Coarseness in Quantization for Stabilization of Linear Systems over Networks

Hideaki Ishii, Koji Tsumura, and Tomohisa Hayakawa

Abstract— **We present an overview on the quantized control approach that was initiated by Elia and Mitter in the context of networked control. In particular, two recent extensions are presented where uncertainties in the channel and the plant are taken into consideration. The general problem setting is that the amount of communication is measured by a notion of coarseness in quantization, and we would like to find the coarsest quantizer for achieving stabilization. Numerical examples are provided to illustrate the differences among the quantized control schemes.**

Index Terms— **Adaptive control, Networked control systems, Quantization, Unreliable channels.**

I. Introduction

One of the main challenges in networked control is the constraint on capacity of the communication channels. When shared channels are used by different system components, the data rate of each signal must be counted to ensure that the total is less than the capacity of the channel. Otherwise, the performance of the overall system would degrade and even worse, the system may lose stability. Effects due to capacity constraints include time delays, data losses, scheduling of transmissions, and quantization of signals; see, e.g., [1], [2].

In this context, the effects of quantization on control systems have recently been studied extensively. By quantization, we mean the conversion of real-valued signals to those taking discrete values. For control systems, it has been the convention to assume that signals within systems take real values. However, signals to be sent over channels with finite capacity must be converted to take discrete values.

The new problems that we must address are (i) to introduce models of the capacity limited channels into control systems and then (ii) to find how much data rate is required for each control signal. A fundamental question unique to these problems comes down to the following: How much capacity is needed for the communication in control systems to guarantee certain stability and/or performance properties? Since the initial problem formulation proposed in [30], these issues have been addressed in various forms; see, e.g., [5], [18], [21]–[24], [28], [31].

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The objective of this paper is to provide an overview on the quantized control approach that was initiated by the influential work of Elia and Mitter in [7]. It is shown there that to achieve closed-loop stability via a Lyapunov approach, a quantizer having the "coarsest" resolution can be characterized under a certain notion of coarseness. Such a quantizer is known as the logarithmic type and has the property that quantization is fine around the origin, but becomes coarser for larger inputs. Moreover, the parameter representing the coarseness can be determined by the unstable poles of the plant. Hence, an important aspect of the result is that one form of fundamental limitation involving control and quantization is clarified. Performance issues based on H^2 and H^∞ norms are incorporated in [8], and sampled-data control results are given in [19], [20].

In this paper, we first discuss the original results in [7] and then present two recent extensions following this approach. One is based on the work [29], where the unreliability in the channel is considered and the transmitted quantized data may become lost in a random manner. We clarify that to achieve stability in a stochastic sense, a certain trade-off exists between the coarseness in the quantization and the rate of losses. The other extension is towards quantized adaptive control for uncertain systems from the work in [13]. To stabilize an uncertain system, logarithmic type quantization is employed. Since the level of instability of the plant is unknown a priori, we show that the coarseness in the quantizer must be time varying by adjusting it to the changes in the adaptive gain.

We emphasize that the schemes in these extensions can be viewed as generalization of the original scheme of [7]. Specifically, if there is no loss in the channel and if the plant is fully known (i.e., with no uncertainty), then the schemes reduce to that of Elia and Mitter. In this respect, the coarse quantized control that we develop is not conservative.

The paper is organized as follows: In Section II, the quantized stabilization problem in [7] is introduced. Then, Section III presents the results for the case where the channel is unreliable and data losses can occur. In Section IV, we turn our attention to the quantized adaptive control scheme for uncertain plants. Numerical examples are given throughout the paper to illustrate the differences in the quantized control schemes. The paper is concluded in Section V.

II. The Coarsest Quantization for Stabilization

In this section, we describe the approach for the design of quantizers proposed by Elia and Mitter in [7].

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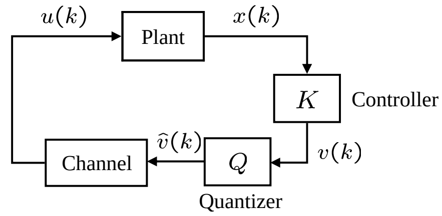

Fig. 1. Stabilization via coarsely quantized signals

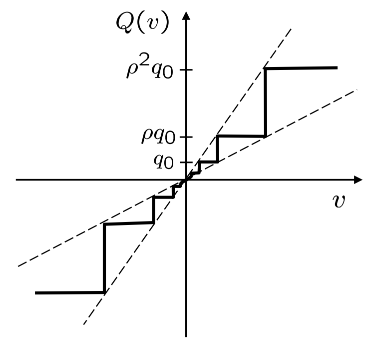

Fig. 2. Logarithmic quantizer

A. Problem Formulation

Consider the system setup shown in Fig. 1. Here, the plant to be controlled has an output that goes to the controller whose output can be sent over the channel after quantization (and coding). It is assumed that the channel is noiseless and there is no communication delay. On the receiver side, the received signal is applied as control input (after proper decoding). This setup is simple having only one channel in the feedback loop. It may represent a large-scale system whose sensors and actuators are separately located. Nevertheless, this is a good starting point in obtaining interesting theoretical results.

In Fig. 1, the plant is a discrete-time linear time-invariant (LTI) system whose state-space equation is given by

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}$ is the control input. We make the following assumptions: (i) The full state $x(k)$ is measured by sensors, (ii) the pair (A, B) is controllable (or, more generally, stabilizable), and (iii) the matrix A is unstable, having at least one eigenvalue with magnitude larger than or equal to 1. Denote by λ_i^u, $i = 1, \dots, n_u$, the unstable eigenvalues of A.

The controller is on the sensor side and its output is given by $v(k) = Kx(k)$, where $K \in \mathbb{R}^{1\times n}$ is the state feedback gain. Hence, the signal to be quantized is the control input.

By a quantizer, we mean a piecewise constant function $Q : \mathbb{R} \to \{q_j\}_{j\in\mathbb{Z}}$. A typical one is the uniform quantizer Q_Δ defined by

$$Q_\Delta(x) = \Delta \left\lceil \frac{x - \frac{\Delta}{2}}{\Delta} \right\rceil, \quad x \in \mathbb{R}, \tag{2}$$

where $\Delta > 0$ is the step size and $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{Z}$ is the ceiling function. This quantizer rounds off a number x to the closest multiple of Δ. Here, however, we do not limit ourselves to this type.

The quantized version of the control input is given by

$$\hat{v}(k) = Q(v(k)) = Q(Kx(k)). \tag{3}$$

Here, the channel is noiseless with no error or delay. Thus, the control input is expressed as

$$u(k) = Q(Kx(k)). \tag{4}$$

The objective is to achieve stability of the overall closed-loop system using quantized information that is as coarse as possible. We formulate the problem under the notion of quadratic stability based on a Lyapunov argument. We begin with a given quadratic function $V(x) = x^T Px$ for $x \in \mathbb{R}^n$, where the matrix $P \in \mathbb{R}^{n\times n}$ is positive definite. Under the control law in (4), the closed-loop system is said to be quadratically stable with respect to $V(x)$ if $V(x)$ becomes a Lyapunov function for the overall system, i.e., it decreases along the state trajectory as

$$V(x(k+1)) - V(x(k)) < 0, \quad \forall k : \; x(k) \neq 0.$$

The inequality above implies that the energy in the system decreases at each time instant and will eventually become 0. We note that in the linear plant case, for $V(x)$ to be a Lyapunov function, it is necessary that the matrix $P > 0$ satisfies the inequality

$$R := P - A^T PA + A^T PB(B^T PB)^{-1}B^T PA > 0 \tag{5}$$

In the interest of reducing the communication rate, we should look for a quantizer in (4) that is coarse. The coarseness of a quantizer Q is measured by how densely the quantized values are distributed. More specifically, define the density of the quantizer Q by

$$d_Q := \limsup_{\epsilon\to 0} \frac{\sharp u[\epsilon]}{-\ln \epsilon}, \tag{6}$$

where $\sharp u[\epsilon]$ denotes the number of quantized values q_j in the interval $[\epsilon, 1/\epsilon]$. This density d_Q can be viewed as the average number of quantization levels that Q has in a certain logarithmic sense.

The problem of finding the coarsest quantizer consists of two steps, which are formulated as follows: The first step is to find the quantizer Q with the minimum density d_Q subject to quadratic stability of the closed-loop system with respect to the given quadratic function $V(x) = x^T Px$. The second step is to find the quantizer that minimizes the density over all quadratic functions.

B. Control with Logarithmic Quantization

In this problem, the class of quantizers having the coarsest structure is shown to be that of the so-called logarithmic quantizers. These have the characteristic that the width of the partition interval becomes larger away from the origin, and finer closer to the origin. For the given quadratic function

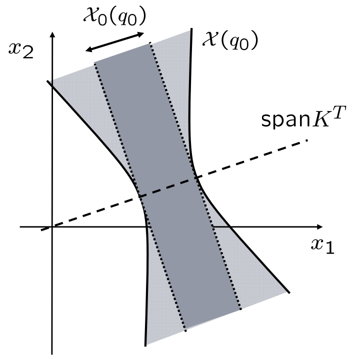

Fig. 3. The sets $\mathcal{X}(q_0)$ and $\mathcal{X}_0(q_0)$ in the state space

$V(x) = x^T Px$, let the state feedback gain be $K := -B^T PA/(B^T PB)$ and let

$$\rho := \frac{1+\delta}{1-\delta} \quad \text{with} \quad \delta := \sqrt{\frac{B^T PB}{B^T PAR^{-1}A^T PB}}. \tag{7}$$

Under the control of (4), the coarsest quantizer is given by

$$Q(v) = \begin{cases} q_i & \text{if } v \in \left(\frac{\rho+1}{2}q_{i-1}, \frac{\rho+1}{2}q_i\right], \\ -q_i & \text{if } v \in \left[-\frac{\rho+1}{2}q_i, -\frac{\rho+1}{2}q_{i-1}\right), \\ 0 & \text{if } v = 0, \end{cases} \tag{8}$$

where the quantized values are $q_i = \rho^i q_0$ for $i \in \mathbb{Z}$ with $q_0 > 0$. This function is presented in Fig. 2. In logarithmic quantizers, the key parameter $\rho > 1$ represents the expansion ratio, and a large ρ means a coarse quantizer.

The idea in finding the logarithmic quantizer above can be roughly explained as follows. For simplicity, assume $R = I$ in (5). First, take a real value q_0 and consider using this particular value for the control as $u = q_0$. Then, we can characterize the states where the function $V(x)$ will decrease in the next time step. The set of all such states is given by

$$\mathcal{X}(q_0) := \left\{x \in \mathbb{R}^n : \ V(Ax + Bq_0) - V(x) < 0\right\}.$$

It can be shown that this set $\mathcal{X}(q_0)$ is symmetric about the line spanned by the gain vector K^T; see Fig. 3. Now, recall that it is Kx that is quantized. Thus, we must assign this control value q_0 to states contained in the set $\mathcal{X}(q_0)$ with the form $\alpha K^T + \beta$, where $\alpha \in \mathbb{R}$ and $\beta \in \mathrm{Ker}K$. The largest set $\mathcal{X}_0(q_0) \subset \mathcal{X}(q_0)$ containing all such states can be found to be a region between two hyperplanes orthogonal to K^T. In fact, it has the structure

$$\mathcal{X}_0(q_0) = \left\{\alpha K^T + \beta : \ \alpha \in [c_0 q_0, c_0 \rho q_0], \ \beta \in \mathrm{Ker}K\right\}$$

for some $c_0 > 0$. Clearly, the expansion ration ρ appears here. This discussion leads us to a logarithmic partitioning in the state space by the sets $\mathcal{X}(\pm q_i)$, $i \in \mathbb{Z}$, which essentially corresponds to the quantization function that is shown in (8).

C. Limitation in Quantizer Coarseness

The second step in the problem is to maximize the parameter ρ over all quadratic Lyapunov functions. Formally, this problem is stated as

$$\rho^* = \sup_{P>0} \rho. \tag{9}$$

This problem will definitely give us the coarsest quantizer. A closed form solution was given in [7]. The following theorem shows that the maximum ρ is determined only by the product of plant unstable poles.

Theorem 2.1: Consider the plant (1) under the quantized control (4). The quantizer Q that minimizes the density d_Q in (6) subject to quadratic stabilization is given by a logarithmic quantizer in (8) whose expansion ratio ρ^* is given by

$$\rho^* = \frac{\prod_i |\lambda_i^u| + 1}{\prod_i |\lambda_i^u| - 1}, \tag{10}$$

where λ_i^u are the unstable eigenvalues of the system matrix A of the plant. In the corresponding Lyapunov function $V(x) = x^T Px$, the matrix $P > 0$ is the solution to the following Riccati equation

$$A^T PA - P - \frac{A^T PBB^T PA}{B^T PB + 1} = 0.$$

This result suggests us that as the plant becomes more unstable, we need finer quantization to guarantee quadratic stabilization. It also provides a form of limitation in the context of networked control involving both control and communication aspects.

D. Numerical Example

We present a numerical example to demonstrate the utility of the quantized control scheme. Consider the second-order system given by

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 1.8 & -0.3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k). \tag{11}$$

The system is unstable with two poles 1.2 and -1.5. To achieve quadratic stabilization, by Theorem 2.1, the maximum coarseness for the logarithmic quantizer is given by $\rho^* = 3.5$. Here, we chose $\rho = 3$ and, as a result, the state feedback gain was found as $K = \begin{bmatrix} -1.80 & 0.520 \end{bmatrix}$. In Fig. 4, the time responses of the state x_2 and the quantized control input u with the initial state $x(0) = \begin{bmatrix} 100 & 100 \end{bmatrix}^T$ are shown. We confirm the closed-loop stability. The responses decay exponentially, resulting from the quadratic stability attained by the control scheme.

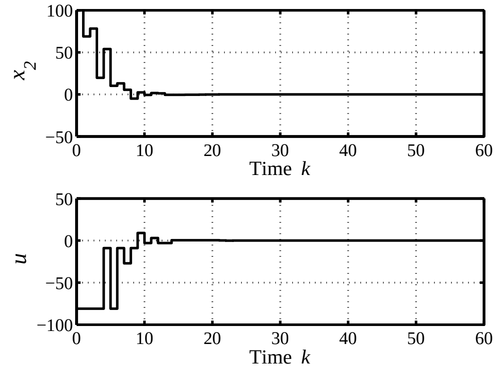

Fig. 4. Time responses: The state x_2 (top) and the control input u (bottom)

III. The Coarsest Quantizer for Stabilization over Lossy Channels

In this section, we extend the results on the coarsest quantizer and consider the case where the channel is unreliable and data loss may occur. We show that the limitation observed for coarse quantization can be generalized. The material is from [29].

When the reliability of the communication channel is insufficient, packets containing control-related signals may get lost during the transmissions. If this occurs often, the control performance degrades and, even worse, stability may not be guaranteed. We employ a packet loss model with the assumption that the probability of a loss is constant for all transmissions. The loss probability will be utilized in the control scheme as a priori known information. This is clearly one of the most simple models for unreliable channels (see, e.g., [4]).

In recent years, much attention has been devoted to the research on issues related to packet losses in networked control. The stochastic stabilization of scalar systems has been reported in [11], and then state estimation problems have been addressed in [27]. For controller design, LQ control schemes are given in [15] for remote control and in [10] for a general network topology setup. Approaches based on H^∞ control are studied in [17], [25].

It is interesting to note that in the presence of probabilistic packet losses, limitations in networked control can be derived. In particular, the following two points are known: (i) To guarantee stability in a stochastic sense, it is necessary and sufficient that the loss probability is smaller than a certain critical bound. (ii) This bound depends only on the level of instability of the plant. The objective of this section is to present an extension of the coarsely quantized control approach to the case when the channels are lossy.

A. Problem Formulation

Consider the networked control system in Fig. 1. The problem setup is almost the same as in the previous case: The plant is as given in (1), where the state x is measured; on the sensor side are the state feedback K and the memoryless quantizer Q, and thus the input to the channel is $\hat{v}(k) = Q(Kx(k))$ as in (3).

However, in the current case, the channel is assumed to be unreliable, and the packet transmitted from the sensor side may not reach the actuator side; when such a loss occurs, we necessarily have $u(k) \neq \hat{v}(k)$. We assume that packet losses occur with probability $\alpha \in (0,1)$ at each time k, independently of other times. The random process that represents the losses is denoted by $\theta(k)$, $k \in \mathbb{Z}_+$, whose probability distribution is given by

$$\mathrm{Prob}\{\theta(k) = i\} = \begin{cases} \alpha & i = 0, \\ 1-\alpha & i = 1. \end{cases} \tag{12}$$

In the case of a loss, the convention is that the control input is set to zero, and otherwise the received data will be applied. Hence, using the process $\theta(k)$, we can write the control input as $u(k) = \theta(k)\hat{v}(k)$.

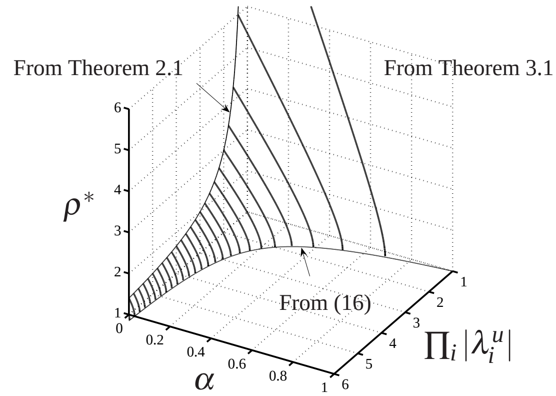

Fig. 5. The relationship between the product of unstable poles $\prod_i |\lambda_i^u|$, the loss probability α, and the coarseness ρ^* of the quantizer.

Consequently, the state-space equation of the closed-loop system becomes

$$x(k+1) = Ax(k) + B\theta(k)\hat{v}(k). \tag{13}$$

This system is a nonlinear system with stochastic switching. Hence, as the notion of stability, we need to employ a stochastic version of quadratic stability defined as follows: For the system (13), the origin is said to be stochastically quadratically stable if there exists a positive-definite function $V(x) = x^T Px$ in a quadratic form and a positive-definite matrix R such that

$$\begin{aligned} \Delta V &:= E[V(x(k+1))|x(k)] - V(x(k)) \\ &\leq -x(k)^T Rx(k), \qquad \forall x(k) \in \mathbb{R}^n. \end{aligned} \tag{14}$$

It is known that the condition above is sufficient to guarantee the origin of the system (13) to be mean-square stable (see, e.g., [3]): For every initial state $x(0)$, it holds that

$$\lim_{k\to\infty} E\left[\|x(k)\|^2\right] = 0. \tag{15}$$

The problem that we consider here is to find the coarsest quantizer such that the quantized control system with random packet losses in (13) is stochastically quadratically stable. As the measure of coarseness for the quantizer Q, we again employ the density d_Q given in (6). Note that this problem is a clear generalization of the one in the last section, where the loss probability α is set to zero; we have seen in Theorem 2.1 the limitation regarding quantization.

On the other hand, another special case of this problem is when quantization is not present. That is, in the system in Fig. 1, $\hat{v}(k) \equiv v(k)$. Then, another form of limitation in networked control is known for the loss probability α. Note that the closed-loop system is now linear, which in turn makes the two stability notions of stochastically quadratic stability and mean-square stability equivalent. It has been shown in [6], [16] that there exists a state feedback gain K such that the closed-loop system in (13) is mean-square stable if and only if

$$\alpha < \frac{1}{\prod_i |\lambda_i^u|^2}. \tag{16}$$

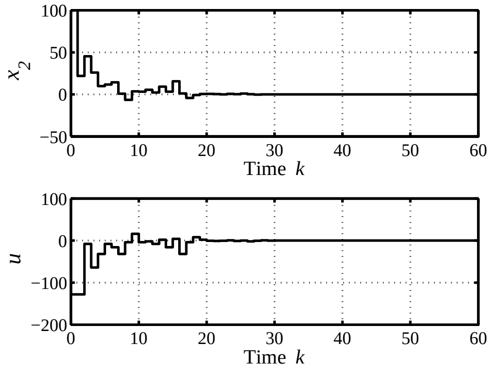

Fig. 6. Time responses of the system designed for the lossy channel: The state x_2 (top) and the control input u (bottom)

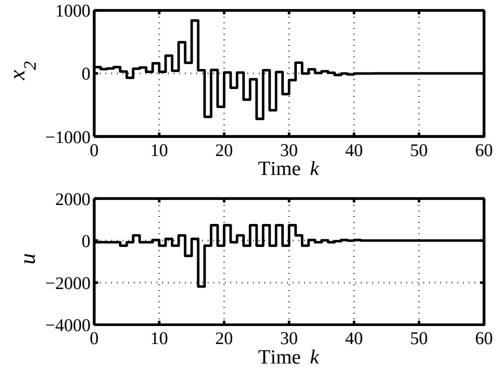

Fig. 7. Time responses of the system designed without accounting for the losses: The state x_2 (top) and the control input u (bottom)

This obviously implies that from the viewpoint of mean-square stability, there is an upper bound on the loss probability, having a tight relation with the unstable poles of the plant. Hence, for more unstable plants, more reliable channels are necessary, which is a natural and reasonable implication.

B. Limitations in Quantization and Packet Loss

Going back to the problem of this section, we consider the effects of both quantization and packet losses. Our goal is to clarify the relationship among the three key parameters: ρ^*, α, and $\prod_i |\lambda_i^u|$. A complete solution to this problem is provided in the theorem below due to [29].

Theorem 3.1: Consider the quantized control system with unreliable communication in (13). The quantizer Q that minimizes the density d_Q in (6) subject to stochastic quadratic stabilization is given by the logarithmic quantizer in (8). Its expansion ratio ρ^* is expressed as

$$\rho^* = \frac{1+\delta^*}{1-\delta^*}, \quad \text{where} \quad \delta^* = \sqrt{\frac{\frac{1}{\prod_i |\lambda_i^u|^2} - \alpha}{1-\alpha}} \qquad (17)$$

and the loss probability α is chosen such that the inequality (16) holds.

Theorem 3.1 generalizes the two previous results in (10) from [7] and (16) from [6], [16]. In particular, as in the case without packet losses, the coarsest quantizer is of logarithmic type represented by the expansion ratio ρ^*.

Fig. 5 summarizes the results mentioned above. The relations in (10) and (16) are found, respectively, on the $\prod_i |\lambda_i^u|$–ρ^* plane and on the $\prod_i |\lambda_i^u|$–α plane. The curved surface represents (17) in Theorem 3.1. It is easy to see that (17) unifies the two previous results. The result also shows a tradeoff between α and ρ^*, i.e., to achieve closed-loop quadratic stability, high packet loss probabilities require quantizers with fine resolution and vice versa.

C. Numerical Example

We continue with the example in Section II-D. As the plant, we employ the unstable one in (11). In Fig. 5, this system corresponds to the cross section obtained by cutting the surface at $\prod_i |\lambda_i^u| = 1.8$. From the upper bound given in (16), we must select a communication channel with a packet loss rate $\alpha < 0.31$. Here, we chose a channel with $\alpha = 0.2$, in which case, the maximum ρ is about $\rho^* = 2.17$. We took the quantizer parameter as $\rho = 2$. The proposed method then yields the feedback gain $K = \begin{bmatrix} -1.80 & 0.651 \end{bmatrix}$. Notice that this gain is similar to the one that was found before.

In Fig. 6, the sample paths of the state x_2 and the quantized control input u with the same initial state $x(0) = \begin{bmatrix} 100 & 100 \end{bmatrix}^T$ are shown. We confirm the closed-loop stability. The responses are somewhat oscillatory, but decay almost exponentially. This good transient is due to the notion of mean-square stability, which takes account of the control performance in the average sense.

To exhibit the difference from the previous design in Section II-D, we applied the control law from there to the current setting with the lossy channel. By simulating the system under the same loss process as above, the sample paths of x_2 and u shown in Fig. 7 were obtained. The difference in the performance is clear. The trajectories for the current case, where the controller does not take account of the losses, start to oscillate with large magnitudes after about 10 time steps.

IV. Quantized Adaptive Control for Uncertain Systems

In this section, we consider the quantized stabilization of an uncertain system whose uncertainty bounds are unknown. In particular, we present an extension of [7] based on adaptive control; the material is from [13]. Under this approach, to ensure system performance, the controller adjusts its feedback gain as well as the coarseness in quantization in response to the plant outputs.

The system setup is depicted in Fig. 8. The plant is LTI with uncertain parameters. The controller is on the sensor side, and the control input is quantized to be sent over the channel; we assume that the channel is noiseless. The quantizer is time varying, and at each time instant, its parameters are determined and adjusted corresponding to the updates in the feedback gain. Here, we employ a logarithmic quantizer and aim at keeping it as coarse as possible at each moment.

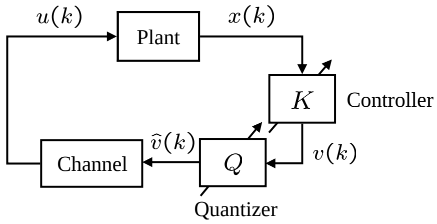

Fig. 8. Quantized adaptive control scheme

The proposed control scheme has three features as follows. First, our adaptive approach is a generalization of Theorem 2.1 from [7] in that if the system matrices are completely known, the controller and quantizer coincide with the optimal ones given in Theorem 2.1. Second, the coarseness in quantization is time varying and, in particular, it must be fine while the controller gain is large, and vice versa. In general, this implies that systems that are more unstable would require more information for stabilization. This may appear contradictory to the fact that the unknown plant parameters, which determine the coarsest quantization levels according to Theorem 2.1, are constant. However, it is noted that the coarsest quantizer requires a specific gain, and in the adaptive case, this gain is unknown. Third, since the coarseness of the quantization varies with time, it is necessary to send over the channel the information specifying the quantizer being used. We also note that an adaptive quantized control method for continuous-time uncertain nonlinear plants is developed in [14].

From the adaptive control viewpoint, we emphasize that the proposed approach is Lyapunov-based. In particular, it employs the method of [12], to which without quantization the scheme in this section reduces. This method guarantees Lyapunov stability of the plant state and the adaptive gain and attraction with respect to the plant state. We note that such an approach is standard in the continuous-time case, but for discrete-time systems, recursive least squares and least mean square algorithms are typically used (e.g., [9]); the primary focus there is on state convergence rather than stability in the sense of Lyapunov.

A. *Problem Formulation*

We introduce the quantized adaptive control problem for linear uncertain plants.

Consider the networked control system in Fig. 8. The state-space equation of the plant is as in (1) given by

$$x(k+1) = Ax(k) + Bu(k), \tag{18}$$

where $x(k) \in \mathbb{R}^n$ is the state, and $u(k) \in \mathbb{R}$ is the control input. We assume that the pair (A, B) is stabilizable, but the matrix A is unknown.

The state $x(k)$ is measured and the control input to be quantized is generated through

$$v(k) = K(k)x(k), \tag{19}$$

where $K(k)$ is the adaptive feedback gain. Then, the signal $v(k)$ is to be sent over the channel and quantized via

$$u(k) = Q_k(v(k)). \tag{20}$$

Here, Q_k represents the time-varying logarithmic quantizer of the form

$$Q_k(v) = \begin{cases} q_i(k) & \text{if } v \in \left(\frac{\rho(k)+1}{2} q_{i-1}(k), \frac{\rho(k)+1}{2} q_i(k)\right], \\ -q_i(k) & \text{if } v \in \left[-\frac{\rho(k)+1}{2} q_i(k), -\frac{\rho(k)+1}{2} q_{i-1}(k)\right), \\ 0 & \text{if } v = 0, \end{cases} \tag{21}$$

where the quantized values are $q_i(k) = \rho(k)^i q_0$ for $i \in \mathbb{Z}$ with $q_0 > 0$. Note that the expansion ratio $\rho(k)$ determines coarseness of the quantizer at time k.

The logarithmic quantizer (21) can be viewed as a time-varying sector-bounded memoryless nonlinearity. Letting

$$\delta(k) := \frac{\rho(k) - 1}{\rho(k) + 1},$$

we can write the sector condition for Q_k as

$$(1 - \delta(k))v^2 \leq Q_k(v)v \leq (1 + \delta(k))v^2, \quad v \in \mathbb{R}. \tag{22}$$

Thus, the parameter $\delta(k)$ specifies the bounds. A notable difference from the static quantizer case is that the information on the quantizer being used at each time k must be shared by both the sensor and the actuator. Here, we realize this by quantizing $\delta(k)$ and then transmitting it over the channel as well. We will later explain more on how to perform this quantization.

Our objective is to design a stabilizing adaptive controller in the form of (19) and a coarse logarithmic quantizer Q_k. We present a control law that ensures stability of the closed-loop system. Here, the results are limited to the case that the matrix B is known, but this assumption may be relaxed; for details, see [13].

The system matrix A is unknown under the following assumption: Let $\mathcal{A} := \{A + BK_{\mathrm{g}}^{(1)} : K_{\mathrm{g}}^{(1)} \in \mathbb{R}^{1\times n}\}$. Assume that there is a known matrix $\tilde{A} \in \mathcal{A}$ in this set, which may be unstable. This assumption holds for the class of systems in the controllable canonical form.

B. *Quantized Adaptive Control Law*

We now provide the design procedure of the adaptive controller and then present the proposed scheme.

Let $R \in \mathbb{R}^{n\times n}$ be a positive-definite matrix and let $\gamma > 0$. Find the positive-definite solution $P \in \mathbb{R}^{n\times n}$ of the Riccati equation

$$P = \tilde{A}^T P \tilde{A} + R - \tilde{A}^T PB(B^T PB)^{-1} B^T P \tilde{A} \tag{23}$$

with $P \geq I$. Such a P always exists. Then, let $A_{\mathrm{s}} := \tilde{A} + BK_{\mathrm{g}}^{(2)}$ with $K_{\mathrm{g}}^{(2)} := -B^T P\tilde{A}/(B^T PB)$. This matrix A_{s} is stable. Finally, take $\sigma \in (0, 2)$.

The proposed adaptive control law is given as

$$v(k) = K(k)x(k).$$

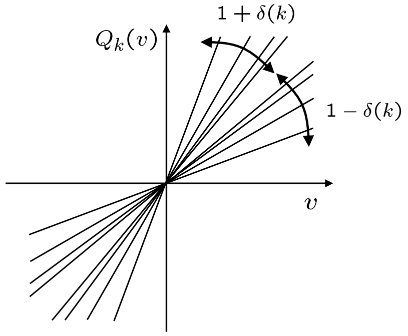

Fig. 9. Sector bounds for a time-varying logarithmic quantizer

Here, the gain $K(k) \in \mathbb{R}^{1\times n}$ is given by the update law

$$\begin{aligned} &K(k+1) \\ &= K(k) - \frac{\sigma}{1+x^T(k)Px(k)} B^\dagger \{x(k+1) - A_s x(k) \\ &\qquad - B\big[Q_k(v(k)) - v(k)\big]\} x^T(k), \quad (24) \end{aligned}$$

where $B^\dagger$ denotes the pseudo inverse of B. The quantizer Q_k in (20) is chosen such that the parameter $\delta(k)$ satisfies

$$R - 2\delta(k)^2 K^T(k) B^T P B K(k) \geq \gamma I. \quad (25)$$

We can establish closed-loop stability under this adaptive control law as follows.

Theorem 4.1: Consider the linear uncertain system in (18), where $A \in \mathbb{R}^{n\times n}$ is an unknown matrix, but $\tilde{A} \in \mathcal{A}$ is known. Then, the adaptive control law given above guarantees that the solution $(x(k), K(k)) \equiv (0, K_g)$, where $K_g := -B^T PA/(B^T PB)$, of the closed-loop system is Lyapunov stable. Furthermore, $x(k) \to 0$ as $k \to \infty$ for each initial state $x(0) \in \mathbb{R}^n$.

This theorem shows that the solution $(x(k), K(k)) \equiv (0, K_g)$ of the overall closed-loop system is Lyapunov stable and that the state $x(k)$ converges to the origin. This stability notion is known as partial asymptotic stability. The convergence of the gain $K(k)$ follows from (24) though the gain may not go to K_g.

We have several remarks on the adaptive quantization scheme. As shown in (22), the coarseness in quantization is determined by the sector bounds and in particular the parameter $\delta(k)$. This must be chosen so that the bound in (25) holds at all times. In (25), we observe that quantization must be fine while the controller gain is large, and vice versa. This is in contrast to the nonadaptive case in the previous sections, where specific (constant) gains are used for a known system to maximize the coarseness in quantization.

To meet the requirement in (25), the information regarding $\delta(k)$ must be communicated to the actuator side and hence must also be quantized. A simple scheme for this quantization is to employ a logarithmic one as follows (see Fig. 9): Let $\delta(k)$ take values of the form $\mu^{-\ell}$ with $\ell \in \mathbb{Z}_+$ and $\mu > 1$. Then, (25) can always be satisfied since $\delta(k)$ can be arbitrarily small. In this scheme, it suffices to send the index ℓ over the channel.

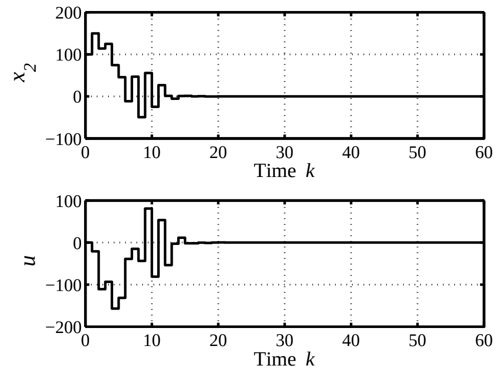

Fig. 10. Time responses: The state x_2 (top) and the control input u (bottom)

The theorem has a close connection with the result in Section II in the special case when perfect knowledge of the plant is available. This can be seen as follows. Since the plant is known, we may take $\tilde{A} = A$ and thus

$$K(k) \equiv K_g = -B^T PA/(B^T PB), \quad (26)$$

where P is the solution of the Riccati equation (23). Then, with $\sigma = 0$, the update law in (24) of the gain is unnecessary. Further, in this case, the sector condition (25) can be expressed as $R - \delta^2 K_g^T B^T PBK_g > 0$. By (26), this condition can be rewritten to provide the maximum δ as

$$\delta = \sqrt{\frac{B^T PB}{B^T PAR^{-1}A^T PB}}. \quad (27)$$

This is precisely the result given in (7). characterizing the coarsest possible quantizer for the given matrices A, B, and R. Moreover, it is shown there that properly choosing R in (27) further leads us to the coarsest possible quantizer, which is determined solely by the unstable eigenvalues of A.

C. Numerical Example

We present a numerical example to demonstrate the utility of the proposed adaptive control scheme. We consider the same plant (11) in Section II-D:

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

where the parameters a_0 and a_1 are assumed to be unknown. The true values are $a_0 = -1.8$ and $a_1 = 0.3$. The plant is in the controllable canonical form. Hence, take $\tilde{A} \in \mathcal{A}$ as

$$\tilde{A} = A + BK_g^{(1)} = \begin{bmatrix} 0 & 1 \\ \theta_0 & \theta_1 \end{bmatrix}$$

with $K_g^{(1)} = \begin{bmatrix} \theta_0 + a_0 & \theta_1 + a_1 \end{bmatrix}$, where $\theta_0, \theta_1 \in \mathbb{R}$ can be arbitrary. Here, we chose them as $\theta_0 = -0.167$ and $\theta_1 = 0.833$. In the design of the adaptive controller, we chose $R = I$, $\sigma = 0.6$, and $\gamma = 0.25$. The coarseness of the quantizer is determined via the parameter $\delta(k)$, which is quantized and takes discrete values as

$$\delta(k) = 1.1^{-\ell(k)} \text{ with } \ell(k) \in \mathbb{Z}_+. \quad (28)$$

Fig. 10 shows the time responses of $x_2(k)$ and the control input $u(k)$ with the initial conditions $x(0) = [100\ 100]^T$ and

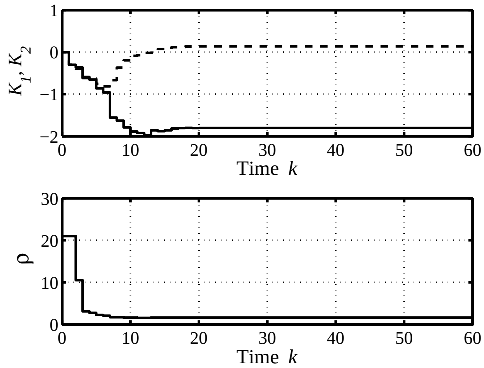

Fig. 11. Time responses: The gains K_1 (top: solid line), K_2 (top: dashed line) and ρ (bottom)

$K(0) = [0\ 0]$. Furthermore, Fig. 11 displays the responses of the gains, where $K(k) = [K_1(k)\ K_2(k)]$, as well as the expansion ratio $\rho(k)$. It is interesting to note that the gain $K(k)$ converges as $\lim_{k\to\infty} K(k) = [-1.80\ 0.138]$; this final vector takes a similar form as the gains used in earlier examples in Sections II and III. Further, it can be seen that $\rho(k)$ remains large for several time steps in the beginning, but then it gradually decreases. This shows that the communication rate for control is low while the adaptive gains are small. The final value of $\rho(k)$ is 1.63. In view of the earlier examples, this value is not small. We also note that for the quantized values of $\delta(k)$, the parameter $\ell(k)$ in (28) ranged between 1 and 15.

V. Conclusion

In this paper, we have discussed the quantized control approach that was initiated in [7]. This is based on the notion of coarseness of static quantizers and the objective is stabilization with a limited amount of information. We then presented two extensions, one for lossy chancels [29] and the other for uncertain plants [13]. Future research should address quantized adaptive control over lossy channels [26].

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