Stabilising Stochastic Linear Plants via Erroneous Channels

Girish N. Nair and Kartik Venkat

Abstract— **In the field of networked control, a powerful methodology for constructing quantised controllers with rigorous stability bounds is the zooming strategy. This approach is attractive for implementation purposes since it yields finite-dimensional schemes. However, most available results somewhat unrealistically require no transmission errors to occur in the channel. In this paper, a novel zooming-like scheme is proposed for controlling a stochastic linear plant over an erroneous digital channel, with neither transmitter nor receiver informed when errors occur. Using a stochastic pseudo-norm technique, it is shown that mean square stability can be achieved, provided that the number of quantisation points is sufficiently larger than the plant dynamical constant and the probability of symbol error is sufficiently small.**

I. INTRODUCTION

Within the field of networked control, there is a considerable and growing literature on the stabilisation of linear dynamical systems over digital channels - see [1] and the references therein. In such systems, the continuous-valued outputs of the plant are sampled and converted into digital format, i.e. *quantised*, for transmission over the channel to a distant controller. At a high bit rate r (bits/sample), the quantisation error can be approximated as additive white noise and thus in practice classical linear stochastic control theory is sufficient for design purposes. However, as r decreases, closed-loop performance deteriorates, until a certain fundamental threshold is reached at which stabilisability is completely lost. This loss of stabilisability, which occurs in both stochastic and deterministic settings and for different stability notions (see e.g. [8], [10]), stresses the importance of a rigorous approach to quantised feedback control.

One powerful methodology for constructing quantised controllers with rigorous stability bounds is the *zooming* strategy of [2]. Similar to *adaptive quantisation* in digital communications [3], the current range-width of the quantiser is multiplicatively decreased or increased, depending on whether the latest plant state is in or out of the current quantiser range. As this update rule requires only the current symbol and not the state value, it can be implemented identically at the decoder side. Due to the multiplicative increase regime, this technique can globally asymptotically stabilise noiseless LTI plants, with no *a priori* bounds needed on the unknown initial state, Variations on this idea have been proven to mean-square-stabilise LTI plants with unbounded, non-Gaussian noise [8] and input-to-state-stabilise LTI plants with disturbances [6].

Zooming-style schemes are attractive for implementation purposes since the encoder and decoder need to remember and update an internal state variable of only finite dimension. However, most existing analyses assume that the encoder and decoder begin from identical initial internal states and that no transmission errors occur in the channel. As the internal state updates depend only on the channel symbols, the encoder and decoder internal states thus remain identical for all time. While this eases analysis, real digital channels always suffer from a non-zero probability of symbol error and, even if only one transmission error occurs, the encoder and decoder states will cease to be equal. Given the discontinuity introduced by the quantiser, continuity cannot be used in an obvious way to argue the existence of an in-built stability margin with respect to errors.

Initial steps towards redressing these drawbacks were made in the recent articles [5], [4], in which the channels were still posited to be errorless but the encoder and decoder internal states were allowed to be initially mismatched. In the former paper, input-to-state stability was proven for disturbance-free LTI plants. In the latter article, a novel, zooming-like scheme of finite dimension was proposed and proven to mean-square-stabilise partially observed LTI plants with unbounded, non-Gaussian noise, with the mismatch between internal states additionally shown to decay mean-absolutely to zero with time. The resynchronisation between internal states suggested that stability could be maintained if the error rate was sufficiently low, a conjecture that was supported by simulations over a binary symmetric channel, but not proven.

In this paper, a simplified, zooming-like scheme is proposed for controlling a fully observed stochastic LTI plant with unbounded noise over an erroneous digital channel. Neither transmitter nor receiver are informed of the occurrences of channel errors. Using a stochastic pseudo-norm, it is argued that as long as the number of quantisation levels exceeds a threshold determined by the plant dynamical parameter, then this scheme can maintain mean-square stability in all closed-loop state variables, for sufficiently small error probabilities. This criterion is not claimed to be tight; rather, the usefulness of this result lies in the construction and analysis of an explicit, finite-dimensional scheme that guarantees stability without employing long, randomised channel codes, which are typically needed to prove tight conditions as in [7], [9].

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II. FORMULATION

For simplicity of exposition, only the scalar, fully observed case is presented here; similar results can be shown for vector states with noisy, partial observations.

Consider the fully observed, discrete-time, stochastic, linear time-invariant (LTI) plant

$$X(t+1) = aX(t) + bU(t) + V(t) \in \mathbb{R}, \quad \forall t \in \mathbb{Z}_{\geq 0}, \tag{1}$$

where at time t, $X(t) \in \mathbb{R}$ is the state, $U(t) \in \mathbb{R}$ is the input, and $V(t)$ is dynamical noise which has uniformly bounded $(2+\varepsilon)$-th absolute moments for some $\varepsilon > 0$.[1]

Suppose that the plant state must be causally encoded and transmitted to the controller over an independent, M-ary, symmetric, discrete memoryless channel. That is, if $S(t), R(t) \in [1,2,\ldots,M]$ respectively denote the channel input and output at time t, then $\forall r \in [1,2,\ldots,M]$

$$\begin{aligned} &\mathrm{P}\{R(t)=r|S(0:t),R(0:t-1),X(0),V\} \\ &= \mathrm{P}\{R(t)=r|S(t)\} \\ &= \begin{cases} p/(M-1) & \text{when } r \neq S(t) \\ 1-p & \text{when } r = S(t). \end{cases}, \end{aligned} \tag{2}$$

where $p \geq 0$. From this it is clear that the conditional and unconditional probabilities of symbol error are both given by

$$\mathrm{P}\{R(t) \neq S(t)\} = \mathrm{P}\{R(t) \neq S(t)|S(t)=s\} = p, \ \forall s \in [1,\ldots,M]. \tag{3}$$

For convenience it is assumed that no delays are introduced by the channel.

The encoder state at time t is defined by $L^{\mathrm{e}}(t) \in \mathbb{R}_{>0}$ and the decoder state by $L^{\mathrm{d}}(t) \in \mathbb{R}_{>0}$. The heart of the scheme is a time-invariant, non-uniform M-point quantiser $q: \mathbb{R} \to \{\pi(1),\ldots,\pi(M)\} \subset \mathbb{R}$, where each point $\pi(s)$ is further associated with a non-uniform *resolution factor* $\kappa(s) > 0$ [8]. At time t, the encoder scales the state $X(t)$, quantises it to yield $q(X(t)/L^{\mathrm{e}}t) \equiv \pi(S(t))$ and transmits the index $S(t)$. It then performs the update

$$L^{\mathrm{e}}(t+1) = \left(|a+bk| + |bk|\gamma + |bk|(1+\gamma)\kappa(S(t))\right)L^{\mathrm{e}}(t) + d, \tag{4}$$

where k is a controller gain, and d and γ are positive constants.

Upon receiving the possibly incorrect symbol $R(t)$, the decoder/controller applies the control signal

$$U(t) = kL^{\mathrm{d}}(t)\pi(R(t)) \tag{5}$$

and then performs an equivalent version of the update (4), but using $R(t)$ in place of $S(t)$, i.e.

$$L^{\mathrm{d}}(t+1) = \left(|a+bk| + |bk|\gamma + |bk|(1+\gamma)\kappa(S(t))\right)L^{\mathrm{d}}(t) + d. \tag{6}$$

Observe that the encoder and decoder are not explicitly informed when a channel error has occurred, nor does the encoder have to know the control input actually applied. The objective of this paper is to investigate whether it is possible to achieve closed loop stability by a zooming-style scheme of this form, with appropriate parameter choices.

[1] Additional knowledge of the noise statistics could improve performance, but is not needed to prove stability.

III. CONDITIONAL PSEUDO-NORM

The possibly unbounded noise support and the nonlinearity of the quantiser make deriving recursive bounds directly in terms of mean square state norms a difficult endeavour. Instead, as in [4], the approach adopted here is to first pair every a random vector $X \in \mathbb{R}^n$ of interest with a scale factor $L \in \mathbb{R}_{>0}$ and then stochastically bound the pair $(X,L) \in \mathbb{R}^n \times \mathbb{R}_{>0}$ by a pseudo-norm. The pseudo-norm used here is based on a functional first introduced in [8], but modified so as to handle channel errors.

Let the random process Φ denote the occurence ($\Phi(t)=0$) or not ($\Phi(t)=1$) of symbol errors over time, noting that by the symmetry of the channel (2)–(3), Φ is a Bernoulli process which is independent of the initial state and driving noise of (1). Now define the conditional pseudo-norm[2]

$$\|X,L\|_{\Phi} := \sqrt{\mathrm{E}\left\{L^2\left(1+\left(\frac{|X|}{L}\right)^{2+\varepsilon}\right)\middle|\Phi\right\}} \tag{7}$$

This functional satisfies the triangle inequality and positive homogeneity. Furthermore, it can be shown that

$$\|X,L\|_{\Phi}^2 \geq \mathrm{E}\{|X|^2|\Phi\}, \ \mathrm{E}\{L^2|\Phi\} \tag{8}$$

and that the pseudo-norm of the quantisation error produced by the non-uniform mappings $q(\cdot)$ and $\kappa(\cdot)$ satisfies

$$\left\|X - Lq\left(\frac{X}{L}\right), L\kappa(S)\right\|_{\Phi} \leq \frac{\zeta}{\mu^{\nu}}\|X,L\|_{\Phi}, \forall \nu \in [2,3,\ldots). \tag{9}$$

In this bound S is the index of the selected quantiser point $\pi(S) \equiv q(X/L)$; $\mu,\nu \in [2,3,\ldots)$ are quantiser parameters such that the number of levels $M = \left(1+\frac{2}{\mu}-\frac{1}{\mu^2}\right)\mu^{\nu}$; and $\zeta > 0$ is a constant that depends only on ε and μ. As ζ depends neither on the distribution of X,L nor on μ, this inequality can be applied recursively and, moreover, the error pseudo-norm can be made as small as pleased by choosing ν (hence M) sufficiently large.

A trivial scaling bound that will also be of use is

$$\|\Gamma X, \Theta L\|_{\Phi} \leq \max\left\{\Theta, \frac{|\Gamma|^{1+\varepsilon/2}}{\Theta^{\varepsilon/2}}\right\}\|X,L\|_{\Phi}, \tag{10}$$

provided that the random variables $\Gamma \in \mathbb{R}$ and $\Theta > 0$ are fully determined given Φ.

IV. STABILITY

Denote the scaling error between the encoder and decoder by

$$Z(t) := L^{\mathrm{e}}(t) - L^{\mathrm{d}}(t), \quad \forall t \in \mathbb{Z}_{\geq 0}, \tag{11}$$

and define the proportional scaling error

$$F(t) := |Z(t)|/L^{\mathrm{e}}(t) > 0. \tag{12}$$

[2] In the analysis of the next section, conditioning over the entire process Φ reduces to doing so over a finite segment $\Phi(0:t)$.

The following lemma will be important.

Lemma 1 ($F(t)$ is Stochastically Bounded): Let the encoding and control scheme (4)–(6) be used on the plant (1) via the channel (2). Then for any realisation of the plant initial state $X(0)$ and dynamical noise process V, the proportional scaling error (12) satisfies

$$F(t) \leq G(t), \quad \forall t \in \mathbb{Z}_{\geq 0}, \tag{13}$$

where $G(t)$ depends only on the sequence of previous channel symbol errors, $\Phi(0:t)$.

Furthermore, if $F(0) = 0$ then $\limsup_{t\to\infty} \mathrm{E}\{G(t)^{2+\varepsilon}\}$ can be made as small as pleased by reducing the probability p of symbol error (3).

Proof: Omitted.

Next, rewrite (1) as

$$\begin{aligned} &X(t+1) \\ &= (a+bk)X(t) + bk\,(L^{\mathrm{e}}(t)\pi(S(t)-X(t))) + V(t) \\ &\quad + bk\left(L^{\mathrm{d}}(t)\pi(R(t)) - (L^{\mathrm{e}}(t)\pi(S(t))\right) \\ &= (a+bk)X(t) + bk\,(L^{\mathrm{e}}(t)\pi(S(t)-X(t))) + V(t) \\ &\quad + bk\,(-Z(t)\pi(S(t)) - Z(t)E(t) + L^{\mathrm{e}}(t)E(t)), \end{aligned} \tag{14}$$

where $E(t) := \pi(R(t)) - \pi(S(t))$. Observe that

$$|E(t)| \leq H(t)|\pi(S(t))|,$$

where H is the iid process given by

$$H(t) := \begin{cases} 0 & \text{if } \Phi(t) = 1 \\ 1 + \pi_{\max}/\pi_{\min} & \text{otherwise} \end{cases}, \tag{15}$$

with $\pi_{\max(\min)} := \max(\min)_{s\in[1,\dots,M]}|\pi(s)|$. Taking moduli on both sides of (14) and substituting (15), (12) and (13) yields

$$\begin{aligned} &|X(t+1)| \\ &\leq |a+bk||X(t)| + |bk|\,|X(t) - L^{\mathrm{e}}(t)\pi(S(t))| + |V(t)| \\ &\quad + |bk|\overbrace{(G(t)(1+H(t)) + H(t))}^{:=\Delta(t)}|L^{\mathrm{e}}(t)\pi(S(t))| \quad (16) \\ &\leq |a+bk||X(t)| + |bk|\,|X(t) - L^{\mathrm{e}}(t)\pi(S(t))| \\ &\quad + |bk|\Delta(t)\,(|X(t)| + |L^{\mathrm{e}}(t)\pi(S(t)) - X(t)|) + |V(t)| \\ &= (|a+bk| + |bk|\Delta(t))\,|X(t)| \\ &\quad + |bk|\,(1+\Delta(t))\,|X(t) - L^{\mathrm{e}}(t)\pi(S(t))| + |V(t)|. \quad (17) \end{aligned}$$

Taking the conditional pseudo-norm (7) of this and (4) and then applying the triangle inequality, monotonicity in the X-argument, and the scaling bound (10) yields

$$\begin{aligned} &\|X(t+1), L^{\mathrm{e}}(t+1)\|_{\Phi} \\ &\leq (|a+bk| + N(t))\|X(t), L^{\mathrm{e}}(t)\|_{\Phi} + c \\ &\quad + |bk|(1+W(t))\|X(t) - L^{\mathrm{e}}(t)\pi(S(t)), L^{\mathrm{e}}(t)\kappa(S(t))\|_{\Phi} \\ &\overset{(9)}{\leq} \left((|a+bk| + N(t)) + \frac{\zeta|bk|}{\mu^{\nu}}(1+W(t))\right)\|X(t), L^{\mathrm{e}}(t)\|_{\Phi} \\ &\quad + c, \end{aligned} \tag{18}$$

where

$$\begin{aligned} c^2 &= \sup_{t\in\mathbb{Z}_{\geq 0}} \mathrm{E}\left\{ d^2 + \frac{|V(t)|^{2+\varepsilon}}{d^{\varepsilon}} \middle| \Phi \right\} \\ &= d^2 + \frac{\sup_{t\in\mathbb{Z}_{\geq 0}} \mathrm{E}\{|V(t)|^{2+\varepsilon}\}}{d^{\varepsilon}} < \infty, \end{aligned} \tag{19}$$

$$N(t) := \max\left\{ |bk|\gamma, \frac{(|a+bk| + |bk|\Delta(t))^{1+\varepsilon/2}}{(|a+bk| + |bk|\gamma)^{\varepsilon/2}} - |a+bk| \right\} \tag{20}$$

$$W(t) := \max\left\{ \gamma, \frac{(1+\Delta(t))^{1+\varepsilon/2}}{(1+\gamma)^{\varepsilon/2}} - 1 \right\}. \tag{21}$$

In other words, the nonnegative random sequence $\{\|X(t), L^{\mathrm{e}}(t)\|^2_{\Phi}\}_{t\geq 0}$, which upper-bounds $\{\mathrm{E}\{|X(t)|^2|\Phi\}\}_{t\geq 0}$ and $\{\mathrm{E}\{L^{\mathrm{e}}(t)^2|\Phi\}\}_{t\geq 0}$ (8), is itself bounded above by a random linear recursion with constant driving term c. If

$$\limsup_{t\to\infty} \mathrm{E}\left\{ \left((|a+bk| + N(t))\frac{\zeta|bk|}{\mu^{\nu}}(1+W(t)) \right)^2 \right\} < 1, \tag{22}$$

then it can be shown that $\mathrm{E}\{\|X(t), L^{\mathrm{e}}(t)\|^2_{\Phi}\}$, and hence $\mathrm{E}\{|X(t)|^2\}$ and $\mathrm{E}\{L^{\mathrm{e}}(t)^2\}$, is uniformly bounded over time. Observe that $\mathrm{E}\{N(t)^2\}$ and $\mathrm{E}\{W(t)^2\}$ (20)–(21) are small if γ and $\mathrm{E}\{\Delta(t)^{2+\varepsilon}\}$ (16) are small. From Lemma 1 and the definition (15), this is achieved by making the probability p of symbol error sufficiently small. By further choosing ν large enough that $\mu^{\nu}\zeta > |a|$, there then exists a controller gain k that ensures that the condition (22) is satisfied. This argument leads to the main result of this paper:

Theorem 1: Suppose the encoding and control scheme (4)–(6) is used on the plant (1) via the symmetric, discrete memoryless channel (2). Assume that $\mu^{\nu}/\zeta > |a|$, where μ and ν are parameters of the quantiser and $\zeta > 1$ is the constant of (9). Then for any sufficiently small probability p of symbol error, $\exists$ an encoder-decoder parameter $\gamma > 0$ and a controller gain k such that mean-square boundedness in *all* closed-loop state variables is guaranteed.

V. SIMULATION RESULTS

The system (1)–(6) was simulated in MATLAB for 500 time steps, with $a = 1.2$, $b = 1$, $x(0) = 0$, V an i.i.d Gaussian sequence with mean zero and variance 1, $p = 0.005$, $k = -0.7$, $\mu = 2$, $\nu = 4$ (so that $M = 16$), $\gamma = 0.1$ and $L^{\mathrm{e}}(0) = L^{\mathrm{d}}(0) = 0.5$. Indicative plots of the trajectories of the squared state (Fig. 1) and scaling factors (Fig. 2) are given in support to the stability results above.

Note that since each channel symbol $S(t)$ can take more than two possible values here ($M = 16$), not all channel errors have the same effect. For instance, 4 errors occured in the realisation depicted here, but only the one when $t \approx 400$ had an appreciable impact on performance; the other 3 errors simply shifted the intended quantiser point to another one nearby.

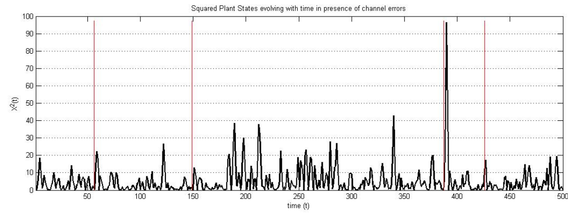

Fig. 1. Squared plant state $X(t)^2$ vs. time t. The vertical lines indicate when symbol errors occur.

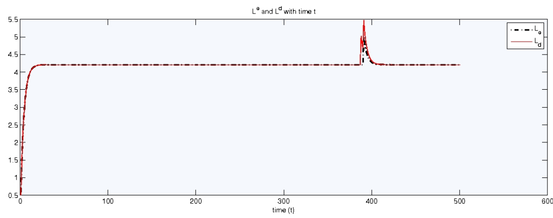

Fig. 2. Scaling factors $L^{\mathrm{e}}(t)$ and $L^{\mathrm{d}}(t)$ vs. time t.

References

[1] P. Antsaklis and J. Baillieul, editors. *Special Issue on the Technology of Networked Control Systems, in Proc. IEEE*, volume 95. IEEE, Jan. 2007.

[2] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Autom. Contr.*, 45(7):1279–89, 2000.

[3] D. J. Goodman and A. Gersho. Theory of an adaptive quantizer. *IEEE Trans. Comms.*, 22:1037–45, 1974.

[4] A. Gurt and G. N. Nair. Internal stability of dynamic quantised control for stochastic linear plants. *Automatica*, 45:1387–96, 2009.

[5] T. Kameneva and D. Nesic. Robustness of quantized control systems with mismatch between coder/decoder initializations. *Automatica*, 45:817–22, 2009.

[6] D. Liberzon and D. Nesic. Input-to-state stabilization of linear systems with quantized state measurements. *IEEE Trans. Autom. Contr.*, 52:767–81, 2007.

[7] A. S. Matveev and A. V. Savkin. An analogue of Shannon information theory for detection and stabilization via noisy discrete communication channels. *SIAM J. Contr. Optim.*, 46(4):1323–67, 2007.

[8] G. N. Nair and R. J. Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM J. Contr. Optim.*, 43(2):413–36, July 2004.

[9] A. Sahai and S. Mitter. The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link part 1: scalar systems. *IEEE Trans. Info. The.*, 52(8):3369–95, 2006.

[10] S. Tatikonda and S. Mitter. Control under communication constraints. *IEEE Trans. Autom. Contr.*, 49(7):1056–68, July 2004.