $\mathcal{L}_2$-gain analysis naturally leads to the intensively studied $\mathcal{H}_\infty$-control methods [7], [11], which is to minimize the $\mathcal{L}_2$-gain from the external disturbance inputs. This conventional $\mathcal{L}_2$-gain with a finite number actually represents a linear upper bound function for the energy amplification from input to output. The fact that the output energy is usually related with the input energy for a general nonlinear system through a nonlinear gain function makes the conventional linear $\mathcal{L}_2$-gain bound function restrictive in many cases. The fact that many nonlinear systems do not possess finite $\mathcal{L}_2$-gain, yet retain asymptotic stability, motivates a generalization to nonlinear $\mathcal{L}_2$-gain bound function to make the gain based methodology applicable to more general nonlinear systems [4], [5], [14].

This paper is concerned with the approximation of the minimum transient bound for which the nonlinear $\mathcal{L}_2$-gain property holds for a specific (fixed) nonlinear gain bound. One way of computing this minimum transient bound is via the solution of a functional fixed-point relation involving a semigroup derived from the dynamic programming principle, or equivalently, the corresponding Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function of an optimal control problem. These equations were established in [14] and are generalizations of the results presented in [10] for linear $\mathcal{L}_2$-gain. The focus of this paper is in applying the max-plus eigenvector method developed in [10] to solve these nonlinear gain equations, hence obtaining an approximation to the minimum transient bound.

The use of max-plus algebra in systems and control has been investigated by several groups of researchers [1], [2], [3], [10]. Many nonlinear optimal control problems exhibit a linear structure in the max-plus algebra, making it an attractive paradigm for solving those problems. The essential idea of the max-plus eigenvector method in [10] is to transform a variational fixed-point relation into a max-plus eigenvector equation in the presence of an appropriately selected set of basis functions. It has been shown that the eigenvector equation can be solved via a simple power method under certain assumptions. The work presented in this paper proposes an extension in application of this max-plus eigenvector method to a more general class of problems. However, serious limitations are evident in the application of this approximation strategy in the problem formulation considered here, due entirely to specific constraints introduced by the consideration of nonlinear gains.

II. A nonlinear $\mathcal{L}_2$-gain property and associated minimum transient bound

Consider a nonlinear system of the form

$$\Sigma : \begin{cases} \dot{x}(t) &= f(x(t)) + g(x(t))w(t), \quad x(0) = x \\ z(t) &= h(x(t)), \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^s$ and $z(t) \in \mathbf{R}^l$ denote the state, input and output respectively, all at time $t \geq 0$. The input space is restricted to locally square integrable mappings from $\mathbf{R}_{\geq 0}$ to $\mathbf{R}^s$,

$$\mathcal{W} \doteq \left\{ w : [0,\infty) \to \mathbf{R}^s \;\middle|\; \begin{array}{c} w_{[0,T]} \in \mathcal{L}_2[0,T] \\ \forall\, T \geq 0 \end{array} \right\}.$$

Standard assumptions are imposed on functions f and g to guarantee existence and uniqueness of solutions given the initial condition $x \in \mathbf{R}^n$ and input $w \in \mathcal{W}$. Properties A4.1I, A4.2I and A4.3I on pages 39–40 of [10] are also assumed.

The following notion of nonlinear $\mathcal{L}_2$-gain [4] is a generalization of the well studied (linear) $\mathcal{L}_2$-gain property considered in [11], [12].

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Definition 2.1: System Σ of (1) has nonlinear $\mathcal{L}_2$-gain with transient / gain bound pair $(\beta,\gamma)\in\mathcal{K}^\infty\times\mathcal{K}^\infty$ if

$$\|z\|^2_{\mathcal{L}_2[0,T]}\leq\beta(|x|)+\gamma\left(\|w\|^2_{\mathcal{L}_2[0,T]}\right)\qquad(2)$$

for all $x\in\mathbf{R}^n$, $w\in\mathcal{L}_2[0,T]$, $T\geq 0$.

Denote $\Pi^\Sigma\subset\mathcal{K}^\infty\times\mathcal{K}^\infty$ to be the set of all possible nonlinear $\mathcal{L}_2$-gain transient / gain bound pairs for system Σ in (1). For any given $\gamma\in\mathcal{K}^\infty$, denote the set of all transient bound with respect to γ to be $\Pi^\Sigma(\gamma)\doteq\{\beta\in\mathcal{K}^\infty \text{ s.t. } (\beta,\gamma)\in\Pi^\Sigma\}$. Note it is possible that $\Pi^\Sigma(\gamma)=\varnothing$ for some $\gamma\in\mathcal{K}^\infty$. For $\gamma\in\mathcal{K}^\infty$, define

$$\beta^*_\gamma(\xi)\doteq\inf\left\{\beta(\xi)\;\middle|\;\beta\in\Pi^\Sigma\right\}.$$

Proposition 2.2: If β^*_γ is continuous, then it is the minimum transient bound with respect to γ.

For a given $\gamma\in\mathcal{K}^\infty$, define

$$\widetilde{W}(x)\doteq\sup_{T\geq 0}\sup_{w\in\mathcal{L}_2[0,T]}\left\{\|z\|^2_{\mathcal{L}_2[0,T]}-\gamma\left(\|w\|^2_{\mathcal{L}_2[0,T]}\right)\right\}.\qquad(3)$$

Proposition 2.3: Given any $\gamma\in\mathcal{K}^\infty$ such that $\Pi^\Sigma(\gamma)\neq\varnothing$,

$$\beta^*_\gamma(s)=\max_{|x|\leq s}W(x),\ \forall\ s\geq 0\,.$$

Hence, the computation of $\widetilde{W}$ is the key to obtaining the minimum gain bound β^*_γ. In attempting to apply dynamic programming as a first step in this regard, an essential difficulty associated with nonlinear gains is immediately apparent. In particular, as the nonlinear gain bound γ of (2) is not distributive with respect to addition, it is clear that

$$\gamma\left(\|w\|^2_{\mathcal{L}_2[0,T]}\right)\neq\gamma\left(\|w\|^2_{\mathcal{L}_2[0,\tau]}\right)+\gamma\left(\|w\|^2_{\mathcal{L}_2[\tau,T]}\right)$$

for $0\leq\tau\leq T$ and $w\in\mathcal{L}_2[0,T]$. Hence a dynamic programming principle (DPP) for $\widetilde{W}$ is not available in general. However, this difficulty can be overcome by the addition of dynamics that accumulate the $\mathcal{L}_2$-norm of the input, and by restricting the gain functions of interest to be differentiable [14]. An optimization problem that characterizes the minimal transient bound whilst admitting the application of dynamic programming then follows. Specifically, the value $W_a:\mathbf{R}^n\times\mathbf{R}_{\geq 0}\mapsto\mathbf{R}$ of this optimization problem,

$$W_a(x,\xi)=\sup_{T\geq 0}\sup_{w\in\mathcal{L}_2[0,T]}\left\{\int_0^T|z(t)|^2-\gamma'(\eta(t))|w(t)|^2dt\right\}\qquad(4)$$

is defined with respect to the augmented dynamics

$$\Sigma^a:\begin{cases}\begin{bmatrix}\dot{x}(t)\\\dot{\xi}(t)\end{bmatrix}=\begin{bmatrix}f(x(t))+g(x(t))w(t)\\|w(t)|^2\end{bmatrix},\\\begin{bmatrix}z(t)\\\eta(t)\end{bmatrix}=\begin{bmatrix}h(x(t))\\\xi(t)\end{bmatrix}\end{cases}\qquad(5)$$

with initial state $\begin{bmatrix}x(0)\\\xi(0)\end{bmatrix}=\begin{bmatrix}x\\\xi\end{bmatrix}$. For any $(x,\xi)\in\mathbf{R}^n\times\mathbf{R}_{\geq}0$, it has been shown [14] that

$$\widetilde{W}(x)=W_a(x,0).$$

Hence, the problem of computing $\widetilde{W}$ is transformed to computing W_a, which is readily interpreted as the available storage function [13] of system Σ^a with the supply rate

$$S(w,z,\eta)=\gamma'(\eta)|w|^2-|z|^2.$$

The DPP for W_a is then standard, requiring that

$$W_a(x,\xi)=S_T[W_a](x,\xi)\qquad(6)$$

hold for all $T\geq 0$. There, $S_T[\phi]$ denotes the corresponding DPP evolution operator applied to function $\phi:\mathbf{R}^n\times\mathbf{R}_{\geq 0}\mapsto\mathbf{R}$ on the interval $[0,T]$,

$$S_T[\phi](x,\xi)\doteq\sup_{w\in\mathcal{L}_2[0,T]}\left\{\int_0^T-S(w(t),z(t),\eta(t))\,dt\right.\\\left.+\phi(x(T),\xi(T))\right\}$$

The corresponding PDE satisfied by W_a is

$$0=|h(x)|^2+\nabla_xW(x,\xi)\cdot f(x)\\+\frac{\nabla_xW(x,\xi)g(x)g'(x)\nabla_xW(x,\xi)'}{4(\gamma'(\xi)-\nabla_\xi W(x,\xi))}$$
$$0>\nabla_\xi W(x,\xi)-\gamma'(\xi)$$

for all $(x,\xi)\in\mathbf{R}^n\times\mathbf{R}_{\geq 0}$, with boundary condition $W_a(0,0)=0$. Rather than solve the PDE (7), the objective of this paper is to solve the DPP directly via a max-plus power method [10], by interpreting W_a as the fixed-point of the DPP (6). The minimum transient gain may then by obtained as

$$\beta^*_\gamma(s)=\max_{|x|\leq s}W_a(x,0).$$

III. MAX-PLUS EIGENVECTOR METHOD

A. Max-plus eigenvalue equation

The principal reason for employing a max-plus method in solving equation (6) is that the semigroup $\mathcal{S}_T$ is linear under the max-plus algebra $(\mathbf{R}^-,\oplus,\otimes)$, where $\mathbf{R}^-=\mathbf{R}\cup-\infty$, $a\oplus b=\max\{a,b\}$ and $a\otimes B=a+b$. This max-plus linearity of $\mathcal{S}_T$, i.e.

$$\mathcal{S}_T[a\oplus W\otimes b]=a\oplus\mathcal{S}_T[W]\otimes b,\ \forall a,b\in\mathbf{R}^-$$

can be shown the same way as in [10] and is omitted here. As proved in [10], a group of (countable) max-plus basis functions for the space of semi-convex functions are

$$\psi_i(x,\xi)=-\frac{1}{2}C(|x-x_i|^2+|\xi-\xi_i|^2)\qquad(7)$$

where $C\in\mathbf{R}_{>0}$ and $\{(x_i,\xi_i),i=1,2,\cdots\}$ forms a dense set of $\mathbf{R}^{n+1}$. (The choice of C will be discussed later.) Hence, if W_a is semi-convex (as is the case in the linear $\mathcal{L}_2$-gain case), W_a may be expressed as a (max-plus) linear combination of these basis functions. Specifically,

$$W_a(x,\xi)=\bigoplus_{i=1}^{\infty}a_i\otimes\psi_i(x,\xi)\approx\bigoplus_{i=1}^{\nu}a_i\otimes\psi_i(x,\xi)$$

where the max-plus coordinates a_i are given by

$$a_i=-\max_{(x,\xi)\in B_R}\{\psi_i(x,\xi)-W_a(x,\xi)\},\ \forall\ i,$$

and $\nu < \infty$ denotes the number of terms used in a truncated series approximation. Denoting the truncated vector of coordinates by $a \in \mathbf{R}^\nu$ and an abuse of notation $\psi = [\psi_1, \psi_2, \cdots, \psi_\nu]'$ yields the compact notation $W_a \approx a' \otimes \psi$, where evaluation of the scalar product employs the standard max-plus operations. Similarly, let $B_j \in \mathbf{R}^\nu$ denote the vector of coordinates corresponding to $S_T[\psi_j]$ (again assuming semi-convexity), so that $S_T[\psi_j] \approx B_j' \otimes \psi$. Then, writing

$$\begin{aligned} S_T[\psi] &= \begin{bmatrix} S_T[\psi_1], & S_T[\psi_2], & \cdots & S_T[\psi_\nu] \end{bmatrix}' \\ &\approx \begin{bmatrix} B_1'\psi, & B_2'\psi, & \cdots B_\nu'\psi \end{bmatrix}' \\ &= B' \otimes \psi \end{aligned} \tag{8}$$

where the j^{th} column of matrix $B \in \mathbf{R}^{\nu\times\nu}$ is $B_j \in \mathbf{R}^\nu$. The DPP (6) then implies that

$$\begin{aligned} a' \otimes \psi \approx W_a &= S_T[W_a] \\ &\approx S_T\left[a' \otimes \psi\right] \\ &= a' \otimes S_T[\psi] \\ &\approx a' \otimes B' \otimes \psi\,. \end{aligned}$$

Uniqueness of the maximizer in the implicit $\oplus$ operations above imply [10] that the truncation approximations a and B are related by $a' = a' \otimes B'$, or more explicitly,

$$0 \otimes a = B \otimes a \tag{9}$$

Equation (9) states that the vector of interest a is an eigenvector of B corresponding to eigenvalue 0. This interpretation motivates the application of a power method to approximate the coordinate vector a.

B. Max-plus power method

The objective here is to show that the eigenvector a may be computed in an analogous way to [10] by

$$a = B^{\otimes N} \otimes 0$$

for some $N > 0$. In order to show this, it is necessary for B to satisfy some specific properties that ensure that the power method converges. Whilst similar in style to [10], the arguments used differ due to the unusual supply rate and dynamics under consideration here. To this end, the first lemma states positive energy must be used to generate a cycle in the state trajectory on a time interval $[0, T]$.

Lemma 3.1: For any $x \neq 0$, $T > 0$ fixed, there exist $M > 0, \kappa > 0$ such that

$$|x^w(T) - x| + M||w||_{\mathcal{L}_2[0,T]} \geq \kappa > 0 \tag{10}$$

for all $w \in \mathcal{L}_2[0, T]$, where $x^w(T)$ denotes the state at T, driven from an initial state $x^w(0) = x$ by input w.

Proof. The terminal state of the unforced trajectory,

$$x(T) = x + \int_0^T f(x(t))\, dt\,,$$

satisfies $|x(T) - x| > 0$ since $x \neq 0$ and $x = 0$ is the unique equilibrium point of f. For any $w \in \mathcal{L}_2[0, T]$, write the corresponding terminal state of the forced trajectory as

$$x^w(T) = x + \int_0^T f(x^w(t))\, dt + \int_0^T g(x^w(t))\, w(t)\, dt\,,$$

so that

$$\begin{aligned} &|x(T) - x^w(T)| \\ &\quad\leq \int_0^T |f(x(t)) - f(x^w(t))|\, dt \\ &\qquad\quad + \int_0^T |g(x^w(t))w(t)|\, dt \\ &\quad\leq K\int_0^T |x(t) - x^w(t)|dt + M_g\int_0^T |w(t)|\, dt\,, \end{aligned}$$

where K is the Lipschitz constant of dynamics f and M_g is the upper bound for g as in Assumption A4.1I-A4.3I in [10]. Applying Grownwall's inequality and Holder's inequality, note that (respectively)

$$|x(T) - x^w(T)| \leq M_g K e^{KT}\int_0^T |w(t)|\, dt\,,$$

$$\int_0^T |w(t)|\, dt \leq \left(\int_0^T |w(t)|^2\, dt\right)^{\frac{1}{2}} \left(\int_0^T 1\, dt\right)^{\frac{1}{2}}\,,$$

so that

$$|x(T) - x^w(T)| \leq \sqrt{\overline{M}}\|w\|_{\mathcal{L}_2[0,T]}\,,$$

where $\overline{M} \doteq (M_g K e^{KT})^2\, T$. Applying the triangle inequality, note that

$$\begin{aligned} |x - x^w(T)| &\geq |x - x(T)| - |x(T) - x^w(T)| \\ &= |x - x(T)| - \sqrt{\overline{M}}||w||_{\mathcal{L}_2[0,T]}\,. \end{aligned}$$

Hence (10) is proved with $\kappa = |x - x(T)| > 0$, $M = \sqrt{\overline{M}}$. □

Next, define the input space associated with a particular pair of initial and terminal states,

$$\mathcal{W}_x^y \doteq \left\{ w \in \mathcal{L}_2[0, T] \;\middle|\; \begin{array}{c} x^w(0) = x,\ x^w(T) = y, \\ \text{(5) holds} \end{array} \right\} \tag{11}$$

and

$$\mathcal{W}_{(x,\xi)}^{(\hat{x},\hat{\xi})} \doteq \left\{ w \in \mathcal{L}_2[0, T] \;\middle|\; \begin{array}{l} (x(0), \xi(0)) = (x, \xi), \\ (x(T), \xi(T)) = (\hat{x}, \hat{\xi}), \\ \text{(5) holds} \end{array} \right\}. \tag{12}$$

Corollary 3.2: For any $x \neq 0$, $T > 0$, there exists $\kappa > 0$ such that

$$||w||_{\mathcal{L}_2[0,T]} \geq \kappa,\ \forall\ w \in \mathcal{W}_x^x.$$

Define

$$Q((x, \xi), (\hat{x}, \hat{\xi})) \doteq \sup_{w\in\mathcal{W}_{(x,\xi)}^{(\hat{x},\hat{\xi})}} \int_0^T |z(t)|^2 - \gamma'(\eta(t))|w(t)|^2 dt \tag{13}$$

Lemma 3.3: 1) For all $(\hat{x}, \hat{\xi}) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$

$$Q((x, \xi), (\hat{x}, \hat{\xi})) \leq W_a(x, \xi), \tag{14}$$

and

$$Q((0, \xi), (0, \xi)) = 0,\ \forall\ \xi \geq 0; \tag{15}$$

2) For any fixed $x \neq 0$, there exist $M > 0, \kappa > 0$ such that

$$Q((x,\xi),(\hat{x},\hat{\xi})) = -\infty \tag{16}$$

for all $|x-\hat{x}| + M\sqrt{|\xi-\hat{\xi}|} < \kappa$.

Proof.

1) From the DPP of W_a

$$\begin{aligned} W_a(x,\xi) &= \sup_{(\hat{x},\hat{\xi})}\{Q((x,\xi),(\hat{x},\hat{\xi})) + W_a(\hat{x},\hat{\xi})\} \\ &\geq Q((x,\xi),(\hat{x},\hat{\xi})) + W_a(\hat{x},\hat{\xi}) \\ &\geq Q((x,\xi),(\hat{x},\hat{\xi})) \end{aligned}$$

where the last inequality follows from the fact $W_a \geq 0$. It follows that $Q((0,\xi),(0,\xi)) \leq W_a(0,\xi) - W_a(0,\xi) = 0$. On the other hand, from the definition

$$Q((0,\xi),(0,\xi)) \geq 0,$$

so $Q((0,\xi),(0,\xi)) = 0$.

2) For fixed $x \neq 0$, Lemma 3.1 says there exist $M > 0, \kappa > 0$ such that

$$|x - x(T)| + M||w||_{\mathcal{L}_{2[0,T]}} \geq \kappa, \ \forall\ w \in \mathcal{L}_{2[0,T]},$$

which means

$$\mathcal{W}_{(x,\xi)}^{(\hat{x},\hat{\xi})} = \varnothing$$

for $|x-\hat{x}| + M\sqrt{|\xi-\hat{\xi}|} < \kappa$. Hence (16) is proved. □

For the set of basis functions (7), we denote $\psi_i^C, i = 1,2,\cdots,\nu$ to be explicit on the Hessian C, and similarly B^C to be the matrix B of (8) depending explicitly on the Hessian C. Let the maximum point of ψ_1^C to be $(x_1,\xi_1) = (0,0)$. Denote $\mathcal{K} \subset \{1,2,\cdots,\nu\}$ to be $\mathcal{K} = \{i | x_i = 0\}$.

Theorem 3.4: There exists $\bar{C} > 0$ such that for all $C \geq \bar{C}$, $B_{11}^C = 0$, $B_{ii}^C \leq 0, i \in \{1,2,\cdots,\nu\}$, furthermore, there exists $\delta > 0$ such that, $B_{ii}^C \leq -\delta, i \notin \mathcal{K}$.

Proof. We prove $B_{ii}^C \leq -\delta$ for $i \notin \mathcal{K}$ first.

$$\begin{aligned} B_{ii}^C &= -\max_{(x,\xi)\in B_R}\{\psi_i^C(x,\xi) - \mathcal{S}_T[\psi_i^C](x,\xi)\} \\ &= \min_{(x,\xi)\in B_R}\{\mathcal{S}_T[\psi_i^C](x,\xi) - \psi_i^C(x,\xi)\} \\ &\leq \mathcal{S}_T[\psi_i^C](x_i,\xi_i) - \psi_i^C(x_i,\xi_i) \\ &= \mathcal{S}_T[\psi_i^C](x_i,\xi_i) \\ &= \sup_{(\hat{x},\hat{\xi})}\left\{ \begin{array}{l} Q((x_i,\xi_i),(\hat{x},\hat{\xi})) \\ -\frac{1}{2}C(|\hat{x}-x_i|^2 + |\hat{\xi}-\xi_i|^2) \end{array}\right\}. \end{aligned}$$

The claim will follow if we prove there exist $\bar{C} > 0, \delta > 0$ such that $\mathcal{S}_T[\psi_i^C](x_i,\xi_i) \leq -\delta, \forall i \notin \mathcal{K}$. Assume for the sake of contradiction, for any $\bar{C}, \delta > 0$, there exists $C > \bar{C}$ such that

$$\mathcal{S}_T[\psi_{i_0}^C](x_{i_0},\xi_{i_0}) > -\delta, \text{ for some } i_0 \notin \mathcal{K}.$$

Hence there is a sequence $C_n \uparrow \infty$ and $i_0 \notin \mathcal{K}$ (note $\mathcal{K}$ is a finite set) such that

$$\mathcal{S}_T[\psi_{i_0}^{C_n}](x_{i_0},\xi_{i_0}) > -\frac{1}{n}, \ \forall\ n \geq 1.$$

Let $(\hat{x}_n,\hat{\xi}_n) \in \mathbf{R}^n$ be such that

$$\begin{aligned} &\mathcal{S}_T[\psi_{i_0}^{C_n}](x_{i_0},\xi_{i_0}) < Q((x_{i_0},\xi_{i_0}),(\hat{x}_n,\hat{\xi}_n)) - \\ &\quad \tfrac{1}{2}C_n(|\hat{x}_n - x_{i_0}|^2 + |\hat{\xi}_n - \xi_{i_0}|^2) + \tfrac{1}{n}, n = 1,2,\cdots, \end{aligned}$$

then $(\hat{x}_n,\hat{\xi}_n) \to (x_{i_0},\xi_{i_0}), n \to \infty$. Since otherwise (for some subsequence)

$$\begin{aligned} &Q((x_{i_0},\xi_{i_0}),(\hat{x}_n,\hat{\xi}_n)) > \mathcal{S}_T[\psi_{i_0}^{C_n}](x_{i_0},\xi_{i_0}) \\ &\quad + \tfrac{1}{2}C_n(|\hat{x}_n - x_{i_0}|^2 + |\hat{\xi}_n - \xi_{i_0}|^2) - \tfrac{1}{n} \to \infty, \end{aligned}$$

which contradicts (14). Since $x_{i_0} \neq 0$, (16) implies that for those C_n such that $|x_{i_0} - \hat{x}_n| + M\sqrt{|\xi_{i_0} - \hat{\xi}_n|} < \kappa$, $Q((x_{i_0},\xi_{i_0}),(\hat{x}_n,\hat{\xi}_n)) = -\infty$ which makes $\mathcal{S}_T[\psi_{i_0}^{C_n}](x_{i_0},\xi_{i_0}) = -\infty$ which is impossible. Hence $\mathcal{S}_T[\psi_i^C](x_i,\xi_i) < -\delta$ for some $\delta > 0$.

We next show $B_{kk}^C \leq 0, \forall\ k \in \mathcal{K}$. For these k, we show $\mathcal{S}_T[\psi_k^C](0,\xi_k) \leq 0$. Assume instead $\mathcal{S}_T[\psi_{k_0}^{C_n}](0,\xi_{k_0}) > 0$ for some $k_0 \in \mathcal{K}$ for a sequence $C_n \uparrow \infty$, as $n \to \infty$. Let $(\hat{x}_n,\hat{\xi}_n) \in \mathbf{R}^n$ be such that

$$\begin{aligned} &\mathcal{S}_T[\psi_{k_0}^{C_n}](0,\xi_{k_0}) < Q((0,\xi_{k_0}),(\hat{x}_n,\hat{\xi}_n)) \\ &\quad - \tfrac{1}{2}C_n(|\hat{x}_n|^2 + |\hat{\xi}_n - \xi_{k_0}|^2) + \tfrac{1}{n}, n = 1,2,\cdots. \end{aligned}$$

By the boundedness of $\mathcal{S}_T[\psi_{k_0}^{C_n}](0,\xi_{k_0})$ and $Q((0,\xi_{k_0}),(\hat{x}_n,\hat{\xi}_n))$, it must be true that

$$\hat{\xi}_n \geq \xi_{k_0}, \hat{x}_n \to 0, \hat{\xi}_n \to \xi_{k_0}, n \to \infty.$$

While on the other hand,

$$Q((0,\xi_{k_0}),(\hat{x}_n,\hat{\xi}_n)) \to 0, \text{ as } \hat{x}_n \to 0, \hat{\xi}_n \to \xi_{k_0}$$

from the definition of Q. Hence there exists $\bar{C}$ such that for all $C_n \geq \bar{C}$

$$\begin{aligned} &\mathcal{S}_T[\psi_{k_0}^{C_n}](0,\xi_{k_0}) \\ &\quad < Q((0,\xi_{k_0}),(\hat{x}_n,\hat{\xi}_n)) - \tfrac{1}{2}C_n(|\hat{x}_n|^2 + |\hat{\xi}_n - \xi_{k_0}|^2) + \tfrac{1}{n} \\ &\quad < \frac{\mathcal{S}_T[\psi_{k_0}^{C_n}](0,\xi_{k_0})}{2}, \end{aligned}$$

which is a contradiction.

We are left to show $B_{11}^C \geq 0$ to prove $B_{11}^C = 0$. For $x \in \mathbf{R}^n$, denote $x(T) = \int_0^T f(x(t)dt)$ the state at T for the unforced dynamics. From the stability of the unforced dynamics, we know there exists $T > 0$ such that

$$|x(T)| \leq |x|, \ \forall\ (x,\xi) \in B_R.$$

Also it is clear from the definition of Q that $Q((x,\xi),(x(T),\xi)) \geq 0$. Then

$$\begin{aligned} B_{11}^C &= \min_{(x,\xi)\in B_R}\{\mathcal{S}_T[\psi_1^C](x,\xi) - \psi_1^C(x,\xi)\} \\ &= \min_{(x,\xi)\in B_R}\sup_{(\hat{x},\hat{\xi})}\left\{ \begin{array}{l} Q((x,\xi),(\hat{x},\hat{\xi})) - \\ \frac{1}{2}C(\hat{x}^2 + \hat{\xi}^2) \\ +\frac{1}{2}C(x^2 + \xi^2) \end{array}\right\} \\ &\geq \min_{(x,\xi)\in B_R}\left\{ \begin{array}{l} Q((x,\xi),(x(T),\xi)) \\ -\frac{1}{2}C((x(T))^2 + \xi^2) \\ +\frac{1}{2}C(x^2 + \xi^2) \end{array}\right\} \\ &\geq \min_{(x,\xi)\in B_R}\left\{ \begin{array}{l} Q((x,\xi),(x(T),\xi)) \\ -\frac{1}{2}C((x(T))^2 - x^2) \end{array}\right\} \\ &\geq 0. \end{aligned}$$

□

Theorem 3.5: Let $N \in \{1, 2, \cdots, \nu\}$, $\{k_i\}_{i=1}^{i=N+1}$ be such that $1 \le k_i \le \nu$ for all i and $k_{N+1} = k_1$. Suppose we are not in the case $k_i = k \in \mathcal{K}$ for all i. Then there exists $\bar{C} > 0, \delta > 0$ such that for all $C > \bar{C}$

$$\sum_{i=1}^{N} B^C_{k_i k_{i+1}} \le -\delta.$$

Proof. We show the case for $N = 2$ first, that is,

$$B^C_{ij} + B^C_{ji} \le -\delta.$$

Since we are not in the case $i = j \in \mathcal{K}$, it must hold

$$|x_i| + |x_j| + |\xi_i - \xi_j| > 0. \tag{17}$$

The case $i = j \notin \mathcal{K}$ has already been proved in Theorem 3.4. We only need to prove the theorem when $i \neq j$. We have shown

$$\begin{aligned} B^C_{ij} &\le \mathcal{S}_T[\psi^C_j](x_i, \xi_i) \\ &= \sup_{(\hat{x},\hat{\xi})} \left\{ \begin{array}{l} Q((x_i, \xi_i), (\hat{x}, \hat{\xi})) \\ -\frac{1}{2}C(|\hat{x} - x_j|^2 + |\hat{\xi} - \xi_j|^2) \end{array} \right\}. \end{aligned}$$

We show there exist $\bar{C} > 0, \delta > 0$ such that

$$\mathcal{S}_T[\psi^C_j](x_i, \xi_i) + \mathcal{S}_T[\psi^C_i](x_j, \xi_j) \le -\delta, \ \forall\ C \ge \bar{C}.$$

Assume for the sake of contradiction, for any $\bar{C}, \delta > 0$, there are i_0, j_0 such that

$$\mathcal{S}_T[\psi^C_{j_0}](x_{i_0}, \xi_{i_0}) + \mathcal{S}_T[\psi^C_{i_0}](x_{j_0}, \xi_{j_0}) > -\delta, \text{ for } C \ge \bar{C}.$$

So there exists a sequence $C_n \uparrow \infty, i_0, j_0$ such that

$$\mathcal{S}_T[\psi^{C_n}_{j_0}](x_{i_0}, \xi_{i_0}) + \mathcal{S}_T[\psi^C_{i_0}](x_{j_0}, \xi_{j_0}) > -\frac{1}{2n}$$

for all C_n. Denote $(\hat{x}_n, \hat{\xi}_n)$ be such that

$$\begin{aligned} \mathcal{S}_T[\psi^{C_n}_{j_0}](x_{i_0}, \xi_{i_0}) &< Q((x_{i_0}, \xi_{i_0}), (\hat{x}_n, \hat{\xi}_n)) \\ &\quad - \tfrac{1}{2}C_n(|\hat{x}_n - x_{j_0}|^2 + |\hat{\xi}_n - \xi_{j_0}|^2) + \tfrac{1}{2n} \end{aligned}$$

Since Q is bounded from above, it must hold $(\hat{x}_n - x_{j_0})^2 + (\hat{\xi}_n - \xi_{j_0})^2 \to 0$, which implies $(\hat{x}_n, \hat{\xi}_n) \to (x_{j_0}, \xi_{j_0})$.

Similarly, if let $(\check{x}_n, \check{\xi}_n)$ be suboptimal for $\mathcal{S}_T[\psi^C_{i_0}](x_{j_0}, \xi_{j_0})$, it holds $(\check{x}_n, \check{\xi}_n) \to (x_{i_0}, \xi_{i_0})$ and subsequently

$$\begin{aligned} &\mathcal{S}_T[\psi^{C_n}_{j_0}](x_{i_0}, \xi_{i_0}) + \mathcal{S}_T[\psi^C_{i_0}](x_{j_0}, \xi_{j_0}) < \\ &\quad Q((x_{i_0}, \xi_{i_0}), (\hat{x}_n, \hat{\xi}_n)) + Q((x_{j_0}, \xi_{j_0}), (\check{x}_n, \check{\xi}_n)) \\ &\quad - \tfrac{1}{2}C_n((|\hat{x}_n - x_{j_0}|^2 + |\hat{\xi}_n - \xi_{j_0}|^2) \\ &\quad + (|\check{x}_n - x_{i_0}|^2 + |\check{\xi}_n - \xi_{i_0}|^2)) + \tfrac{1}{n}. \end{aligned}$$

We show it must hold $\xi_{i_0} = \xi_{j_0}$. Otherwise (without loss of generality assume $\xi_{i_0} > \xi_{j_0}$) there exist $\bar{C}$ such that $\hat{\xi}_n < \xi_i,\ C_n > \bar{C}$. Hence $Q((x_{i_0}, \xi_{i_0}), (\hat{x}_n, \hat{\xi}_n)) = -\infty$, which is impossible. Then from (17) one obtains $|x_{i_0}| + |x_{j_0}| > 0$. Assume without loss of generality $x_{i_0} \neq 0$.

From Lemma 3.1, one knows it holds for $M = (M_g K e^{2KT})\sqrt{2T}$ and $\kappa = |x_{i_0} - x(2T)|$

$$|x_{i_0} - \check{x}_n| + M||w||_{\mathcal{L}_{2[0,2T]}} \ge \kappa$$

for all $w \in \mathcal{W}^{\check{x}_n}_{x_{i_0}}$. On the other hand, since $\hat{x}_n \to x_{j_0}$, there exist T_n and w^n_2 satisfying $T_n \to 0$ and $||w^n_2|| \to 0$ such that w^n_2 drives $\hat{x}_n$ to x_{j_0} and $T_n \to 0$. Take any $w^n_1 \in \mathcal{W}^{(\hat{x}_n,\hat{\xi}_n)}_{(x_{i_0},\xi_{i_0})}$ and $w^n_3 \in \mathcal{W}^{(\check{x}_n,\check{\xi}_n)}_{(x_{j_0},\xi_{j_0})}$, define $w^n \in \mathcal{W}^{\check{x}_n}_{x_{i_0}}$ on interval $[0, 2T + T_n]$ to be

$$w^n(t) = \begin{cases} w^n_1(t), & 0 \le t < T \\ w^n_2(t - T), & T \le t < T + T_n \\ w^n_3(t - T - T_n), & T + T_n \le t \le 2T + T_n. \end{cases}$$

So for all n, from Lemma 3.1 again

$$\begin{aligned} ||w^n_1||_{\mathcal{L}_{2[0,T]}} &+ ||w^n_2||_{\mathcal{L}_{2[0,T_n]}} + ||w^n_3||_{\mathcal{L}_{2[0,T]}} \\ &= ||w^n||_{\mathcal{L}_{2[0,2T+T_n]}} \\ &\ge \frac{|x_{i_0} - x(2T + T_n)| - |x_{i_0} - \check{x}_n|}{(M_g K e^{K(2T+T_n)})\sqrt{2T + T_n}}, \end{aligned}$$

Now $|x_{i_0} - x(2T + T_n)| \to \kappa$ and $(M_g K e^{K(2T+T_n)})\sqrt{2T + T_n} \to M$ as $n \to \infty$ since $T_n \to 0$. Hence if taking $\bar{C}$ such that for all $C_n \ge \bar{C}$

$$|x_{i_0} - x(2T + T_n)| > \frac{3}{4}\kappa,$$

$$|x_{i_0} - \check{x}_n| < \frac{1}{4}\kappa,$$

$$(M_g K e^{K(2T+T_n)})\sqrt{2T + T_n} < 2M$$

and

$$||w^n_2||_{\mathcal{L}_{2[0,T_n]}} < \frac{\kappa}{8M}.$$

One obtains

$$||w^n_1||_{\mathcal{L}_{2[0,T]}} + ||w^n_3||_{\mathcal{L}_{2[0,T]}} \ge \frac{1}{8}\frac{\kappa}{M}$$

which contradicts the fact

$$||w^n_1||_{\mathcal{L}_{2[0,T]}} + ||w^n_3||_{\mathcal{L}_{2[0,T]}} \le (\hat{\xi}_n - \xi_{i_0}) + (\check{\xi}_n - \xi_{j_0}) \to 0.$$

For the case $N > 0$ can be shown exactly the same as done in [10].

□

Denote a directed graph $G(B)$ to be the communication graph of B, i.e. the transition cost from node i to j is B_{ij}. For a loop $\vartheta = (k_1, k_2, \cdots, k_{N+1}), k_1 = k_{N+1}, 1 \le N \le \nu$ with the length $\lambda(\vartheta) = N$, denote the transition cost

$$\rho(\vartheta) = \sum_{i=1}^{N+1} B_{k_i k_{i+1}}.$$

Denote Θ the set of all loops in $G(B)$ with maximum length ν, that is

$$\Theta = \{\lambda(\vartheta) \le \nu,\ \vartheta \text{ is a loop}\}.$$

Lemma 3.6: Given B, assume $\max_{i \in \{1,2,\cdots,\nu\}} B_{ii} = 0$ and $\rho^* = \max_{\vartheta \in \Theta, \lambda(\vartheta) > 1} \rho(\vartheta) < 0$, then there exists $N^\dagger > 0$ such that

$$B^{\otimes N} = B^{\otimes N^\dagger},\ \forall\ N \ge N^\dagger.$$

Proof. Denote $\mathcal{K} = \{0 \le i \le \nu : B_{ii} = 0\}$, Ψ_i be the maximum cost of all paths from node i to any other node, then

$$\Psi(i) \ge B_{ik},\ k \in \mathcal{K},$$

hence

$$\min_i \Psi(i) \geq \min_i \max_{k\in\mathcal{K}} B_{ik} > -\infty.$$

We first show the number of nontrivial loops in all optimal paths without trivial loops are uniformly bounded. Let C_B be the maximum total transition cost of all paths from any node to any other node without loops. It is clear $C_B < \infty$. Denote N_ν be the largest integer less than or equal to N/ν. For an optimal path of length N, there are at least N_ν loops. So the cost for any optimal path without trivial loops is at most $C_B - N_\nu \rho^*$. Hence

$$C_B - N_\nu \rho^* \geq \min_i \Psi(i) \geq \min_i \max_{k\in\mathcal{K}} B_{ik},$$

which implies

$$N_\nu \leq \frac{(C_B - \min_i \max_{k\in\mathcal{K}} B_{ik})}{\rho^*}.$$

So there must be trivial loops for any optimal path of length $N \geq N^\dagger$ such that

$$N_\nu^\dagger > \frac{(C_B - \min_i \max_{k\in\mathcal{K}} B_{ik})}{\rho^*} + 1.$$

That is, the path has to pass some $k \in \mathcal{K}$. Since all nontrivial loops has negative cost, we can delete all the nontrivial loops and add more trivial loops to make a bigger total cost with the same path length. That is, the optimal path will only have trivial loops. Since $(B^\otimes N)_i$ is the maximum cost of length N starting from node i, we know

$$B^{\otimes N} = B^{\otimes N^\dagger}, \ N \geq N^\dagger.$$

□

Theorem 3.7: There exists $\bar{C} > 0, N^\dagger > 0$ such that for all $C \geq \bar{C}, N > N^\dagger$, $e = B^{\otimes N} \otimes 0$ satisfies $e = B \otimes e$.

Proof. It has been established in Theorem 3.4 and Theorem 3.5 that there exists $\bar{C} > 0$ such that for all $C \geq \bar{C}$, the communication graph of B matrix satisfies all the conditions in Lemma 3.6. There is $N^\dagger$ such that for all $N > N^\dagger$, $B^{\otimes N} = B^{\otimes(N-1)}$, so

$$e = B^{\otimes N} \otimes 0 = B \otimes B^{\otimes(N-1)} \otimes 0 = B \otimes e.$$

□

IV. Computation of Matrix B

A. *Towards an approximation for $\mathcal{S}_T[\psi_j]$*

The main computational effort associated with the max-plus eigenvector method is in approximating the B matrix according to (8). This is due to the fact that the computation of the j^{th} column of B requires an approximate solution to the finite horizon optimal control problem defined by $\mathcal{S}_T[\psi_j]$. The nature of the approximation used largely determines the computational effort. Motivated by the approximations employed in [10], a Taylor series approach is considered here. Defining $V(T,x,\xi) \doteq \mathcal{S}_T[\psi_j](x,\xi)$, standard dynamic programming arguments point to a non-stationary PDE of the form

$$\frac{\partial V}{\partial T}(T,x,\xi) = |h(x)|^2 + \nabla_x V(T,x,\xi) + \frac{\nabla_x V(T,x,\xi) g(x) g(x)' \nabla_x V(T,x,\xi)'}{4\,(\gamma'(\xi) - \nabla_\xi V(T,x,\xi))} \tag{18}$$

subject to the constraints

$$\begin{aligned} 0 &> \nabla_\xi V(T,x,\xi) - \gamma'(\xi)\,, \\ 0 &= V(0,x,\xi) - \psi_j(x,\xi)\,. \end{aligned} \tag{19}$$

Assuming for a moment that V is adequately smooth, a first order Taylor series approximation for V with respect to T yields that

$$\begin{aligned} \mathcal{S}_T[\psi_j](x,\xi) = V(T,x,\xi) &\approx V(0,x,\xi) + T\left(\frac{\partial V}{\partial T}(0,x,\xi)\right) \\ &\approx \psi_j(x,\xi) + T\Bigg(|h(x)|^2 + \nabla_x \psi_j(x,\xi) \\ &\quad + \frac{\nabla_x \psi_j(x,\xi) g(x) g(x)' \nabla_x \psi_j(x,\xi)'}{4\,(\gamma'(\xi) - \nabla_\xi \psi_j(x,\xi))}\Bigg) \end{aligned}$$

The inequality constraint of (19) implies the additional requirement that

$$0 > \nabla_\xi \psi_j(x,\xi) - \gamma'(\xi)\,. \tag{20}$$

In the case of a quadratic basis,

$$\psi_j(x,\xi) = -\frac{1}{2}(x-\bar{x}_j)' C_j (x-\bar{x}_j) - \frac{q_j}{2}(\xi - \bar{\xi}_j)^2$$

with $C_j > 0$, $q_j \in \mathbf{R}_{>0}$, this condition obviously requires that $\gamma'(\xi) + q_j(\xi - \bar{\xi}_j) > 0$ for all ξ in the domain of interest. With γ strictly increasing, this condition is trivially satisfied for $\xi \geq \bar{\xi}_j$. However, for $\xi < \bar{\xi}_j$, an upper bound is imposed on $q_j \in \mathbf{R}_{>0}$. Denoting $I \subset \mathbf{R}_{>0}$ to be the interval of interest for computation of V in the ξ direction,

$$q_j < q_j^* \doteq \inf_{\xi\in I} \left\{ \frac{\gamma'(\xi)}{\bar{\xi}_j - \xi} \right\}. \tag{21}$$

In the linear gain case where $\gamma(s) = \bar{\gamma}^2 s$, this upper bound is $q_j^* = \bar{\gamma}^2/\bar{\xi}_j$. This upper bound is particularly significant in any hypothesis concerning the smoothness of V. To see this, consider the cost function J, where $V(T,x,\xi) = \sup_{w\in\mathcal{L}_2[0,T]} J(T,x,\xi;w)$,

$$\begin{aligned} J(T,x,\xi;w) &= \int_0^T -S(w(t),z(t),\eta(t))\,dt + \psi_j(x(T),\xi(T)) \\ &= \|z\|^2_{\mathcal{L}_2[0,T]} - \gamma(\xi(T)) + \gamma(\xi) + \psi_j(x(T),\xi(T))\,, \end{aligned}$$

so that

$$V(T,x,\xi) - \gamma(\xi) = \sup_{w\in\mathcal{L}_2[0,T]} \left\{ \|z\|^2_{\mathcal{L}_2[0,T]} + \theta_j(x(T),\xi(T)) \right\}$$

where $\theta_j(x,\xi) \doteq \psi_j(x,\xi) - \gamma(\xi)$. Note that $\theta_j(x,\xi) \leq 0$ for all $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$. Furthermore, where $\xi \geq \bar{\xi}_j$, this terminal cost is non-increasing in ξ, so that the input is penalized. However, if $\xi < \bar{\xi}_j$ and θ_j is increasing with respect to ξ, no such penalty is applied. This leads to a lack of continuity in V with respect to T, with

$$\lim_{T\to 0^+} V(T, x, \xi) \neq \psi_j(x, \xi)\,,$$

rendering application of the proposed Taylor series approximation invalid. However, this problem is avoided if θ_j is decreasing with respect to ξ – a property that is guaranteed by the inequality constraint of (19), (20).

B. Example

Notwithstanding the serious limitations highlighted above, a simple example is considered for illustration. In particular, a scalar linear system with nonlinear gain is considered, with $f(x) = -2\,x$, $g(x) = 1$, $h(x) = 1$, and a nonlinear gain bound γ as illustrated in Figure 1. An approximation for W_a is computed using 189 quadratic basis functions defined on the rectangular region $\{(x, \xi) \mid |x| \le 10,\, \xi \in [0, 1]\}$, where the q_j's are selected in view of (21). The resulting approximation is shown in Figure 2. Should a linear gain bound be used, $W_a(x, \xi)$ recovers the standard ARE solution in the x direction, whilst being invariant in the ξ direction.

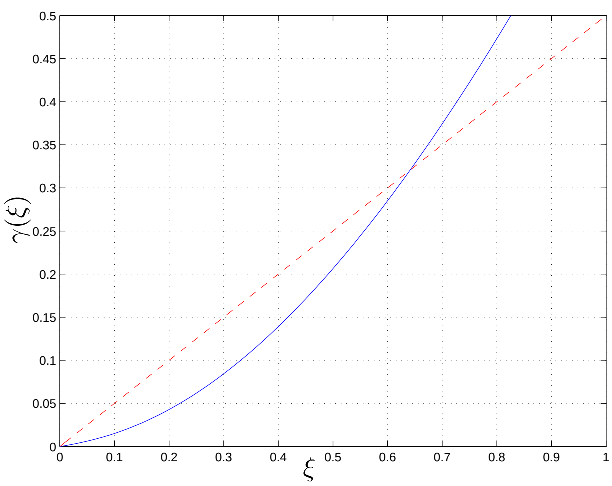

Fig. 1. Nonlinear gain bound γ

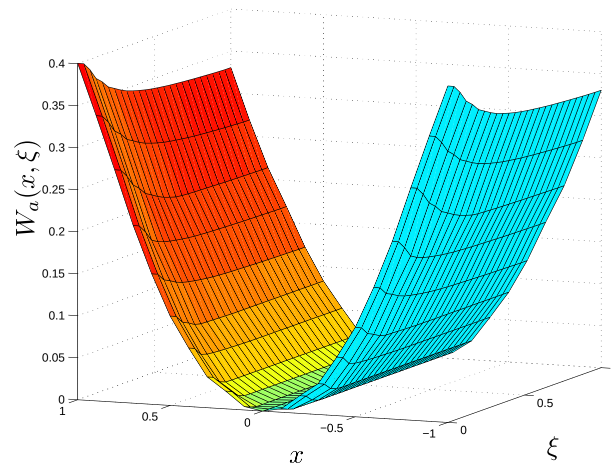

Fig. 2. Approximation for W_a

V. Conclusion

A max-plus eigenvector method for the approximation of transient bounds for systems with nonlinear $\mathcal{L}_2$-gain was explored. Specific convergence properties concerning the underlying power method were demonstrated. However, significant hurdles remain to be addressed in the approximation of the dynamic programming evolution operator associated with the attendant underlying finite horizon optimization problems.

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