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Control of the Observation matrix for Control Purposes

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***Abstract*— How to control the activiation of an expensive observation channel of a stochastic system? The control objective is to reduce the conditional error variance of state estimation but a cost is to be paid for acquiring a reading of the channel. The optimal control law depends only on the conditional error variance and has to be determined computationally. A second problem is to use the state estimate for control of the conditional mean. The solution method is stochastic control with partial observations.**

I. INTRODUCTION

This paper presents a partial information control problem for a stochastic system in which the input also directly influences the conditional error variance of the state estimate. The contribution of the paper is towards the fundamental issues relating information and control.

The motivation for this research comes from several control engineering problems for distributed systems. At the University of Porto the coordination of several underwater and aerial vehicles is studied. One underwater vehicle can request information of another such vehicle, of a surface vehicle, or of an aerial vehicle. The information requested may include the position and speed of the vehicle, but it may also involve recent observations by the vehicles, and it may have an influence on the trajectories followed by the vehicles in the near future. The control objective of limiting the use of battery power is very important in this situation for some references on this application of coordination of multi-vehicle systems see [8]. Similar issues of information exchange and coordination control of several aerial vehicles are studied at the University of Cyprus. In ad hoc networks each node may request from neighboring nodes information on their state, including information on the number of messages waiting and on its estimate of the different arrival intensities.

In each of the examples mentioned above, the controller of a vehicle or node may influence the system behavior in two ways: by control action it modifies the evolution of the state of plant itself, and moreover it can request request additional observations of the system it is controlling. Both control actions are costly. But the additional information may improve the state estimate and thus decreases the achievable minimal cost of the partial information LQG problem [10]. The goal of the paper is to study the trade-off between the additional cost due to taking measurements versus the gains achievable thanks to the improvement in the state estimate.

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The problem is thus: When should additional observations be requested so as to better achieve the control objectives? In this paper a canonical example of this problem will be formulated and solved. Attention is restricted to a discrete time linear Gaussian stochastic control system. The control value $u(t)$ influences both the state update equation through the standard $Bu(t)$ term as well as the quality of the observations through the control dependent observation matrix $C(u(t))$ In this paper we first deal with the control objective of minimizing the sum of the trace of the conditional error variance of the state estimate and the cost of requesting the observations. In a second problem the obtained information is also used to control the system, and we consider a cost that also depends on how well the state of the linear plant follows its reference trajectory.

The problem of feedback control of the observations matrix was first treated in the Ph.D. thesis of Khanna [9]. Rajesh Bansal and Tamer Basar established closely related results, see [3], [4]. Relevant contributions were developed by D. Teneketzis and his students. The paper [2] discusses a case in which the feedback is open loop. The problem treated in this paper was not found in the literature.

In this paper the following problem will be discussed. Consider a discrete-time linear Gaussian stochastic control system starting at an initial state with a Gaussian probability distribution and driven by a linear feedback control input $Bu(t)$ which depends on the state estimate $\hat{x}(t)$ and driven by a Gaussian white noise process. The system differs from the standard one in that the observation matrix $C(u(t))$ depends on the input. In other words the control action can improve the signal to noise ratio by activating an expensive sensor.

There follows a summary of the paper by section. The next section, Section II, provides motivation of the problem including examples of how the results of this paper can be used for the coordination control two interacting systems with limited communication capabilities. Section III presents results for control of the conditional variance based on the past outputs and the past inputs. Section IV presents results on the combined control of the conditional mean and the conditional variance. Concluding remarks are stated in Section V.

II. MOTIVATION

In this section we will try to illustrate how the problems treated in the remainder of the paper can help in solving coordination control problems. Consider several autonomously controlled agents that each execute a task assigned by a supervisor. These agents could control autonomous vehicles taking measurements $y(t, x(t))$ describing the environment in an area A they travel through. Typically the supervisor assigns a reference trajectory $x_{ref}(t) \in A$ that is expected to provide as much information as possible in as short a time as possible. The autonomous agents must collect as much information as possible at the same time avoiding collisions with obstacles and with other autonomous vehicles. These autonomous agents might for example be UAVs detecting forest fires, or AUVs observing oil spills in the sea.

Based on the information obtained, the reference trajectories may have to be adjusted. However communication with the supervisor may not always be possible and therefore the autonomous agents must from time to time (say at the possibly controllable times T_n) exchange the large information set $z(T_n; y(\tau, x(\tau)), T_{n-1} \leq \tau \leq T_n))$ (e.g. pictures or other high dimensional data) with other agents operating in the same area. Based on this information exchange (which might involve several iterations for data fusion) they agree on new reference trajectories $x_{ref}(t), t \geq T_n$. The quality of these new reference trajectory, and thus the performance measure of the task execution, depends how much information has been exchanged. Since bandwidth can be very expensive (especially in the underwater vehicle example) the amount of information that can be exchanged may then depend on how far apart the two autonomous vehicles are. Vehicles can exchange more information by moving closer together, but this requires a deviation from their reference trajectory and thus a waste of time for the execution of their task. There is a clear trade-off between executing the assigned task with minimal deviation from the assigned reference path (and thus minimal delay for the short term performance) versus the achievable improved future performance obtainable via better new reference trajectories.

The dynamic model of the trajectory of each agent, and the information exchange among autonomous agents, are very complicated non-linear phenomena, to which the decomposition result of this paper is not directly applicable. Moreover the energy used for controlling the steering force, and the energy used by the transmitter and the receiver come from the same battery, and are tightly coupled by upper bounds on the instantaneously available power. Nevertheless in many cases some simple first order linear approximation may provide insight in how a good coordination control should work.

Consider two autonomous vehicles (AVs), $AV_i, i = 1, 2$, executing a task that requires that they follow two closely coupled, predefined, paths. They must tightly couple their speed and their path (e.g. because their measurement equipment interfere with each other, leading to a very high cost when their distance and speed difference deviate too much from the assigned values). Their trajectories are defined by:

$$
\begin{aligned}
\dot{x}_i(t) & \\
= & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_i(t) + \\
& + \begin{pmatrix} 0 & 0 \\ \frac{1}{m_i} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_i} \end{pmatrix} \cdot \left[\begin{pmatrix} u_{i,1}(t) \\ u_{i,2}(t) \end{pmatrix} + \begin{pmatrix} w_{i,1}(t) \\ w_{i,2}(t) \end{pmatrix} \right].
\end{aligned}
$$

where $x_{i,1}(t)$, resp. $x_{i,3}(t)$, represent the position of the i-th AV in a plane at a fixed level under the water surface, $x_{i,2}(t)$, resp. $x_{i,4}(t)$, represent the speed of the i-th AV in this same plane, $u_{i,1}(t)$, resp. $u_{i,2}(t)$, represent the forces acting on the i-th AV in this same plane, and $w_{i,1}(t)$, resp. $w_{i,2}(t)$, represent the noise acting on the i-th AV.

Under normal underwater operation the only sensor available to AV_i for controlling its position is a speed sensor (e.g. using an inertial navigation system). Of course other sensors related to the task to be executed by the AVs will be present, but these do not in general contribute to the location control. Initially both AVs have a reasonably accurate estimate of their own position, of the position of the other AV, and they have some information on the trajectory the other AV tries to follow in the near future, but since the system models are pure integrators their estimation error for their own position will quickly increase, and the error in their (open loop) estimate of the position of the other AV will grow even faster. Hence the quality of the task execution by the AVs will quickly deteriorate. This problem can be resolved by allowing the AVs to turn on an expensive additional position sensor from time to time. This could e.g. be implemented by allowing the AVs to climb to the surface from time to time, at the cost of a lot of energy consumption, and at the cost of extra delay because they cannot perform their task while surfacing. This can be modeled by having a sensor control input $u_i(t) = 0$ or $= 1$, leading to a linear sensor model for AV_i with

$$
y_i(t) = C_i(u_i(t)) \begin{pmatrix} x_i(t) \\ x_{3-i}(t) \end{pmatrix} + v_i(t) \tag{1}
$$

where $C_i(0) \in \Re^{6 \times 8}$ has all rows identically 0 except for a 1 in position $(1, 2)$ and in position $(2, 4)$, measuring the 2 components of the speed of AV_i. However when the expensive control decision $u_i(t) = 1$ is activated, then $C_i(1)$ also has non-zero elements on the other rows: row 3 and 4 measure the position of the AV_i by defining $y_{i,\ell}(t) = c_{own}.x_{i,\ell}(t) + v_{own}(t)$ where $\ell = 1, 3$ refers to the first and the second component of the speed of AV_i, and with $v_{own}(t)$ a white noise vector with known variance. The 5-th and 6-th component of $y_i(t)$ is defined by similar rows of $C_i(1)$, but now with $c_{other}.x_{3-i,\ell}(t) + v_{other}(t)$ with a smaller signal to noise ratio, measuring less accurately the position of the other AV.

A quadratic cost criterion

$$
\sum_{t=t_0 \ldots t_1} \begin{pmatrix} ((x_{1,1}(t) - x_{2,1}(t))^2 + (x_{1,2}(t) - x_{2,2}(t))^2 + \\ +\rho_1.u_1^2(t) + \rho_2.u_2^2(t) \end{pmatrix} \tag{2}
$$

will lead to an optimal control where from time to time, when the error estimate of the own position and of the position of the other AV has become too high, the AVs switch on the expensive sensor (e.g. by surfacing). After this more accurate measurement has been taken, the AVs can proceed with their task, until at some later time their position error becomes too large again for performing their task successfully. Note that the model assumes a common control decision for both AVs: they have to surface at the same time in order to take measurements on the position of the other AV. This is actually feasible without communication between the AVs because the dynamic equation for the quadratic error matrices $Q_i(t)$ are deterministic. Hence the decision on when to surface next can be agreed on by the AVs at each time they surface.

The proposed problem solution can be interpreted as a lower layer optimization as part of a higher layer problem optimizing the way in which 2 AVs coordinate their trajectories for executing a joint task. The non-linear optimization problem that must be solved when also taking into account the quality of the performed task is a topic for future research.

III. Control of the observation channel for filtering purposes

The approach to the problem is based on optimal stochastic control theory. Books on control of stochastic systems with partial observations include [5], [10].

The notation is fairly standard. The integers are denoted by $\mathbb{Z}$ and the positive integers by $\mathbb{Z}_+$. The real numbers are denoted by $\mathbb{R}$ and the set of n-tuples of the real numbers by $\mathbb{R}^n$. A matrix which is positive definite is denoted by $Q \geq 0$ and one which is strictly positive definite is denoted by $Q > 0$. Denote the set of $n \times n$ matrices with entries in the real numbers and which are symmetric and positive definite by $\mathbb{R}^{n\times n}_{spd}$ and the subset of this which are strictly positive defined by $\mathbb{R}^{n\times n}_{sspd}$. A Gaussian random variable of dimension $n \in \mathbb{Z}_+$ is a random variable $x : \Omega \to \mathbb{R}^n$ of which the probability distribution is of Gaussian type. If the parameters of this distribution function are the mean $m_0 \in \mathbb{R}^n$ and the variance $Q_0 \in \mathbb{R}^{n\times n}_{spd}$ then denote this random variable as $x \in G(m_0, Q_0)$.

Definition III.1 *Consider the following* Gaussian stochastic control system *with a controllable* observation matrix.

$$x(t+1) = Ax(t) + Mv(t), \; x(t_0) = x_0, \quad (3)$$
$$y(t) = C(u(t))x(t) + Nv(t), \quad (4)$$

where (Ω, F, P) is a complete probability space, $T = \{t_0, t_0+1, \ldots, t_1\} \subset \mathbb{Z}_+$ is the time index set, $n, p \in \mathbb{Z}_+$, $X = \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$, $Y = \mathbb{R}^p$, $x_0 : \Omega \to X$, is a Gaussian random variable with $x_0 \in G(m_0, Q_0)$, $v : \Omega \times T \to \mathbb{R}^{m_v}$, is Gaussian white noise, an independent sequence with $v(t) \in G(0, Q_v)$, for all $t \in T$, $A \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times m_v}$, $N \in \mathbb{R}^{p\times m_v}$ satisfying $NQ_vN^T > 0$, $C : U \to \mathbb{R}^{p\times n}$, $u : \Omega \times T \to U$, $x : \Omega \times T \to X$, $y : \Omega \times T \to Y$, denote respectively the input, the state and the output process.

Example III.2 Consider a Gaussian stochastic control system with a controllable observation matrix. A specific observation matrix is provided by the formulas

$$U = \{0,1\}, \quad C : U \to \mathbb{R}^{2\times 4},$$
$$C(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5)$$
$$C(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (6)$$

The observation matrix is such that if $u(t) = 0$ then only the first output component provides information while if $u(t) = 1$ then both the first and the second components of the output provide state information. The model could be modified further by requiring the matrix of the observation noise N to be dependent on the input in a way corresponding to that of the C matrix.

Definition III.3 *Define the* set of control laws *for the System III.1*

$$G = \left\{ \begin{array}{l} g : T \times Y^T \times U^T | \\ g(t,.,.) : Y^{t-t_0} \times U^{t-t_0} \to U, \\ \textit{satisfying conditions below} \end{array} \right\}$$

$g(t,.,.)$, is Borel measurable;

the control law is causal *if*

$$g(t,y,u) = g(t,\overline{y},\overline{u}), \; \forall t \in T, \quad (7)$$
$$\forall \; y_{\{t_0,t\}} = \overline{y}_{\{t_0,t\}}, \; u_{\{t_0,t\}} = \overline{u}_{\{t_0,t\}}. \quad (8)$$

The closed-loop system *is denoted by*

$$x^g(t+1) = Ax^g(t) + Mv(t), \; x^g(t_0) = x_0, \quad (9)$$
$$y^g(t) = C(g(t, y^g_{\{t_0,t-1\}}, u^g_{\{t_0,t-1\}}))x^g(t) + Nv(t),$$
$$u^g(t) = g(t, y^g_{\{t_0,t-1\}}, u^g_{\{t_0,t-1\}}), \quad (10)$$
$$y^g_{\{t_0,t\}} = (y^g(0), y^g(1), \ldots, y^g(t)), \quad (11)$$
$$u^g_{\{t_0,t\}} = (u^g(0), u^g(1), \ldots, u^g(t)). \quad (12)$$

$\{H^g_t, t \in T\}$, a filtration,

$$H^g_t = \sigma(y^g_{\{t_0,t-1\}}, u^g_{\{t_0,t-1\}}). \quad (13)$$

Definition III.4 *Define the cost function,*

$$J : G \to \mathbb{R}_+,$$
$$J(g) = E\left[\sum_{s=t_0}^{t_1-1} \left(\begin{array}{l} b_c(g(s, y^g_{\{t_0,s-1\}}, u^g_{\{t_0,s-1\}}))+ \\ +b_f(Q^g_f(s)) \end{array} \right) \right.$$
$$\left. +b_1(Q^g_f(t_1)) \right], \quad (14)$$
$$Q^g_f(t) = E[(x^g(t) - \hat{x}^g(t))(x^g(t) - \hat{x}^g(t))^T | H^g_t], \quad (15)$$

see Theorem III.8 for the definition,

$$b_c : U \to \mathbb{R}_+, \; b_f : \mathbb{R}^{n\times n}_{spd} \to \mathbb{R}_+,$$
$$b_1 : \mathbb{R}^{n\times n}_{spd} \to \mathbb{R}_+, \quad (16)$$

where b_c is the cost rate of using the observation channel, b_f the cost rate of the estimation-error variance, and b_1 is the terminal cost on the estimation-error variance.

The cost function forces a trade off between the reduction of the conditional error variance Q_f and the costs to be paid for sampling the extra observation channel. The problem was formulated in a general setting by the second named author, see [14].

Example III.5 Consider the cost function defined above. A particular example of the functions appearing in the cost function is:

$$b_f(Q) = b_1(Q) = \text{trace}(WQW^T),\ W \in \mathbb{R}^{n\times n}. \quad (17)$$

The choice of a term of the rate of the cost function (14), $b_f(Q_f)$, requires comments. The conditional error variance function Q_f defined in Theorem III.8 is a symmetric and positive-definite matrix function. In the discussion of the remainder of this paragraph we consider only its value at a particular time, a positive-definite matrix $Q_f \in \mathbb{R}^{n\times n}_{spd}$. This matrix admits a decompostion as,

$$Q_f = SDS^T,$$

where $S \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, hence satisfies $SS^T = I = S^TS$, and $D = \text{Diag}(d_1,\ldots,d_n) \in \mathbb{R}^{n\times n}_{spd}$ is a diagonal matrix The characteristics of the conditional variance matrix Q_f are thus specified by the intensities along the main axes and by the orthogonal matrix $S \in \mathbb{R}^{n\times n}$ which describes the directions of the main axes in $\mathbb{R}^n$. A measure of the sum of the intensities along the main axes is thus

$$\sum_{i=1}^n d_i = \text{tr}(D) = \text{tr}(S^TSD) = \text{tr}(SDS^T) = \text{tr}(Q_f), \quad (18)$$

as follows from [13, Th. 9.1]. For these reasons the cost rate and the terminal cost may be chosen to be

$$b_f(Q_f(t)) = \text{tr}(W_fQ_f(t)W_f^T), \quad (19)$$
$$b_1(Q_f(t_1)) = \text{tr}(W_1Q_f(t_1)W_1^T), \quad (20)$$

where $W_f,\ W_1 \in \mathbb{R}^{n\times n}$ are weighing matrices which allow for a scaling of the intensities and of the directions.

Problem III.6 *The* problem of control of the observation matrix for state estimation. *Consider the stochastic control system of Definition III.1, the class of control laws of Definition III.3, and the cost function of Definition III.4. Solve the problem*

$$J^* = \inf_{g\in G} J(g) = J(g^*). \quad (21)$$

This involves determining the value J^* and an optimal control law $g^* \in G$, if one exists.

The approach to the problem is based on optimal stochastic control theory in particular on the following control synthesis procedure. First the filtering problem for the system is solved. Second an optimal stochastic control problem with complete observations is solved. This approach is followed below.

A. Filtering problem

Problem III.7 *The* filtering problem *for the stochastic control system of Definition III.1. Determine for any control law $g \in G$ the conditional distribution of the state based on past observations and past inputs, equivalently, determine,*

$$E[\exp(iw^Tx^g(t))|H_t^g],\ \forall w \in \mathbb{R}^n,\ \forall t \in T. \quad (22)$$

In particular, determine recursions for the parameters of this conditional distribution.

Problem III.7 differs from the classical Kalman filtering problem because the observation matrix depends on the control variable $u^g(t)$. Note that $u^g(t)$ depends only on H_t^g so that $C(u^g(t))$ only depends on past outputs $y^g(s)$ and past inputs $u^g(s)$ for $s = t_0,\ldots,t-1$.

Theorem III.8 *Consider System III.1, a control law $g \in G$, the closed-loop system of Definition III.3, and the filtration $\{H_t^g, t \in T\}$.*

(a) *The conditional distribution of the state conditioned on the past outputs and the past inputs is Gaussian with the expression,*

$$\begin{aligned} &E[\exp(iw^Tx^g(t))|H_t^g] \quad (23)\\ &= \exp(iw^T\hat{x}^g(t) - \frac{1}{2}w^TQ_f^g(t)w),\ \forall\ w \in \mathbb{R}^n,\\ &\hat{x}^g : \Omega\times T \to \mathbb{R}^n,\ Q_f^g : \Omega\times T \to \mathbb{R}^{n\times n}_{spd}.\end{aligned}$$

(b) *The parameters of the conditional distribution can be recursively calculated by the formulas,*

$$\begin{aligned} &\hat{x}^g(t+1) = f_{KF}(\hat{x}^g(t), Q_f(t), y^g(t), u^g(t))\\ &= A\hat{x}^g(t) +\\ &+K(Q_f^g(t), u^g(t))[y^g(t) - C(u^g(t))\hat{x}^g(t)], \quad (24)\\ &\hat{x}^g(t_0) = E[x_0],\\ &Q_f^g(t+1) = f_{FR}(Q_f^g(t), u^g(t))\\ &= AQ_f^g(t)A^T + MQ_vM^T +\\ &-[AQ_f^g(t)C(u^g(t))^T + MQ_vN^T]\times\\ &\times[C(u^g(t))Q_f^g(t)C(u^g(t))^T + NQ_vN^T]^{-1}\times\\ &\times[AQ_f^g(t)C(u^g(t))^T + MQ_vN^T]^T, \quad (25)\\ &Q_f^g(t_0) = Q_0,\end{aligned}$$

called the Filter Riccati Recursion,

$$\begin{aligned} &Q_{\overline{v}}^g(Q_f(t), u^g(t))\\ &= C(u^g(t))Q_f^g(t)C(u^g(t))^T + NQ_vN^T, \quad (26)\\ &K(Q_f^g(t), u^g(t)) \quad (27)\\ &= [AQ_f^g(t)C(u^g(t))^T + MQ_vN^T]\times\\ &\times Q_{\overline{v}}^g(Q_f(t), u^g(t)),\quad K : \mathbb{R}^{n\times n}_{spd}\times U \to \mathbb{R}^{n\times p}.\end{aligned}$$

(c) *Define the* innovation process *by the formulas*

$$\overline{v}^g : \Omega\times T \to \mathbb{R}^p,\ \forall g \in G, \quad (28)$$
$$\overline{v}^g(t) = y^g(t) - C(u^g(t))\hat{x}^g(t). \quad (29)$$

Then the innovation process is a conditional Gaussian white noise process with conditional distribution,

$$E[\exp(iw^T \overline{v}^g(t))|H_t^g] \quad (30)$$
$$= \exp(-\frac{1}{2} w^T Q_{\overline{v}}^g(t) w), \ \forall w \in \mathbb{R}^p, \quad (31)$$
$$\overline{v}(t) \in CG(0, Q_{\overline{v}}(Q_f(t), u^g(t))|H_t^g). \quad (32)$$

The proof of the above theorem is analogous to that of the paper [7]. However, in that paper another representation of a Gaussian stochastic control system is used so another version of the Kalman filter is obtained. The theorem above follows along similar lines.

The filtering problem defined above is non trivial. Note that in general neither the state process x nor the output process y is Gaussian because the input process u is allowed to be an arbitrary nonlinear function of the past outputs and the past inputs. However, by considering the stochastic system as a conditional Gaussian system the filtering problem becomes tractable.

B. Control of the information system

According to the control synthesis method adopted, after the solution of the filtering problem one obtains a stochastic control problem with complete observation. The state of this stochastic control system consists of $(\hat{x}, Q_f)$, encompassing the estimate of the state of the original system, $\hat{x}$, and the conditional error variance matrix Q_f.

Definition III.9 *Define the* information system *with complete observations associated with the stochastic control system of Definition III.1 as the following stochastic control system*

$$\begin{aligned}
&(\hat{x}(t+1), Q_f(t+1)) \\
&= f_{is}(\hat{x}(t), Q_f(t), y(t), u(t)), \quad (33)\\
&\quad (\hat{x}(t_0), Q_f(t_0)) = (m_0, Q_0), \\
&f_{is}(\hat{x}, Q_f, y(t), u(t)) \\
&= (f_{KF}(\hat{x}, Q_f, y(t), u(t)), f_{FR}(Q_f, u(t))). \quad (34)
\end{aligned}$$

Define the class of Markov control laws based only on the current state as

$$G_m = \left\{ \begin{array}{l} g_{cv} : T \times X \times \mathbb{R}_{spd}^{n \times n} \to U \\ \text{Borel measurable} \end{array} \right\}, \quad (35)$$
$$u^g(t) = g_{cv}(t, \hat{x}^g(t), Q_f^g(t)). \quad (36)$$

The cost function of the control problem with complete observations is identical to that with partial observations because the cost rate is already adapted to the filtration.

Problem III.10 *Solve the optimal control problem for the stochastic control system of Definition III.9 with the cost function (14). In mathematical symbols, solve*

$$J^* = \inf_{g \in G} J(g) = J(g^*). \quad (37)$$

C. The optimal control law

Problem III.10 is an optimal stochastic control problem for the information system which thus has complete observations and a state $(\hat{x}(t), Q_f(t)) \in \mathbb{R}^n \times \mathbb{R}_{spd}^{n \times n}$ (not just in $\mathbb{R}^n$ as in the classical LQG problem). This stochastic control problem can be solved using the dynamic programming approach.

Definition III.11 *Define the* conditional cost-to-go *process*

$$\begin{aligned}
&J_{c2g}(g, t) \quad (38)\\
&= E\left[\sum_{s=t}^{t_1-1} [b_c(u^g(s)) + b_f(Q_f(s))] + b_1(Q_f(t_1))|H_t^g\right], \\
&J_{c2g} : G \times T \to \mathbb{R}_+.
\end{aligned}$$

Theorem III.12 *Consider the optimal stochastic control Problem III.10 with the stochastic control system of Definition III.9 and with the cost function (14). Define the value function by the backward recursion,*

$$\begin{aligned}
&V : T \times X \times \mathbb{R}_{spd}^{n \times n} \to \mathbb{R}, \\
&V(t_1, (\hat{x}, Q_f)) = b_1(Q_f), \\
&V(t, (\hat{x}, Q_f)) \\
&= \min_{u \in U} [b_c(u) + b_f(Q_f) + \\
&\quad + E[V(t+1, f_{is}(\hat{x}, Q_f, y(t), u))|H_t^g]], \quad (39)\\
&\forall \ t \in \{t_1 - 1, \ldots, t_0\}.
\end{aligned}$$

(a) *The function V is a lower bound of the conditional cost-to-go for all time and for all control laws,*

$$V(t, (\hat{x}^g(t), Q_f^g(t))) \leq J_{c2g}(g, t), \quad a.s. \quad (40)$$
$$\forall t \in T, \ \forall \ g \in G. \quad (41)$$

(b) *Define the optimal control law as follows*

$$\begin{aligned}
&\forall \ (t, \hat{x}, Q_f) \in T \times X \times \mathbb{R}_{spd}^{n \times n} \quad (42)\\
&\text{assume } \exists \ u^* \in U \text{ such that,} \\
&b_c(u^*) + b_f(Q_f) + \\
&+V(t+1, f_{is}(\hat{x}, Q_f, y(t), u^*)) \\
&= \inf_{u \in U} \left[\begin{array}{l} b_c(u) + b_f(Q_f) + \\ +V(t+1, f_{is}(\hat{x}, Q_f, y(t), u)) \end{array} \right], \quad (43)\\
&g^*(t, y_{\{t_0-1\}}^g, u_{\{t_0, t-1\}}^g)) = u^*, \\
&g_{cv}^*(t, (\hat{x}, Q_f)) = u^*, \quad (44)\\
&g_{cv}^* : T \times X \times \mathbb{R}_{spd}^{n \times n} \to U, \ g_{cv}^* \in G_m.
\end{aligned}$$

Assume that $g^* \in G$ and that $g_{cv}^* \in G_{cv}$. Then $g^* \in G$ is a conditionally optimal control law for the considered problem, the function V defined above is the true value function, and

$$\begin{aligned}
&V(t, (\hat{x}^{g^*}(t), Q_f^{g^*}(t))) \quad (45)\\
&= J_{c2g}(g^*, t), \ \forall \ t \in T, \ \text{where}, \\
&(\hat{x}^{g^*}(t+1, Q_f^{g^*}(t+1)) \quad (46)\\
&= f_{is}(\hat{x}^{g^*}(t), Q_f^{g^*}(t), y^{g^*}(t), u^{g^*}(t)), \quad (47)\\
&(\hat{x}^{g^*}(0), Q_f^{g^*}(0)) = (m_0, Q_0), \quad (48)
\end{aligned}$$

$$
\begin{aligned}
y^{g^*}(t) &= C(g^*(t, y^{g^*}_{\{t_0,t-1\}}, u^{g^*}_{\{t_0,t-1\}}))x^{g^*}(t) + \\
&\quad +Nv(t), \quad (49)\\
u^{g^*}(t) &= g^*_{cv}(t, (\hat{x}^{g^*}(t), Q^{g^*}_f(t))). \quad (50)
\end{aligned}
$$

(c) *There is no loss in cost by restricting attention to control laws depending on the conditional error variance only. Equivalently,*

$$\inf_{g \in G} J(g) = \inf_{g_{cv} \in G_{cv}} J(g). \quad (51)$$

That the optimal control law and the value function depend only on the state Q_f of the information system and neither on the state $\hat{x}$ of the information system nor on the output y is due to the following factors: (1) The terminal cost b_1 depends only on the state component Q_f and not on $\hat{x}$. (2) The cost rate depends on the input u and on the state component Q_f of the information system but not on the state component $\hat{x}$. (3) The dynamics of the Filter Riccati recursion is deterministic and depends only on the state Q_f and on the input u.

D. Minimization of the dynamic programming equation

The computation of the value function and of the optimal control law is addressed next. The optimal control law can be written as

$$
\begin{aligned}
&g^*_{cv}(t, (\hat{x}(t), Q_f(t))) = u^*(t)\\
&= \text{argmin}_{u(t) \in U} \left\{ \begin{array}{l} b_c(u(t)) + b_f(Q_f(t)) + \\ +v(t+1, \\ (0, f_{FR}(Q_f(t), u(t)))) \end{array} \right\}.
\end{aligned}
$$

Note that in the above expression, the argument of $\hat{x}$ in $v(t+1,.)$ is set to the zero vector as explained below. The computation of the value function is described, it is a backward recursion with

$$
\begin{aligned}
&v(t_1, (0, Q_f)) = b_1(Q_f).\\
&v(t_1 - 1, (\hat{x}, Q_f))\\
&= \text{argmin}_{u \in U}\\
&\quad \left\{ \begin{array}{l} b_c(u) + b_f(Q_f) + \\ +E[v(t_1, f_{is}(\hat{x}, Q_f, y(t_1 - 1), u))|H^g_t] \end{array} \right\}\\
&= \text{argmin}_{u \in U} \left\{ \begin{array}{l} b_c(u) + b_f(Q_f) + \\ +v(t_1, (0, f_{FR}(Q_f, u))) \end{array} \right\},
\end{aligned}
$$

which is true because the value function at the terminal time, $v(t_1, (\hat{x}, Q_f))$ depends only on Q_f, and because the dynamics of the Filter Riccati recursion is deterministic given Q_f and u. Because the $v(t, (\hat{x}, Q_f))$ depends only on Q_f, the argument of $\hat{x}$ is set to the zero vector in $\mathbb{R}^n$. The minimization is aided by the use of notation,

$$
\begin{aligned}
&\Delta(t_1 - 1, Q_f)\\
&= b_c(0) + b_f(Q_f) + v(t_1, (0, f_{FR}(t, Q_f, 0))) + \\
&\quad -[b_c(1) + b_f(Q_f) + v(t_1, (0, f_{FR}(t, Q_f, 1)))],
\end{aligned}
$$

$$
\begin{aligned}
&u^*(t_1 - 1)\\
&= \left\{ \begin{array}{ll} 1, & \text{if } \Delta(t_1 - 1, Q_f) > 0, \\ 0, & \text{if } \Delta(t_1 - 1, Q_f) \le 0. \end{array} \right.
\end{aligned}
$$

In the scalar case it has been proven that the map $Q_f \mapsto \Delta(t_1 - 1, Q_f)$ is monotonically increasing. Whether or not for very large scalar Q_f the condition $\Delta(t, Q_f) > 0$ or $\Delta(t, Q_f) \le 0$ holds depends on the parameters of that map.

At the time $t = t_1 - 2$ the formula for the control law is

$$
\begin{aligned}
&v(t_1 - 2, (\hat{x}, Q_f))\\
&= \min_{u(t_1 - 2) \in U}\\
&\quad \left\{ \begin{array}{l} b_c(u(t_1 - 2)) + b_f(Q_f) + \\ +v(t_1 - 1, (0, f_{FR}(t, Q_f, u(t_1 - 2)))) \end{array} \right\}.
\end{aligned}
$$

The computation thus proceeds by backward recursion. However, since the function $v(t_1 - 1, (0, f_{FR}(Q_f, u(t))))$ is not available in explicit form, it has to be computed for the particular value of $f_{FR}(Q_f, u_2)$. Further research on this topic is in progress and will be published elsewhere.

It is of interest to study control of the conditional error variance Q_f described by the Filter Riccati recursion. Controllability of the Riccati differential equation was studied in the paper [6]. Control of teams of objects leading to control of the Riccati equation is developed in the paper [11]. The book [12] provides a framework for these questions.

IV. CONTROL OF THE OBSERVATION CHANNEL FOR CONTROL PURPOSES

In this section the optimal control problem is formulated and solved for the joint control of the conditional mean and of the conditional error variance. The stochastic control system is different in only one term compared to that of Definition III.1 and the cost function has an additional term.

Definition IV.1 *Consider a time-invariant Gaussian stochastic control system with the representation,*

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu_1(t) + Mv(t),\\
&\quad x(t_0) = x_0 \in G(m_0, Q_0),\\
y(t) &= C(u_2(t))x(t) + Nv(t), \quad v(t) \in G(0, Q_v),\\
&\quad u_1 : \Omega \times T \to U_1 = \mathbb{R}^m,\\
&\quad u_2 : \Omega \times T \to U_2 \subseteq \mathbb{R},\\
&\quad u(t) = (u_1(t), u_2(t))^T.
\end{aligned}
$$

Definition IV.2 *Consider the Gaussian stochastic system with the controllable observation matrix and the above defined class of control laws. Define the cost function,*

$$
\begin{aligned}
&J : G \to \mathbb{R}_+,\\
&J(g)\\
&= E\left[\sum_{s=t_0}^{t_1-1} \left[\begin{array}{l} b_c(g_2(s, y^g_{\{t_0,s-1\}}, u^g_{\{t_0,s-1\}})) + \\ +b_f(Q^g_f(s)) + \\ + \left(\begin{array}{c} x^g(s) \\ u^g_1(s) \end{array} \right)^T L \left(\begin{array}{c} x^g(s) \\ u^g_1(s) \end{array} \right) + \end{array} \right] + \right.\\
&\quad \left. +b_1(Q^g_f(t_1)) + x^g(t_1)^T L_1 x^g(t_1) \right]. \quad (52)\\
&L = \left(\begin{array}{cc} L_{11} & L_{12} \\ L^T_{12} & L_{22} \end{array} \right) = L^T \ge 0, \quad L_{22} > 0,\\
&L_1 = L^T_1 \ge 0.
\end{aligned}
$$

Problem IV.3 *The* control of the conditional mean and of the conditional error variance. *Consider the stochastic control system of Definition IV.1 the class of control laws, and the cost function of Definition IV.2. Solve the problem*

$$J^* = \inf_{g \in G} J(g) = J(g^*). \tag{53}$$

The filtering problem for the closed-loop system is solved by a filter which differs from the Kalman filter of Equation (25) *only* in the presence of the input in the recursion of the conditional mean. That equation is stated here as

$$\begin{aligned} & \hat{x}(t+1) = f_{KF2}(\hat{x}(t), Q_f(t), y(t), u(t)) & (54)\\ & \quad = \quad A\hat{x}(t) + Bu_1(t) + K(Q_f(t), u_2(t))\overline{v}(t), \\ & \quad\quad\quad \hat{x}(t_0) = m_0, \\ & f_{is}(\hat{x}, Q_f, y(t), u(t)) \\ & \quad = \quad (f_{KF2}(\hat{x}, Q_f, y(t), u(t)), f_{FR}(Q_f, u_2(t))). & (55) \end{aligned}$$

The cost function for this control problem can be rewritten as follows,

$$\begin{aligned} & J(g) & (56)\\ & = E\Bigg[\sum_{s=t_0}^{t_1-1} b_c(u_2^g(s)) + b_f(Q_f^g(s)) + \mathrm{tr}(L_{11}Q_f^g(s)) + \\ & \quad + \left(\begin{array}{c} \hat{x}^g(s) \\ u_1^g(s) \end{array}\right)^T L \left(\begin{array}{c} \hat{x}^g(s) \\ u_1^g(s) \end{array}\right) + \\ & \quad + \hat{x}^g(t_1)^T L_1 \hat{x}^g(t_1) + \\ & \quad + b_1(Q_f^g(t_1)) + \mathrm{tr}(L_1 Q_f(t_1)) \Bigg]. & (57) \end{aligned}$$

Problem IV.4 *Consider the closed-loop stochastic control system of the information system for a control law* $g = (g_1, g_2) \in G$

$$\begin{aligned} & (\hat{x}^g(t+1), Q_f^g(t+1)) & (58)\\ & \quad = f_{is}(\hat{x}^g(t), Q_f^g(t), y^g(t), u^g(t)), \\ & \quad\quad (\hat{x}^g(0), Q_f^g(0)) = (m_0, Q_0), & (59)\\ y^g(t) &= C(u_2^g(t))\hat{x}^g(t) + \overline{v}^g(t), & (60)\\ u_1^g(t) &= g_1(y^g_{t_0,t-1}, u^g_{t_0,t-1}), & (61)\\ u_2^g(t) &= g_2(y^g_{t_0,t-1}, u^g_{t_0,t-1}). & (62) \end{aligned}$$

and the cost function of Equation (57). Solve the optimal stochastic control problem

$$J^* = \inf_{g \in G} J(g) = J(g^*).$$

Below use is made of the optimal control problem of Gaussian systems with quadratic criteria (LQG) in the partial observations setting. That theory may be found in the references [1, Section 8.5] and [10, Sections 6.7 and 7.5].

Theorem IV.5 *Consider the optimal stochastic control problem IV.4. Define the function*

$$\begin{aligned} & v : T \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_{spd} \rightarrow \mathbb{R}_+, \\ & v(t_1, (\hat{x}, Q_f)) \\ & \quad = \hat{x}^g(t_1)^T L_1 \hat{x}^g(t_1) + \\ & \quad\quad + b_1(Q_f^g(t_1)) + \mathrm{tr}(L_1 Q_f(t_1)), & (63)\\ & v(t, (\hat{x}, Q_f)) \\ & \quad = \inf_{u_1 \in U_1, u_2 \in U_2} \{ b_c(u_2) + b_f(Q_f) + \\ & \quad\quad + \mathrm{tr}(L_{11}Q_f) + \left(\begin{array}{c} \hat{x} \\ u_1 \end{array}\right)^T L \left(\begin{array}{c} \hat{x} \\ u_1 \end{array}\right) + \\ & \quad\quad + E[v(t+1, f_{is}(\hat{x}, Q_f, y(t), u)) | H_t] \}. & (64) \end{aligned}$$

(a) *The function* v *is a lower bound for the conditional cost-to-go for any control law* $g \in G$*, or, equivalently,*

$$v(t, \hat{x}^g(t), Q_f^g(t)) \leq J_{c2g}(g, t), \quad \forall t \in T, \ \forall g \in G. \tag{65}$$

(b) *Assume that*

$$\begin{aligned} & \forall (t, \hat{x}, Q_f) \in T \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_{spd}, & (66)\\ & \exists\ (u_1^*, u_2^*) \in U_1 \times U_2, \text{ such that} \\ & \text{the infimum is attained in Equation (64).} \\ & \text{Define then,} \\ & g_1^*(t, (\hat{x}, Q_f)) = u_1^*, \quad g_2^*(t, (\hat{x}, Q_f)) = u_2^*. & (67) \end{aligned}$$

Assume that $g_1^* \in G_{cm}$ *and* $g_2^* \in G_{cv}$*. Then* (g_1^*, g_2^*) *is an optimal control law, equality in Equation (65) is attained, and* v *is the value function.*

(c) *The computation of the value function* v *of (b) decomposes additively into one over the input set* U_1 *and one over the communication input set* U_2 *according to the formulas,*

$$\begin{aligned} & v(t, (\hat{x}, Q_f)) \\ & \quad = v_1(t, (\hat{x}, Q_f)) + v_2(t, (\hat{x}, Q_f)), & (68)\\ & v_1(t_1, (\hat{x}, Q_f)) \\ & \quad = \hat{x}^T Q_c(t_1) \hat{x}, & (69)\\ & \quad\quad Q_c(t_1) = L_1, & (70)\\ & v_1(t, (\hat{x}, Q_f)) & (71)\\ & \quad = \hat{x}^T Q_c(t) \hat{x}, & (72)\\ Q_c(t) &= A^T Q_c(t+1) A + L_{11} + \\ & \quad -[A^T Q_c(t_1) B + L_{12}] \times \\ & \quad \times [B^T Q_c(t+1) B + L_{22}]^{-1} \times \\ & \quad \times [A^T Q_c(t+1) B + L_{12}]^T, & (73)\\ & v_2(t_1, (\hat{x}, Q_f)) & (74)\\ & \quad = b_1(Q_f) + \mathrm{tr}(L_1 Q_f) = v_2(t_1, (0, Q_f)), \\ & v_2(t, (\hat{x}, Q_f)) & (75)\\ & \quad = \inf_{u_2 \in U_2} \\ & \quad\quad \left\{ \begin{array}{l} b_c(u_2) + b_f(Q_f) + \\ + \mathrm{tr}(L_{11} Q_f) + \\ + \mathrm{tr}(Q_c(t+1) f_{FR}(Q_f(t), u_2)) + \\ + \mathrm{tr}(Q_c(t+1) K(Q_f, u_2) \times \\ \times Q_{\overline{v}}(Q_f, u_2) K(Q_f, u_2)) + \\ + v_2(t+1, (0, f_{FR}(Q_f, u_2(t))) \end{array} \right\} \\ & \quad = v_2(t, (0, Q_f)). & (76) \end{aligned}$$

Proof: (c) The result will be proven by backward induction. That equation (68) holds for t_1 follows from the definitions provided above in (c). Choose $t \in T$. Suppose that Equation (68) holds for $s = t+1, t+2, \ldots, t_1$. It will be proven to hold for t.

In Equation (64) the conditional expectation is first calculated.

$$\begin{aligned}
& E[v(t+1, f_{is}(\hat{x}, Q_f, y(t), u))|H_t] \\
= \; & E[v_1(t+1, f_{is}(\hat{x}, Q_f, y(t), u))|H_t] + \\
& +E[v_2(t+1, f_{is}(\hat{x}, Q_f, y(t), u))|H_t] \\
& \text{by the induction assumption,} \\
= \; & E[f_{KF2}(\hat{x}, Q_f, y(t), u)^T Q_c(t+1) \times \\
& \times f_{KF2}(\hat{x}, Q_f, y(t), u))|H_t] + \\
& +v_2(t+1, (0, f_{FR}(Q_f, u_2))), \\
& \text{by the induction assumption,} \\
= \; & (A\hat{x} + Bu_1)^T Q_c(t+1)(A\hat{x} + Bu_1) + \\
& +\text{tr}(Q_c(t_1) Q_f(t+1)) + \\
& +\text{tr}(L_{11} K(Q_f, u_2) Q_{\overline{v}}(Q_f, u_2) K(Q_f, u_2)^T) \\
& +v_2(t+1, (0, f_{FR}(Q_f, u_2))).
\end{aligned}$$

Then the recursion of the value function can be decomposed.

$$\begin{aligned}
& v(t, (\hat{x}, Q_f)) \\
& = \inf_{u_1 \in U_1, u_2 \in U_2} \left\{ \begin{array}{l} b_c(u_2) + b_f(Q_f) + \text{tr}(L_{11} Q_f) \\ + \left(\begin{array}{c} \hat{x} \\ u_1 \end{array} \right)^T L \left(\begin{array}{c} \hat{x} \\ u_1 \end{array} \right) + \\ +E[v(t+1, f_{is}(\hat{x}, Q_g, y(t), u))|H_t] \end{array} \right\} \\
& = \inf_{u_1 \in U_1} \left\{ \left(\begin{array}{c} \hat{x} \\ u_1 \end{array} \right)^T H(t+1) \left(\begin{array}{c} \hat{x} \\ u_1 \end{array} \right) + \right\} \\
& \quad + \inf_{u_2 \in U_2} \left\{ \begin{array}{l} b_c(u_2) + b_f(Q_f) + \text{tr}(L_{11} Q_f) + \\ +\text{tr}(Q_c(t+1) f_{FR}(Q_f, u_2)) + \\ +\text{tr}(Q_c(t+1) K(Q_f, u_2) \times \\ \times Q_{\overline{v}}(Q_f, u_2) K(Q_f, u_2))^T + \\ +v_2(t+1, (0, f_{FR}(Q_f, u_2)) \end{array} \right\}, \\
& = \hat{x}^T Q_c(t) \hat{x} + v_2(t, (0, Q_f)) \\
& = v_1(t, (\hat{x}, Q_f)) + v_2(t, (0, Q_f)), \\
H(t) & = \left(\begin{array}{cc} H_{11}(t) & H_{12}(t) \\ H_{12}^T(t) & H_{22}(t) \end{array} \right), \\
& H_{11}(t+1) = A^T Q_c(t+1) A + L_{11}, \\
& H_{12}(t+1) = A^T Q_c(t+1) B + L_{12}, \\
& H_{22}(t+1) = B^T Q_c(t+1) B + L_{22}, \\
u_1^*(t) & = -H_{22}(t+1)^{-1} H_{12}(t+1)^T, \\
Q_c(t) & = H_{11}(t+1) + \\
& \quad -H_{12}(t+1) H_{22}(t+1)^{-1} H_{12}(t+1)^T.
\end{aligned}$$

■

The conclusion of Theorem IV.5 is that the optimal control problem separates into two optimal control problems, (1) for the problem of when to sample the extra observation channel and (2) for the optimal control of the state $\hat{x}$. The computation of the value functions separate additively. However, the cost function of the second problem, that for determining when to sample the extra observation channel, depends on the cost of control of the state via the term $\text{tr}(L_{11} Q_f)$. The remaining control decision problem of when to take additional observations can be solved using the results of Section III adding to the cost function the term $\text{tr}(L_{11} Q_f(t)$.

V. CONCLUDING REMARKS

The paper presents a solution to two optimal stochastic control problems in which the observation matrix is subject of control. It has been shown that in the case of Gaussian systems there is a separation into the problem of control for minimizing the conditional error variance and the problem of control of the state.

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