Monodromy Operator Approach to Time-Delay Systems: Numerical Method for Solving Operator Lyapunov Inequalities

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***Abstract*— This paper extends the technique for stability analysis of time-delay systems based on the operator Lyapunov inequalities about (fast-lifted) monodromy operators by establishing a numerical method for solving the operator Lyapunov inequalities. First, quasi-finite-rank approximation is applied to the fast-lifted monodromy operator, and an operator Lyapunov inequality is considered with respect to the resulting approximated operator. Then, a numerically tractable method is developed for finding a solution to the approximated operator Lyapunov inequality out of a special class of operators that is known to be nonconservative (with respect to the original operator Lyapunov inequality before quasi-finite-rank approximation) as long as the fast-lifting parameter N is large enough. The above special class is described by two finite-dimensional matrices, and thus solving the operator Lyapunov inequality amounts to solving a finite-dimensional LMI. In particular, due to the discrete-time viewpoint intrinsic to the (fast-lifted) monodromy operator approach, the resulting LMI is a discrete-time one, which is suitable for extension to the case with discrete-time controllers. A method is also provided to confirm that the solution to the approximated operator Lyapunov inequality does solve the original operator Lyapunov inequality. Furthermore, it is shown that the overall procedure gives an asymptotically exact numerical method for stability analysis of time-delay systems. A numerical example illustrating the arguments of the paper is also given. A brief sketch is also provided on the extension to the use of generalized hold and sampling operators $\mathbf{J}_{Hk}$ and $\mathbf{J}_{Sk}$ based on Legendre polynomials.**

I. INTRODUCTION

Stimulated by the study in [1], we have developed in our recent studies [2],[3] a fundamental framework for dealing with time-delay systems, which we call a monodromy operator approach. In particular, we considered the feedback system in Fig. 1, denoted by Σ, consisting of the finite-dimensional linear time-invariant (FDLTI) system F and the pure delay H, where F has the state-space representation

$$\frac{dx}{dt} = Ax + Bu,\ y = Cx + Du \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times \mu}$, $C \in \mathbb{R}^{\mu\times n}$ and $D \in \mathbb{R}^{\mu\times \mu}$, and the input-output relation of the pure delay H is given by $u(t) = y(t-h),\ h > 0$. By introducing the monodromy operator $\mathbf{T}$ associated with Σ and then taking a positive integer (fast-lifting parameter) N to introduce the fast-lifted monodromy operator $\mathbf{T}_N$, we have related the exponential stability of Σ to an operator Lyapunov inequality with respect to $\mathbf{T}$ (or $\mathbf{T}_N$). In particular, we have introduced a special class $\mathcal{P}_N$ of operators described by two finite-dimensional matrices P and Π (see (14)), and showed that a solution to the operator Lyapunov inequality exists within that class whenever Σ is exponentially stable, as long as N is large enough. While the studies in [2],[3] are thus devoted to a theoretical side of the monodromy operator approach to time-delay systems (TDS), this paper extends the studies so that the operator Lyapunov inequality can be solved numerically, and establishes a numerical method for dealing with the stability of TDSs.

The contents of this paper are as follows. In Section II, we review the preliminary results developed in [2],[3] regarding the stability analysis of Σ via the monodromy operator approach as well as the fast-lifting technique [4]–[6]. Roughly speaking, we recall that the stability analysis problem of Σ reduces to finding $\mathbf{P} \in \mathcal{P}_N$ such that the operator Lyapunov inequality $\mathbf{T}_N^*\mathbf{P}\mathbf{T}_N \prec \mathbf{P}$ holds with respect to the fast-lifted monodromy operator $\mathbf{T}_N$. Even though it has been established theoretically that whenever Σ is stable, such a solution $\mathbf{P}$ does exist provided that N is large enough [2],[3], it has not been discussed how to find such $\mathbf{P}$ numerically. To facilitate the numerical computation for finding $\mathbf{P}$, Section III introduces what we call quasi-finite-rank approximation $\mathbf{T}_{NX}$ of $\mathbf{T}_N$ by applying a recent technique developed for sampled-data systems [7], and gives a theoretical approach to solving the operator Lyapunov inequality approximately by dealing with $\mathbf{T}_{NX}$ instead of $\mathbf{T}_N$. Furthermore, a method is provided that confirms if the approximate solution indeed solves the original inequality without quasi-finite-rank approximation. It is further discussed that such a "two-stage approach" to stability analysis of Σ is ensured to be asymptotically exact (i.e., whenever Σ is stable, a solution to the operator Lyapunov inequality can indeed be found also numerically in such a way) as N is increased, provided that some appropriate steps are followed in the two-stage procedure. Carrying out the overall procedure numerically involves the optimization of the matrices P and Π involved in $\mathbf{P} \in \mathcal{P}_N$, together with the computations of $\|\mathbf{S}\|$ and $\|\mathbf{S}^{-1}\|$, where $\mathbf{S}$ denotes the square root of $\mathbf{P}$. Section IV gives methods for reducing the optimization of P and Π and the computations of $\|\mathbf{S}\|$ and

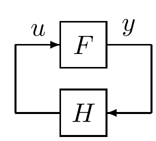

Fig. 1. Feedback system Σ with delay H.

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$\|\mathbf{S}^{-1}\|$ to finite-dimensional ones. In particular, we show that all the associated computations can be reduced to discrete-time LMIs despite the fact that the system $\varSigma$ we deal with is a continuous-time system. This leads to an important feature of our approach that it can readily be extended also to the case with discrete-time controllers. Section V gives a numerical example illustrating the arguments of the paper. Section VI gives a brief sketch on the extension to the use of generalized hold and sampling operators studied in [3] to extend the class $\mathcal{P}_N$.

We use the following notation in this paper. $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and positive integers, respectively. $\mathcal{K}_m$ is a shorthand notation for the Hilbert space $(L_2([0,h);\mathbb{R}))^m$ with an underlying $h > 0$. The symbol $\otimes$ denotes the Kronecker product of matrices, and for a matrix $(\cdot)$, we use the shorthand notations $\mathbf{I}(\cdot)$, $\mathbf{O}(\cdot)$ and $\overline{(\cdot)}$ to mean $\mathrm{diag}[I,(\cdot)]$, $\mathrm{diag}[0,(\cdot)]$, and $I_N \otimes (\cdot)$, respectively. These notations are also used in a parallel fashion for an operator, too. $\mathbf{G}_1 \succ \mathbf{G}_2$ (or $\mathbf{G}_2 \prec \mathbf{G}_1$) means that $\mathbf{G}_1 - \mathbf{G}_2$ is a strictly positive-definite operator [2],[3].

II. Review on Monodromy Operator and Operator Lyapunov Inequality

In this section, we review the preliminary results derived in [2],[3] regarding the stability analysis of $\varSigma$ via the (fast-lifted) monodromy operator and the associated operator Lyapunov inequality.

We first introduce the matrix $A_d \in \mathbb{R}^{n\times n}$ and the operators $\mathbf{B} : \mathcal{K}_\mu \to \mathbb{R}^n$, $\mathbf{C} : \mathbb{R}^n \to \mathcal{K}_\mu$ and $\mathbf{D} : \mathcal{K}_\mu \to \mathcal{K}_\mu$ defined by

$$A_d = \exp(Ah) \tag{2}$$

$$\mathbf{B}f = \int_0^h \exp(A(h-\tau))Bf(\tau)d\tau \tag{3}$$

$$(\mathbf{C}v)(\theta) = C\exp(A\theta)v \tag{4}$$

$$\begin{aligned}(\mathbf{D}f)(\theta) &= ((\mathbf{D}_0 + D)f)(\theta)\\ &= \int_0^\theta C\exp(A(\theta-\tau))Bf(\tau)d\tau + Df(\theta)\end{aligned} \tag{5}$$

and the monodromy operator of the system $\varSigma$ defined by

$$\mathbf{T} = \begin{bmatrix} A_d & \mathbf{B} \\ \mathbf{C} & \mathbf{D}\end{bmatrix} : \mathcal{M} \to \mathcal{M} \tag{6}$$

where $\mathcal{M} := \mathbb{R}^n \oplus \mathcal{K}_\mu$. Roughly speaking, $\mathbf{T}$ represents the discrete-time transition of the state $[x(kh)^T, u(kh+\cdot)^T]^T$ of the system $\varSigma$ (note that $u(kh+\tau)$, $0 \le \tau < h$ is not an input of $\varSigma$ any more, but is a part of the state of $\varSigma$ that has no external input). To follow the "pseudo-discretization" treatment of $\mathbf{T}$ developed in [2],[3], we take a positive integer N and consider the fast-lifting $\mathbf{L}_N : \mathcal{K}_\mu \to (\mathcal{K}'_\mu)^N$ [4]–[6], where $\mathcal{K}'_\mu$ denotes the Hilbert space $\mathcal{K}_\mu$ with h replaced by $h' := h/N$. Under the notation $\mathbf{I}(\cdot) = \mathrm{diag}[I,\ (\cdot)]$ for an operator $(\cdot)$, the fast-lifted monodromy operator of the system $\varSigma$ is given by

$$\mathbf{T}_N = \mathbf{I}(\mathbf{L}_N)\mathbf{T}\,\mathbf{I}(\mathbf{L}_N)^{-1} =: \begin{bmatrix} A_d & \mathbf{B}_N \\ \mathbf{C}_N & \mathbf{D}_N\end{bmatrix} : \mathcal{M}'_N \to \mathcal{M}'_N \tag{7}$$

where $\mathcal{M}'_N$ is a shorthand notation for $\mathbb{R}^n \oplus (\mathcal{K}'_\mu)^N$. Regarding the representation (7), we have

$$\mathbf{B}_N = \begin{bmatrix} (A'_d)^{N-1}\mathbf{B}' & \cdots & A'_d\mathbf{B}' & \mathbf{B}'\end{bmatrix} \tag{8}$$

$$\mathbf{C}_N = \begin{bmatrix} \mathbf{C}' \\ \mathbf{C}'A'_d \\ \vdots \\ \mathbf{C}'(A'_d)^{N-1}\end{bmatrix} \tag{9}$$

$$\mathbf{D}_N = \begin{bmatrix} \mathbf{D}' & 0 & \cdots & 0 \\ \mathbf{C}'\mathbf{B}' & \mathbf{D}' & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mathbf{C}'(A'_d)^{N-2}\mathbf{B}' & \cdots & \mathbf{C}'\mathbf{B}' & \mathbf{D}'\end{bmatrix} \tag{10}$$

$$\mathbf{D}' = \mathbf{D}'_0 + D : \mathcal{K}'_\mu \to \mathcal{K}'_\mu \tag{11}$$

where A'_d, $\mathbf{B}'$, $\mathbf{C}'$, $\mathbf{D}'$ and $\mathbf{D}'_0$ are defined as A_d, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{D}_0$, respectively, with h replaced by h'.

It has been shown in [2],[3] that $\varSigma$ is exponentially stable if and only if there exists a solution $\mathbf{P} \succ 0$ to the following operator Lyapunov inequality about the fast-lifted monodromy operator $\mathbf{T}_N$:

$$\mathbf{T}_N^*\mathbf{P}\mathbf{T}_N \prec \mathbf{P} \tag{12}$$

It has been further shown that whenever $\varSigma$ is exponentially stable, such a solution $\mathbf{P}$ can actually be found within some easily tractable class of operators described by two finite-dimensional matrices P and $\varPi$. To describe the details, we first need to recall the following definition of the operators $\mathbf{J}_{S0} : \mathcal{K}'_\mu \to \mathbb{R}^\mu$ and $\mathbf{J}_{H0} : \mathbb{R}^\mu \to \mathcal{K}'_\mu$:

$$\begin{aligned}\mathbf{J}_{S0}f &= \frac{1}{\sqrt{h'}}\int_0^{h'} f(\theta')d\theta',\\ (\mathbf{J}_{H0}v)(\theta') &= \frac{1}{\sqrt{h'}}v, \quad 0 \le \theta' < h'\end{aligned} \tag{13}$$

We then have $\mathbf{J}_{H0} = \mathbf{J}_{S0}^*$, and $\mathbf{J}_{S0}\mathbf{J}_{H0} = I$ on $\mathbb{R}^\mu$. Under this notation, it was shown in [2],[3] that, provided that N is large enough, $\varSigma$ is exponentially stable if and only if there exists a solution $\mathbf{P}$ to the operator Lyapunov inequality (12) with the form

$$\begin{aligned}\mathbf{P} = {}&\mathrm{diag}[I, \underbrace{\mathbf{J}_{H0},\cdots,\mathbf{J}_{H0}}_{N}]P\,\mathrm{diag}[I,\underbrace{\mathbf{J}_{H0},\cdots,\mathbf{J}_{H0}}_{N}]^*\\ &+\mathrm{diag}[0,\underbrace{\varPi,\cdots,\varPi}_{N}]\end{aligned} \tag{14}$$

where $P := (P_{ij})_{i,j=0}^N \in \mathbb{R}^{(n+\mu N)\times(n+\mu N)}$ and $\varPi \in \mathbb{R}^{\mu\times\mu}$ are matrices such that

$$P + \mathrm{diag}[0,\varPi,\cdots,\varPi] > 0, \quad \varPi > 0 \tag{15}$$

It should be noted that (15) is a necessary and sufficient condition for $\mathbf{P} \succ 0$. The class $\mathcal{P}_N$ is defined as the set of operators $\mathbf{P}$ satisfying (14) and (15).

III. Quasi-Finite-Rank Approximation of Fast-Lifted Monodromy Operator and Stability Analysis by Scaling

As we have summarized the idea developed in our recent studies [2],[3] in the above, the stability analysis problem reduces to finding $\mathbf{P} \in \mathcal{P}_N$ (and thus $\mathbf{P} \succ 0$ by definition) such that $\mathbf{T}_N^*\mathbf{P}\mathbf{T}_N \prec \mathbf{P}$. The operator $\mathbf{D}$ involved in $\mathbf{T}$, however, is always an infinite-rank operator (if we rule out the trivial case when the FDLTI system F is zero), and this is the case also for the operator $\mathbf{D}'$ that appears in $\mathbf{D}_N$ in the definition of $\mathbf{T}_N$. Hence, $\mathbf{T}_N$ also has infinite rank, which generally makes it hard to find $\mathbf{P} \in \mathcal{P}_N$ such that $\mathbf{T}_N^* \mathbf{P} \mathbf{T}_N \prec \mathbf{P}$. This paper is devoted to resolving this issue by reducing the problem to a finite-dimensional convex one and thus giving a numerically tractable procedure for stability analysis. We also establish that the analysis method developed here is asymptotically exact as N tends to infinity. More precisely, whenever Σ is stable, we can confirm the stability of Σ numerically with a finite N that is large enough.

A. Quasi-Finite-Rank Approximation and Stability Analysis by Scaling

To facilitate numerical computations for the solution $\mathbf{P}$, we consider approximating $\mathbf{T}_N$ with a more tractable operator. More specifically, we approximate the compact operator $\mathbf{D}_0' = \mathbf{D}' - D$ by the finite-rank operator $\mathbf{C}' X \mathbf{B}'$ with a matrix $X \in \mathbb{R}^{n \times n}$ and denote the approximation error by $\mathbf{E}_X' = \mathbf{D}_0' - \mathbf{C}' X \mathbf{B}'$. Then we define the operator $\mathbf{T}_{NX}$, which approximates the fast-lifted monodromy operator $\mathbf{T}_N$ by replacing the operators $\mathbf{D}'$ in the diagonal entries of $\mathbf{D}_N$ (see (10)) with $\mathbf{C}' X \mathbf{B}' + D$. That is, we define

$$\mathbf{T}_{NX} = \begin{bmatrix} I & 0 \\ 0 & \overline{\mathbf{C}'} \end{bmatrix} \begin{bmatrix} A_d & B_{dN} \\ C_{dN} & D_{dN} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \overline{\mathbf{B}'} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \overline{D} \end{bmatrix} \quad (16)$$

with

$$B_{dN} = \begin{bmatrix} (A_d')^{N-1} & \cdots & A_d' & I \end{bmatrix}, \; C_{dN} = \begin{bmatrix} I \\ A_d' \\ \vdots \\ (A_d')^{N-1} \end{bmatrix},$$

$$D_{dN} = \begin{bmatrix} X & 0 & \cdots & 0 \\ I & X & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A_d')^{N-2} & \cdots & I & X \end{bmatrix} \quad (17)$$

and $\overline{\mathbf{B}'} = \mathrm{diag}[\mathbf{B}', \cdots, \mathbf{B}']$, $\overline{\mathbf{C}'} = \mathrm{diag}[\mathbf{C}', \cdots, \mathbf{C}']$ and $\overline{D} = \mathrm{diag}[D, \cdots, D]$; we henceforth use the notation $\overline{(\cdot)}$ to denote N copies of $(\cdot)$ given by $\mathrm{diag}[(\cdot), \cdots, (\cdot)]$ for an operator or matrix $(\cdot)$. We refer to $\mathbf{T}_{NX}$ as quasi-finite-rank approximation of $\mathbf{T}_N$ since the first term on the right hand side of (16) is a finite-rank operator while the second term is an infinite-rank multiplication operator. Defining $\mathbf{E}_{NX} = \mathrm{diag}[0, \overline{\mathbf{E}_X'}]$, we have $\mathbf{T}_N = \mathbf{T}_{NX} + \mathbf{E}_{NX}$.

Obviously, it is important to evaluate the effect of quasi-finite-rank approximation on the stability analysis of Σ. To facilitate such treatment, here we introduce the (strictly positive-definite) square root $\mathbf{S}$ of $\mathbf{P}$, and consider the scaling approach given in [3]; the class of $\mathbf{S} = \mathbf{S}^* \succ 0$ such that $\mathbf{S}^*\mathbf{S} = \mathbf{P} \in \mathcal{P}_N$ is denoted by $\mathcal{S}_N$ (although $\mathcal{S}_N$ is in fact identical with $\mathcal{P}_N$, referring to $\mathcal{S}_N$ would be often helpful to stress that the arguments are about $\mathbf{S}$ rather than $\mathbf{P}$). We will also use the notation $\mathbf{P}_N$ and $\mathbf{S}_N$ for $\mathbf{P}$ and $\mathbf{S}$, respectively, when we wish to stress that $\mathbf{P} \in \mathcal{P}_N$ and $\mathbf{S} \in \mathcal{S}_N$.

Note that $\mathbf{P} \in \mathcal{P}_N$ solves the operator Lyapunov inequality with respect to the quasi-finite-rank approximation $\mathbf{T}_{NX}$, i.e., $\mathbf{T}_{NX}^* \mathbf{P} \mathbf{T}_{NX} \prec \mathbf{P}$, if and only if $\mathbf{T}_{NX}$ scaled with $\mathbf{S}$ is contractive, i.e., $\|\mathbf{S}\mathbf{T}_{NX}\mathbf{S}^{-1}\| < 1$. In view of this, suppose that we have an operator $\mathbf{S} \in \mathcal{S}_N$ such that

$$\|\mathbf{S}\mathbf{T}_{NX}\mathbf{S}^{-1}\| \le \alpha_0 \quad (18)$$

for some $\alpha_0 > 0$. Since we have

$$\begin{aligned} \|\mathbf{S}\mathbf{T}_N\mathbf{S}^{-1}\| &\le \|\mathbf{S}\mathbf{T}_{NX}\mathbf{S}^{-1}\| + \|\mathbf{S}\mathbf{E}_{NX}\mathbf{S}^{-1}\| \\ &\le \alpha_0 + \|\mathbf{S}\| \cdot \|\mathbf{S}^{-1}\| \cdot \|\mathbf{E}_X'\| \end{aligned} \quad (19)$$

it follows that if $\alpha_0 + \|\mathbf{S}\| \cdot \|\mathbf{S}^{-1}\| \cdot \|\mathbf{E}_X'\| < 1$, then we have $\mathbf{T}_N^* \mathbf{P} \mathbf{T}_N \prec \mathbf{P}$ and thus the exponential stability of Σ is ensured even though we only took the solution $\mathbf{P} = \mathbf{S}^*\mathbf{S}$ with respect to the quasi-finite-rank approximation $\mathbf{T}_{NX}$ (rather than the original $\mathbf{T}_N$).

Note that the above arguments are successful by considering $\mathbf{S} \in \mathcal{S}_N$ instead of $\mathbf{P} \in \mathcal{P}_N$; it would not be easy, without referring to $\mathbf{S}$, to have an (only modestly conservative) sufficient condition under which $\mathbf{T}_{NX}^* \mathbf{P} \mathbf{T}_{NX} \prec \mathbf{P}$ implies $\mathbf{T}_N^* \mathbf{P} \mathbf{T}_N \prec \mathbf{P}$. This is why we introduce the scaling treatment with $\mathbf{S}$ for the arguments in this section, so that we can deal only with the scalar inequalities (18) and $\alpha_0 + \|\mathbf{S}\| \cdot \|\mathbf{S}^{-1}\| \cdot \|\mathbf{E}_X'\| < 1$. In the following arguments, however, we will also keep referring to $\mathbf{P}$ whenever we talk about operator Lyapunov inequalities.

By summarizing the above arguments, we arrive at the following procedure for the stability analysis of Σ:

(i) Fix a positive integer N $(=: N_e)$. Determine X with which $\mathbf{D}_0'$ is approximated by $\mathbf{C}' X \mathbf{B}'$, and compute the approximation error $\|\mathbf{E}_X'\| = \|\mathbf{D}_0' - \mathbf{C}' X \mathbf{B}'\|$.

(ii) For $\mathbf{T}_{NX}$ associated with X determined in (i), minimize $\alpha = \|\mathbf{S}\mathbf{T}_{NX}\mathbf{S}^{-1}\|$ with respect to $\mathbf{S} \in \mathcal{S}_N$.

(iii) Check if $\alpha + \|\mathbf{S}\| \cdot \|\mathbf{S}^{-1}\| \cdot \|\mathbf{E}_X'\| < 1$. Exponential stability of Σ is ensured if this condition holds.

If we observe this procedure, it is reasonable to choose X in (i) such that $\|\mathbf{E}_X'\|$ is small. This is a well-studied problem by now [6],[8],[9] (see also the related arguments in [10]); (an upper bound of) $\|\mathbf{E}_X'\|$ can be computed for each fixed X, and in particular, X (approximately) minimizing $\|\mathbf{E}_X'\|$ can also be computed.

Hence the remaining problems are how to optimize the operator $\mathbf{S} \in \mathcal{S}_N$ and compute the corresponding α in (ii), and how to compute the condition number $\kappa(\mathbf{S}) := \|\mathbf{S}\| \cdot \|\mathbf{S}^{-1}\|$. These problems can actually be reduced to finite-dimensional problems, whose details will be deferred to the section to follow.

In the remaining part of this section, we aim at giving some remarks to show that the above procedure can give an asymptotically exact method for stability analysis as $N \to \infty$. More precisely, whenever Σ is stable, we can confirm its stability numerically with some finite fast-lifting parameter N if it is large enough. To confirm this claim, we first recall that as $N \to \infty$, restricting $\mathbf{S}$ to belong to $\mathcal{S}_N$ (i.e., $\mathbf{P}$ to belong to $\mathcal{P}_N$) leads to no loss of generality, as already established in [2],[3]. We also recall that $\|\mathbf{E}_X'\|$ can be made arbitrarily small under a suitable choice of X if we let $N \to \infty$ [6],[9]. We could thus arrive at the claim since we can make the term $\kappa(\mathbf{S})\|\mathbf{E}_X'\|$ in (iii) tend to zero as $N \to \infty$ and thus the condition therein virtually reduces to $\alpha < 1$ with respect to the optimization problem in (ii).

Rigorously speaking, however, we need one more step to establish the claim, since $\mathbf{S} \in \mathcal{S}_N$ is obtained by solving an optimization problem in (ii) *that depends on* N and thus $\mathbf{S}$ is also dependent on N. Thus, it could be the case that the sequence of $\kappa(\mathbf{S})$ for optimal $\mathbf{S}$ diverges as $N \to \infty$ in Step (iii). To get around the difficulties related to such possibilities, we can slightly modify the above procedure as follows. That is, we take $N = N_e$ in Step (i) and then proceed to (ii) to get $\mathbf{S} = \mathbf{S}_{N_e}$. For this $\mathbf{S}$, simply following Step (iii) would lead us to the process of checking the inequality $\alpha + \kappa(\mathbf{S})\|\mathbf{E}'_X\| < 1$. However, it should be noted that this inequality is only a sufficient condition for $\left(\mathbf{S}\mathbf{T}_{N_e}\mathbf{S}^{-1}\right)^* \mathbf{S}\mathbf{T}_{N_e}\mathbf{S}^{-1} \prec I$ for the given $\mathbf{S}$, or equivalently for $\mathbf{T}^*_{N_e}\mathbf{P}\mathbf{T}_{N_e} \prec \mathbf{P}$ for $\mathbf{P} = \mathbf{S}^*\mathbf{S} \succ 0$, and thus we have a chance to apply a less conservative sufficient condition. The modified step we will follow (if the condition in (iii) fails) is indeed to consider an alternative sufficient condition, as will be summarized as Step (iv) later. The ideas leading to such a step, as well as the asymptotic exactness of stability analysis ensured by following the step, will be explained in the following subsection.

B. Asymptotic Exactness of Stability Analysis by Scaling

The key idea is to take a positive integer $\nu \geq 2$ and then repeat Step (i) with $N = \nu N_e$ (hence we obtain a new X for $N = \nu N_e$, where we hope that the associated new $\|\mathbf{E}'_X\|$, denoted by $\|\mathbf{E}'_{X,\nu}\|$, would become smaller compared with the case of $N = N_e$). Basically, we once again apply Step (iii) under the new N $(= \nu N_e)$ and new X, together with "the same $\mathbf{S}$" as that obtained in Step (ii). Regarding the loose wording "the same $\mathbf{S}$," however, it should be noted that the underlying domain $\mathcal{M}'_N$ of the scaling operator $\mathbf{S}$ changes actually with ν (since the domain depends on $N = \nu N_e$). Thus, "the same $\mathbf{S}$" should actually mean embedding the operator $\mathbf{S} = \mathbf{S}_{N_e}$ on $\mathcal{M}'_{N_e}$ to an equivalent operator on $\mathcal{M}'_{\nu N_e}$; the latter equivalent operator will be denoted by $\mathbf{S}^{\uparrow\nu}$ $(= \mathbf{S}^{\uparrow\nu}_{N_e})$. More precisely, $\mathbf{S}^{\uparrow\nu}$ is defined as

$$\mathbf{S}^{\uparrow\nu} := \mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)\mathbf{S}\,\mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)^{-1} = \mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)\mathbf{S}\,\mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)^{*} \tag{20}$$

where $\mathbf{L}'_\nu$ denotes the fast-lifting operator with the fast-lifting parameter ν defined on the interval $[0, h') = [0, h/N_e)$. The associated $\mathbf{P}^{\uparrow\nu}$, which corresponds to an operator equivalent to $\mathbf{P}$ under the introduction of ν, is given by $\mathbf{P}^{\uparrow\nu} = (\mathbf{S}^{\uparrow\nu})^*\mathbf{S}^{\uparrow\nu} = \mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)\mathbf{P}\,\mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)^*$.

Here, let us introduce $\mathcal{K}''_m$, which is given by $\mathcal{K}'_m$ with the underlying h' replaced by $h'' := h'/\nu$ $(= h/(\nu N_e))$. We further introduce $\mathbf{J}'_{H0} : \mathbb{R}^m \to \mathcal{K}''_m$ and $\mathbf{J}'_{S0} := (\mathbf{J}'_{H0})^*$ defined by replacing h' in $\mathbf{J}_{H0}$ and $\mathbf{J}_{S0}$ by h''. Then, it is easy to see that

$$\begin{aligned}\mathbf{L}'_\nu\mathbf{J}_{H0} &= \frac{1}{\sqrt{\nu}}[\underbrace{I, \cdots, I}_{\nu}]^T\mathbf{J}'_{H0} \\ &= \mathrm{diag}[\underbrace{\mathbf{J}'_{H0}, \cdots, \mathbf{J}'_{H0}}_{\nu}]\left(\frac{1}{\sqrt{\nu}}[\underbrace{I, \cdots, I}_{\nu}]^T\right) \\ &=: \mathrm{diag}[\underbrace{\mathbf{J}'_{H0}, \cdots, \mathbf{J}'_{H0}}_{\nu}]H'_\nu\end{aligned} \tag{21}$$

It is also obvious that $\mathbf{L}'_\nu \varPi (\mathbf{L}'_\nu)^{-1} = I_\nu \otimes \varPi$. Hence, it follows from (14) that $\mathbf{P}^{\uparrow\nu} = \mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)\mathbf{P}\,\mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)^*$ can be described as

$$\begin{aligned}\mathbf{P}^{\uparrow\nu} &= \mathrm{diag}[I, \underbrace{\overline{\mathbf{J}'_{H0}, \cdots, \mathbf{J}'_{H0}}}_{\nu}]P^{\uparrow\nu}\mathrm{diag}[I, \underbrace{\overline{\mathbf{J}'_{H0}, \cdots, \mathbf{J}'_{H0}}}_{\nu}]^* \\ &\quad + \mathrm{diag}[0, \overline{\varPi^{\uparrow\nu}}]\end{aligned} \tag{22}$$

where

$$P^{\uparrow\nu} = \mathbf{I}\left(\overline{H'_\nu}\right)P\mathbf{I}\left(\overline{H'_\nu}\right)^T \tag{23}$$

$$\varPi^{\uparrow\nu} = I_\nu \otimes \varPi \tag{24}$$

This in particular implies that $\mathbf{P}^{\uparrow\nu}_{N_e} \in \mathcal{P}_{\nu N_e}$. That is, $\mathbf{P}^{\uparrow\nu}$ can be handled within our special class $\mathcal{P}_N$, and similarly for $\mathbf{S}^{\uparrow\nu}$. It is also obvious that we are led to the following result.

Proposition 1: Suppose $\nu \in \mathbb{N}$. $\mathbf{P} \in \mathcal{P}_{N_e}$ satisfies (12) with $N = N_e$ if and only if $\mathbf{T}^*_{\nu N_e}\mathbf{P}^{\uparrow\nu}\mathbf{T}_{\nu N_e} \prec \mathbf{P}^{\uparrow\nu}$. Furthermore, $\mathbf{P}^{\uparrow\nu} \succ 0$ if and only if $\mathbf{P} \succ 0$, and also $\kappa(\mathbf{P}^{\uparrow\nu}) = \kappa(\mathbf{P})$ holds, where $\kappa(\cdot)$ denotes the condition number.

It is also obvious that $\kappa(\mathbf{S}) = \kappa(\mathbf{S}^{\uparrow\nu})$. This implies that in the modified procedure suggested in the beginning of this subsection by introducing ν (i.e., that based on $\mathbf{S}^{\uparrow\nu}$ equivalent to $\mathbf{S}$), the condition number $\kappa(\mathbf{S}^{\uparrow\nu})$ never grows with ν but stays the same as $\kappa(\mathbf{S})$, while $\|\mathbf{E}'_{X,\nu}\|$ can be made arbitrarily small by letting $\nu \to \infty$. That is, the term $\kappa(\mathbf{S}^{\uparrow\nu})\|\mathbf{E}'_{X,\nu}\|$ (which we would have in the condition in Step (iii) modified by the introduction of ν) can be made arbitrarily small by letting $\nu \to \infty$, and thus the possible problem raised in the preceding subsection (about the behavior of $\kappa(\mathbf{S})\|\mathbf{E}'_X\|$ as N gets larger) can be circumvented by the introduction of ν for each fixed $N = N_e$. In other words, we can claim that if the operator $\mathbf{S}$ determined in Step (ii) (under a fixed integer $N = N_e$) leads to $\mathbf{P} = \mathbf{S}^*\mathbf{S}$ that satisfies the operator Lyapunov inequality (12) (rather than $\mathbf{T}^*_{NX}\mathbf{P}\mathbf{T}_{NX} \prec \mathbf{P}$ that is obviously satisfied if $\alpha < 1$), then the feasibility of (12) by this $\mathbf{P}$ can indeed be established numerically, by taking large enough ν and checking the condition $\alpha_\nu + \kappa(\mathbf{S})\|\mathbf{E}'_{X,\nu}\| < 1$ (where the methods for computing $\alpha_\nu := \|\mathbf{S}^{\uparrow\nu}\mathbf{T}_{NX}(\mathbf{S}^{\uparrow\nu})^{-1}\|$ with $N = \nu N_e$ and $\kappa(\mathbf{S})$ numerically are the topics in the following section, i.e., Subsections IV-A and IV-B, respectively).

For the sake of full rigor, however, we must call further attention to the following fact, before we can conclude that taking ν large enough makes it possible to judge if $\mathbf{P} = \mathbf{S}^*\mathbf{S}$ indeed solves (12) under the $\mathbf{S}$ obtained in Step (ii): the first term α_ν in the above condition $\alpha_\nu + \kappa(\mathbf{S})\|\mathbf{E}'_{X,\nu}\| < 1$ also depends obviously on ν. Fortunately, the dependence of α_ν on ν does not cause any problem because, when $N = \nu N_e$,

$$\|\mathbf{T}_N - \mathbf{T}_{NX}\| = \|\mathbf{E}_{NX}\| = \|\mathbf{E}'_{X,\nu}\| \to 0 \quad (\nu \to \infty) \tag{25}$$

under the choice of X that we consider here [6], [8],[9], and thus $\alpha_\nu = \|\mathbf{S}^{\uparrow\nu}\mathbf{T}_{NX}(\mathbf{S}^{\uparrow\nu})^{-1}\| = \|\mathbf{S}\,\mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)^{-1}\mathbf{T}_{NX}\mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)\mathbf{S}^{-1}\|$ is convergent as $\nu \to \infty$ (to $\|\mathbf{S}_{N_e}\mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)^{-1}\mathbf{T}_{\nu N_e}\mathbf{I}\left(\overline{\mathbf{L}'_\nu}\right)\mathbf{S}^{-1}_{N_e}\| = \|\mathbf{S}_{N_e}\mathbf{T}_{N_e}\mathbf{S}^{-1}_{N_e}\|$).

To summarize the above arguments, we are led to the following step to be taken if the condition in Step (iii) fails.

(iv) Take a positive integer $\nu \geq 2$, let $N = \nu N_e$, and repeat Step (i) for this new N to get new X and the associated new approximation error $\|\mathbf{E}'_{X,\nu}\|$. Construct the operator $\mathbf{S}^{\uparrow\nu}$ from $\mathbf{S}$, and compute $\alpha_\nu = \|\mathbf{S}^{\uparrow\nu}\mathbf{T}_{NX}(\mathbf{S}^{\uparrow\nu})^{-1}\|$ for the new $\mathbf{T}_{NX}$. Check if $\alpha_\nu + \kappa(\mathbf{S})\|\mathbf{E}'_{X,\nu}\| < 1$. Exponential stability of Σ is ensured if this condition holds.

Following Step (iv), we can virtually check the existence of the Lyapunov solution $\mathbf{P} = \mathbf{S}^*\mathbf{S} \in \mathcal{P}_N$ to (12), in principle (by taking large enough ν). Since this is true for each fixed $N = N_e$, the approach developed here gives an asymptotically exact stability analysis method for Σ (even though the discussions on numerically finding an appropriate $\mathbf{S} \in \mathcal{S}_N$ in Step (ii) and computing the condition number $\kappa(\mathbf{S})$ and so on have not been completed and are still deferred to the following section). More precisely, whenever Σ is exponentially stable, this method can always prove it numerically by taking a finite N (and then ν) sufficiently large.

Remark 1: We once again summarize and emphasize some important points in the above arguments. In the above procedure, approximate minimization of $\|\mathbf{S}\mathbf{T}_N\mathbf{S}^{-1}\|$ is carried out with respect to $\mathbf{S} \in \mathcal{S}_N$ by (exactly) minimizing $\|\mathbf{S}\mathbf{T}_{NX}\mathbf{S}^{-1}\|$ instead under the quasi-finite-rank approximation $\mathbf{T}_{NX}$ of $\mathbf{T}_N$ at the parameter $N = N_e$. The resulting "approximately optimal" $\mathbf{S}$ with respect to $\mathbf{T}_N$, however, may not be close enough to "actually optimal" $\mathbf{S}$ in $\mathcal{S}_N$ at the parameter $N = N_e$ due to the optimization only with the approximated operator $\mathbf{T}_{NX}$ rather than $\mathbf{T}_N$. However, since we will eventually use the quasi-finite-rank approximation at $N = \nu N_e$ at the same time if we are to take Step (iv), it might sound reasonable to optimize $\mathbf{S} \in \mathcal{S}_N$ at the parameter $N = \nu N_e$ rather than $N = N_e$ (so that the side effect of approximation can be reduced and $\mathbf{S}$ that is "closer to exact optimality" can hopefully be obtained). It should be recalled, however, that it is what we have avoided since the resulting $\kappa(\mathbf{S})$ might then diverge as $N \to \infty$, in which case the analysis might get spoiled. This is why we have given a sort of two-stage procedure as above by taking ν. To make a compromise between positive and negative sides of making N larger, however, we could alternatively consider a "one-stage" procedure, in which we take $N = \nu N_e$ in Step (i), (exactly) optimize $\|\mathbf{S}\mathbf{T}_{NX}\mathbf{S}^{-1}\|$ with respect to $\mathbf{S} = \mathbf{S}_{N_e}^{\uparrow\nu}$ for some $\mathbf{S}_{N_e} \in \mathcal{S}_{N_e}$, and then follow Step (iii). Restricting to the class $\{\mathbf{S}_{N_e}^{\uparrow\nu} \,|\, \mathbf{S}_{N_e} \in \mathcal{S}_{N_e}\}$ has an advantage of reducing the number of variables in the LMI problem derived in the following section regarding the optimization of $\mathbf{P} = \mathbf{S}^*\mathbf{S}$, and thus can contribute also to keeping the computational load to within a reasonable amount.

IV. Reduction to Finite-Dimensional Discrete-Time LMI Condition

By summarizing the preceding arguments, we can say that we have given an approach to solving the operator Lyapunov inequality $\mathbf{T}_N^*\mathbf{P}\mathbf{T}_N \prec \mathbf{P}$ approximately and then confirming if the approximate solution indeed solves the inequality. This process involves the optimization of $\mathbf{P}$ (or $\mathbf{S}$ such that $\mathbf{S}^*\mathbf{S} = \mathbf{P}$) and the computations of $\|\mathbf{S}\| = \|\mathbf{P}\|^{1/2}$ and $\|\mathbf{S}^{-1}\| = \|\mathbf{P}^{-1}\|^{1/2}$. This section gives methods for reducing these computations to finite-dimensional ones. The optimization of $\mathbf{P}$ has a lot common with the recent development in the theory of sampled-data systems, in particular, modified fast-sample/fast-hold approximation and noncausal linear periodically time-varying scaling [4]–[7],[11]. On the other hand, the computations of $\|\mathbf{P}\|$ and $\|\mathbf{P}^{-1}\|$ get different from the corresponding computations in the sampled-data setting due to different and more complicated structure of $\mathbf{P}$ that involves a "compact part."

A. Computations of $\|\mathbf{S}\|$ and $\|\mathbf{S}^{-1}\|$

It follows readily from the arguments in Appendix of [3] that $\|\mathbf{S}^{-1}\|$ can be computed with a bisection method as the infimum of $\gamma > 0$ such that

$$\gamma^2 \Pi - I > 0, \quad \gamma^2(P + \mathrm{diag}[0, \overline{\Pi}]) - I > 0 \tag{26}$$

In a similar fashion, we can easily show that $\|\mathbf{S}\|$ can be computed with a bisection method as the infimum of $\gamma > 0$ such that

$$\gamma^2 I - \Pi > 0, \quad \gamma^2 I - (P + \mathrm{diag}[0, \overline{\Pi}]) > 0 \tag{27}$$

B. Optimization of $\mathbf{P}$

We give a method for finding $\mathbf{S} \in \mathcal{S}_N$ (or the associated matrices P and Π) that minimizes $\alpha = \|\mathbf{S}\mathbf{T}_{NX}\mathbf{S}^{-1}\|$. We introduce

$$\mathbf{K}_1 = \begin{bmatrix} I & 0 \\ 0 & \overline{\mathbf{C}'} \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} I & 0 \\ 0 & \overline{\mathbf{B}'} \end{bmatrix},$$
$$\Xi = \begin{bmatrix} A_d & B_{dN} \\ C_{dN} & D_{dN} \end{bmatrix}, \quad \mathbf{K} = \mathbf{K}_1 \Xi \mathbf{K}_2 \tag{28}$$

so that $\mathbf{T}_{NX}$ given by (16) can be rearranged as $\mathbf{T}_{NX} = \mathbf{K} + \mathbf{O}(\overline{D})$, where $\mathbf{O}(\cdot)$ is a shorthand notation for $\mathrm{diag}[0, (\cdot)]$. Obviously, $\mathbf{K}$ is a compact (in fact, finite-rank) operator. Minimizing the above α with respect to $\mathbf{S} \in \mathcal{S}_N$ is equivalent to minimizing α subject to

$$\mathbf{T}_{NX}^*\mathbf{P}\mathbf{T}_{NX} \prec \alpha^2\mathbf{P}, \quad \mathbf{P} \in \mathcal{P}_N \tag{29}$$

which in turn can be rearranged as

$$\alpha^2\mathbf{P} - \big(\mathbf{O}(\overline{D})^*\mathbf{P}\mathbf{O}(\overline{D}) + \mathbf{K}^*\mathbf{P}\mathbf{O}(\overline{D}) + \mathbf{O}(\overline{D})^*\mathbf{P}\mathbf{K} + \mathbf{K}^*\mathbf{P}\mathbf{K}\big) \succ 0 \tag{30}$$

Here we introduce

$$\mathbf{P}_0 = \mathbf{I}(\overline{\mathbf{J}_{H0}})P\mathbf{I}(\overline{\mathbf{J}_{H0}})^*, \quad \mathbf{P}_1 = \mathbf{O}(\overline{\Pi}) \tag{31}$$

under the notation $\mathbf{I}(\cdot) = \mathrm{diag}[I, (\cdot)]$. Note that $\mathbf{P}_0$ is a compact (in fact, finite-rank) operator while $\mathbf{P}_1$ is not (i.e., a multiplication operator). It follows from (14) that $\mathbf{P} = \mathbf{P}_0 + \mathbf{P}_1$, and substituting it into the first and second $\mathbf{P}$ in (30) leads to an equivalent inequality $\mathbf{Y}_1 - \mathbf{Y}_0 \succ 0$ with $\mathbf{Y}_0$ and $\mathbf{Y}_1$ defined by

$$\mathbf{Y}_0 = \mathbf{I}(\overline{0}) - \alpha^2\mathbf{P}_0 + \mathbf{O}(\overline{D})^*\mathbf{P}_0\mathbf{O}(\overline{D}) + \mathbf{K}^*\mathbf{P}\mathbf{O}(\overline{D}) + \mathbf{O}(\overline{D})^*\mathbf{P}\mathbf{K} + \mathbf{K}^*\mathbf{P}\mathbf{K} \tag{32}$$

$$\mathbf{Y}_1 = \mathbf{I}(\overline{0}) + \alpha^2\mathbf{P}_1 - \mathbf{O}(\overline{D})^*\mathbf{P}_1\mathbf{O}(\overline{D}) = \mathbf{I}(\overline{\Psi_\alpha}) \tag{33}$$

where

$$\Psi_\alpha := \alpha^2\Pi - D^T\Pi D \tag{34}$$

Since $\mathbf{Y}_0$ is a compact operator, we can see from (33) that $\mathbf{Y}_1 - \mathbf{Y}_0 \succ 0$ only if $\varPsi_\alpha > 0$. Thus, when we seek for the solution of (29), we may assume $\varPsi_\alpha > 0$ without loss of generality, which in particular implies $\mathbf{Y}_1 \succ 0$. Therefore, $\mathbf{Y}_1 - \mathbf{Y}_0 \succ 0$ if and only if $I - \mathbf{Y}_1^{-1/2}\mathbf{Y}_0\mathbf{Y}_1^{-1/2} \succ 0$, where $\mathbf{Y}_1^{-1/2} = \mathbf{I}\left(\overline{\varPsi_\alpha^{-1/2}}\right)$. What we do next is to introduce a factorization $\mathbf{Y}_1^{-1/2}\mathbf{Y}_0\mathbf{Y}_1^{-1/2} = \mathbf{H}_1\mathbf{H}_2$ with appropriate compact (in fact, finite-rank) operators $\mathbf{H}_1$ and $\mathbf{H}_2$. To do this in the general situation $D \neq 0$ is highly involved, and hence we begin with the case $D = 0$.

1) The case of retarded TDS with $D = 0$: If we introduce $\underline{\mathbf{J}} = [I_n, 0] : \mathcal{M}'_N \to \mathbb{R}^n$ and $\varOmega := \mathbf{K}_1^*\mathbf{P}\mathbf{K}_1 \in \mathbb{R}^{(n+nN)\times(n+nN)}$, we can take $\mathbf{H}_1$ and $\mathbf{H}_2$ as follows:

$$\mathbf{H}_1 = \mathbf{Y}_1^{-1/2}\left[\underline{\mathbf{J}}^* \quad \alpha\mathbf{I}\left(\overline{\mathbf{J}_{H0}}\right) \quad \mathbf{K}_2^*\varXi^T\right] \tag{35}$$

$$\mathbf{H}_2^* = \mathbf{Y}_1^{-1/2}\left[\underline{\mathbf{J}}^* \ -\alpha\mathbf{I}\left(\overline{\mathbf{J}_{H0}}\right)P \ \ \mathbf{K}_2^*\varXi^T\varOmega\right] \tag{36}$$

Since $\mathbf{H}_1\mathbf{H}_2$ is a compact operator, $I - \mathbf{H}_1\mathbf{H}_2 \succ 0$ if and only if the eigenvalues of $I - \mathbf{H}_1\mathbf{H}_2$ (and thus $I - \mathbf{H}_2\mathbf{H}_1$) are all positive. Since $\mathbf{Y}_1^{-1} = \mathbf{I}\left(\overline{\varPsi_\alpha^{-1}}\right)$, a direct computation of $\mathbf{H}_2\mathbf{H}_1$ leads to the matrix

$$\mathbf{H}_2\mathbf{H}_1 = \begin{bmatrix} I & \alpha J_1 & J_2\varXi^T \\ -\alpha P J_1^T & -\alpha^2 P\,\mathbf{I}\left(\overline{\mathbf{J}_{H0}^*\varPsi_\alpha^{-1}\mathbf{J}_{H0}}\right) & \mathbf{I}\left(\overline{\mathbf{J}_{H0}^*\varPsi_\alpha^{-1}(\mathbf{B}')^*}\right)\varXi^T \\ \varOmega\varXi J_2^T & \alpha\varOmega\varXi\,\mathbf{I}\left(\overline{\mathbf{B}'\varPsi_\alpha^{-1}\mathbf{J}_{H0}}\right) & \varOmega\varXi\,\mathbf{I}\left(\overline{\mathbf{B}'\varPsi_\alpha^{-1}(\mathbf{B}')^*}\right)\varXi^T \end{bmatrix} \tag{37}$$

where $J_1 = [I, 0] \in \mathbb{R}^{n\times(n+\mu N)}$ and $J_2 = [I, 0] \in \mathbb{R}^{n\times(n+nN)}$. In view of the above representation, we introduce the matrix factorization

$$\begin{bmatrix} \mathbf{J}_{H0}^* \\ \mathbf{B}' \end{bmatrix} \varPsi_\alpha^{-1} \begin{bmatrix} \mathbf{J}_{H0} & (\mathbf{B}')^* \end{bmatrix} = \begin{bmatrix} (Z'_d)^T \\ W'_d \end{bmatrix} \left((\varPsi_\alpha)^{-1}\otimes I_p\right) \begin{bmatrix} Z'_d & (W'_d)^T \end{bmatrix} \tag{38}$$

Such factorization is ensured to exist by the arguments in [7] (Lemma 1 therein), with appropriately defined $p \in \mathbb{N}$ and matrices W'_d and Z'_d that are independent of $\varPsi_\alpha$; see [7] for an explicit procedure to determine them. We are now ready to define the matrices $J_0 = [I, 0] \in \mathbb{R}^{n\times(n+\mu p N)}$,

$$H_1 = Y_1^{-1/2}\left[\ J_0^T \quad \alpha\mathbf{I}\left(\overline{Z'_d}\right) \quad \mathbf{I}\left(\overline{(W'_d)^T}\right)\varXi^T\ \right] \tag{39}$$

$$H_2^T = Y_1^{-1/2}\left[J_0^T \ \ -\alpha\mathbf{I}\left(\overline{Z'_d}\right)P \ \ \mathbf{I}\left(\overline{(W'_d)^T}\right)\varXi^T\varOmega\right] \tag{40}$$

together with $Y_1 := \mathbf{I}\left(\overline{\varPsi_\alpha \otimes I_p}\right)$ to see that $\mathbf{H}_2\mathbf{H}_1 = H_2H_1$. It would be helpful to find some similarity between $\mathbf{H}_k$ $(k = 1, 2)$ and H_k $(k = 1, 2)$ to see what we have established in the above arguments (i.e., a sort of discretization, or more adequately, precise reduction of operators to finite-dimensionality).

Since H_1H_2 is a symmetric matrix, we see that all the eigenvalues of $I - \mathbf{H}_2\mathbf{H}_1 = I - H_2H_1$ (and thus $I - H_1H_2$) are positive if and only if $I - H_1H_2 > 0$. Hence, combining all the above arguments, we are led to the conclusion that (30) holds if and only if $\varPsi_\alpha > 0$ and $I - H_1H_2 > 0$; the latter inequality holds if and only if

$$Y_1 - \Big[J_0^TJ_0 - \alpha^2\mathbf{I}\left(\overline{Z'_d}\right)P\,\mathbf{I}\left(\overline{(Z'_d)^T}\right) + \mathbf{I}\left(\overline{(W'_d)^T}\right)\varXi^T\varOmega\varXi\,\mathbf{I}\left(\overline{W'_d}\right)\Big] > 0 \tag{41}$$

Note that $\varOmega$ admits an affine representation in terms of P and $\varPi$ as given in Appendix (see (57)), and that $Y_1 = \mathbf{I}\left(\overline{(\alpha^2\varPi - D^T\varPi D)\otimes I_p}\right)$, which is also affine with respect to $\varPi$. Hence, we see that the standing assumption

$$\varPsi_\alpha = \alpha^2\varPi - D^T\varPi D > 0 \tag{42}$$

and (41) are both LMIs with respect to P and $\varPi$, once we fix α. Thus, with these two LMIs, together with the LMI (15) for the strict positive-definiteness of $\mathbf{P}$, we can apply a bisection method with respect to α to get optimal P and $\varPi$ (and thus $\mathbf{P}$) together with the optimal α. It would be helpful to note some similarity between the original inequality $\mathbf{Y}_1 - \mathbf{Y}_0 \succ 0$ (see (32) and (33) with $D = 0$), equivalent to (29), and the derived finite-dimensional inequality (41) to see what we have established in the above; it corresponds to the reduction of the infinite-dimensional operator inequality (29) to a finite-dimensional LMI in an exact fashion. Note that the quasi-finite-rank approximation $\mathbf{T}_{NX}$ of $\mathbf{T}_N$, together with the structure of $\mathbf{P} \in \mathcal{P}_N$, has enabled the exact reduction.

2) The case of neutral TDS with $D \neq 0$: When $D \neq 0$, the construction of $\mathbf{H}_1$ and $\mathbf{H}_2$ and in particular an appropriate factorization of the matrix $\mathbf{H}_2\mathbf{H}_1$ get highly involved, but essentially the same arguments can be applied. Tedious manipulations show that we can arrive at an equivalent condition to (30) (i.e., $\mathbf{Y}_1 - \mathbf{Y}_0 \succ 0$) that is in the form

$$Y_1 - Y_0 > 0 \tag{43}$$

where Y_0 and Y_1 are given by

$$\begin{aligned} Y_0 = {} & \mathbf{I}\left(\overline{0_{\mu q,\mu q}}\right) - \alpha^2\mathbf{I}\left(\overline{Z'_d}\right)P\,\mathbf{I}\left(\overline{(Z'_d)^T}\right) \\ & + \mathbf{O}\left(\overline{D^T\otimes I_q}\right)\mathbf{I}\left(\overline{Z'_d}\right)P\,\mathbf{I}\left(\overline{(Z'_d)^T}\right)\mathbf{O}\left(\overline{D}\otimes I_q\right) \\ & + \mathbf{O}\left(\overline{D^T}\otimes I_q\right)\Big\{\mathbf{I}\left(\overline{Z'_d}\right)P\,\mathbf{I}\left(\overline{(Z'_d)^T}\right) \\ & \qquad + \mathbf{O}\left(\overline{\varPi}\otimes I_q\right)\Big\}\mathbf{I}\left(\overline{V'_d}\right)\varXi\,\mathbf{I}\left(\overline{W'_d}\right) \\ & + \mathbf{I}\left(\overline{(W'_d)^T}\right)\varXi^T\mathbf{I}\left(\overline{(V'_d)^T}\right)\Big\{\mathbf{I}\left(\overline{Z'_d}\right)P\,\mathbf{I}\left(\overline{(Z'_d)^T}\right) \\ & \qquad + \mathbf{O}\left(\overline{\varPi}\otimes I_q\right)\Big\}\mathbf{O}\left(\overline{D}\otimes I_q\right) \\ & + \mathbf{I}\left(\overline{(W'_d)^T}\right)\varXi^T\varOmega\varXi\,\mathbf{I}\left(\overline{W'_d}\right) \end{aligned} \tag{44}$$

$$Y_1 = \mathbf{I}\left(\overline{(\alpha^2\varPi - D^T\varPi D)\otimes I_q}\right) \tag{45}$$

provided that the matrices W'_d, Z'_d and V'_d and $q \in \mathbb{N}$ are determined by (58) in Appendix rather than (38), and $\varOmega$ represented as (59) rather than (57) is substituted to (44).

V. Numerical Example

Let us consider the neutral delay-differential equation

$$\frac{dx}{dt} = Ax(t) + (C - DA)x(t-h) + D\frac{dx}{dt}(t-h) \tag{46}$$

with $A = -1/2$, $C = -1$, and $D = -1/2$. Stability of this equation can be studied as that of the time-delay system Σ in Fig. 1 with the FDLTI system F given by the state-space representation $B = 1$ and the above A, C and D. By the Nyquist stability criterion, we can show that Σ is stable for $0 < h < \bar{h} := \mathrm{Arg}\{(\beta_1 + j\beta_2)/(\beta_3 + j\beta_4)\}/\omega_0$, where

$$\beta_1 = -\sqrt{7}/4\sqrt{2},\ \beta_2 = 5/4\sqrt{2} \tag{47}$$

$$\beta_3 = \sqrt{7}/2\sqrt{2},\ \beta_4 = -1/2\sqrt{2},\ \omega_0 = \sqrt{7}/2 \tag{48}$$

and $\mathrm{Arg}(\cdot)$ denotes the principal value of the argument of a complex number restricted to $(0, 2\pi]$. We consider the case $h = 0.9\bar{h} \approx 0.9 \times 1.8285$ and examine the stability of Σ with the numerical method developed in this paper.

To this end, we first take $N = N_e = 3$ in Step (i) in Section III, where X is determined to (approximately) minimize $\|\mathbf{E}'_X\|$ with the one-stage quasi-finite-rank approximation method in [9]. For the optimal X, we have an upper bound e'_X of $\|\mathbf{E}'_X\|$ given by $e'_X = 0.1805$. We next proceed to Step (ii), which corresponds to minimizing $\alpha > 0$ by solving the LMIs (15), (42), (43) for P and Π (these are the LMI conditions for the case $D \neq 0$). We then have $\alpha = 0.8932$, and for the resulting $\mathbf{P}$ (and thus $\mathbf{S}$) we can compute $\|\mathbf{S}\|$ and $\|\mathbf{S}^{-1}\|$ with (26) and (27) to have $\kappa(\mathbf{S}) = 3.4042$. Even though $\alpha < 1$ and thus Σ is likely to be stable (i.e., the obtained $\mathbf{P} \succ 0$ is likely to satisfy (12)), the condition in Step (iii) cannot be ensured to be fulfilled since $\alpha + \kappa(\mathbf{S})e'_X \geq 1$. Hence we cannot conclude stability of Σ at this stage.

We thus proceed to applying a less conservative stability condition by introducing the integer ν as in Step (iv) to confirm that $\mathbf{P} \succ 0$ obtained in Step (ii) is indeed a solution to the operator Lyapunov inequality (12) for $N = N_e = 3$. Let us take $\nu = 10$ and follow Step (iv); that is, we repeat Step (i) with $N = \nu N_e = 30$ to get new X and the associated $\|\mathbf{E}'_{X,\nu}\|$ (in fact, its upper bound $e'_{X,\nu}$) given by $e'_{X,\nu} = 0.0182$. We construct $P^{\uparrow\nu}$ given by (23) where $\overline{(\cdot)}$ is with respect to $N = N_e$, reconstruct the matrices Z'_d, W'_d and V'_d in (58) with respect to $h' = h/N = h/(\nu N_e)$, and consider the LMI (43) with P replaced by $P^{\uparrow\nu}$ where $\overline{(\cdot)}$ is with respect to $N = \nu N_e$. Minimizing α under the LMI constraint (43) corresponds exactly to the computation of $\alpha_\nu = \|\mathbf{S}^{\uparrow\nu}\mathbf{T}_{NX}(\mathbf{S}^{\uparrow\nu})^{-1}\|$ with $N = \nu N_e$ required in Step (iv), and we have $\alpha_\nu = 0.9372$. The condition in Step (iv) is now fulfilled since $\gamma_\nu := \alpha_\nu + \kappa(\mathbf{S})e'_{X,\nu} = 0.9991 < 1$ (taking a larger ν, we can further confirm, e.g., $\gamma_\nu < 0.95$). We also remark that also when $h = 0.98\bar{h}$, taking the parameters $N_e = 5$ and $\nu = 30$ leads to ($\alpha = 0.9722$, $\alpha_\nu = 0.9880$ and) $\gamma_\nu = 0.9998 < 1$.

The above two-stage procedure demonstrates that if $\mathbf{P} \succ 0$ of the form (14) solves the operator Lyapunov inequality (12), then we can confirm it numerically by taking ν sufficiently large. It should be recalled that, as stated in Section II, there always exists such $\mathbf{P} \succ 0$ represented by the matrices P and Π in the form (14), provided that the fast-lifting parameter N is sufficiently large [2],[3]. Hence, the numerical method developed in this paper provides an asymptotically exact stability analysis for time-delay systems by taking N and ν sufficiently large.

VI. Extension to the Use of More General Hold and Sampling Operators $\mathbf{J}_{Hk}$ and $\mathbf{J}_{Sk}$

The preceding arguments are restricted to the use of the hold operator $\mathbf{J}_{H0}$ and the sampling operator $\mathbf{J}_{S0} = \mathbf{J}^*_{H0}$ to construct $\mathbf{P}$ as in (14), as originally studied in [2]. In our recent study [3], the use of more general operators $\mathbf{J}_{Hk}$ and $\mathbf{J}_{Sk} = \mathbf{J}^*_{Hk}$ ($k \in \mathbb{N}$) has been discussed based on the Legendre polynomials. Here we give a brief sketch about how the preceding results can be extended to accommodate the use of $\mathbf{J}_{Hk}$ and $\mathbf{J}_{Sk}$. A key in the success in the preceding arguments lies in the reduction of operators to matrices used in (38) or (58). Behind the feasibility of the reduction is the fact that $\mathbf{J}_{H0} = \mathbf{J}^*_{S0}$ has a "state-space representation" similar to $\mathbf{C}$ in (4) (or more precisely, similar to $\mathbf{C}'$ and $(\mathbf{B}')^*$). Hence, to show that the preceding arguments apply basically to the case with $\mathbf{J}_{Hk}$ and $\mathbf{J}_{Sk}$, it suffices to show that $\mathbf{J}_{Hk}$ also has such a "state-space representation." Due to the lack of space, we only state (without proof) that $(\mathbf{J}_{Hk}v)(\theta') = C'_{Hk}\exp(A'_{Hk}\theta')v,\ v \in \mathbb{R}^{(k+1)\mu},\ \theta' \in [0, h')$ for

$$A'_{Hk} = (\Psi'_k)^{-1}A_{Jk}\Psi'_k,\ C'_{Hk} = C_{Jk}\Psi'_k \tag{49}$$

Here, $A_{Jk} \in \mathbb{R}^{(k+1)\mu\times(k+1)\mu}$ and $C_{Jk} \in \mathbb{R}^{\mu\times(k+1)\mu}$ are given by

$$A_{Jk} = \begin{bmatrix} 0 & I_k \\ 0 & 0 \end{bmatrix} \otimes I_\mu,\ C_{Jk} = [1\ \underbrace{0 \cdots 0}_{k}] \otimes I_\mu \tag{50}$$

and $\Psi'_k \in \mathbb{R}^{(k+1)\mu\times(k+1)\mu}$ is an invertible matrix defined by

$$\Psi'_k = \begin{bmatrix} \psi'_{00} & \psi'_{10} & \cdots & \psi'_{k0} \\ 0 & \psi'_{11} & \cdots & \psi'_{k1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \psi'_{kk} \end{bmatrix} \otimes I_\mu \tag{51}$$

where ψ'_{ij} is determined by rearranging into $\sum_{j=0}^{i}\frac{\psi'_{ij}}{j!}(\theta')^j$ the shifted and normalized Legendre polynomial $\psi_i(\theta')$ introduced in [3], represented as

$$\psi_i(\theta') = \sqrt{\frac{2i+1}{h'}}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\binom{i+j}{j}\left(-\frac{\theta'}{h'}\right)^j \tag{52}$$

The remaining issue is the representation of $\mathbf{P}^{\uparrow\nu}$ when $\mathbf{P}$ is represented by (14) with $\mathbf{J}_{H0}$ replaced by $\mathbf{J}_{Hk}$ (and thus $P \in \mathbb{R}^{(n+(k+1)\mu N)\times(n+(k+1)\mu N)}$). We only state that it is again given by (22) with $\mathbf{J}'_{H0}$ replaced by $\mathbf{J}'_{Hk}$, together with (23) and (24) with H'_ν replaced by

$$H'_\nu = H^{(k)'}_\nu = \begin{bmatrix} (\Psi''_k)^{-1} \\ (\Psi''_k)^{-1}A''_{Jkd} \\ \vdots \\ (\Psi''_k)^{-1}(A''_{Jkd})^{\nu-1} \end{bmatrix}\Psi'_k \tag{53}$$

where $A''_{Jkd} := \exp(A_{Jk}h''),\ h'' := h'/\nu = h/(\nu N_e)$, and Ψ''_k is defined as Ψ'_k with h' replaced by h''.

VII. Conclusion

In our recent studies [2],[3], we have developed a fundamental framework for dealing with linear time-delay systems. The fast-lifted monodromy operator played a key role there, and an operator Lyapunov inequality was introduced for stability analysis based on this operator. A class of candidates to the solutions to the Lyapunov inequality was then introduced, which is represented by two finite-dimensional matrices P and Π together with hold and sampling operators $\mathbf{J}_{H0}$ and $\mathbf{J}_{S0}$, and it was further shown that a solution does exist within this class whenever the system is stable, provided that the integer N for fast-lifting is taken sufficiently large.

Following and extending the theoretical advances therein, the present paper has developed a numerical procedure for stability analysis. Quasi-finite-rank approximation was first applied to the fast-lifted monodromy operator, and it was shown that finding (from the above mentioned candidate class) a solution to the operator Lyapunov inequality associated with the operator after approximation reduces to solving a discrete-time LMI with respect to P and Π. A method was also provided for confirming that the obtained LMI solution does solve the operator Lyapunov inequality with respect to the fast-lifted monodromy operator without approximation. Thus this paper together with our recent studies [2],[3] has successfully established an asymptotically exact method for stability analysis both theoretically and numerically. A numerical example was also given to illustrate the arguments of the paper, and a brief sketch was given about how the method can be extended to the case with generalized operators $\mathbf{J}_{Hk}$ and $\mathbf{J}_{Sk}$ introduced in [3].

As remarked in our recent studies that has provided a fundamental framework to time-delay systems [2],[3], the technique can readily be extended to the stability problem for the situation with a discrete-time controller. Furthermore, the analysis technique developed in this paper can be extended, in principle, to dealing with a discrete-time controller design problem for time-delay systems, and this is the most important feature of the present new approach.

APPENDIX

Factorization of the Matrix $\Omega = \mathbf{K}_1^*\mathbf{P}\mathbf{K}_1$

We give a method for factorizing the matrix $\Omega = \mathbf{K}_1^*\mathbf{P}\mathbf{K}_1$ *in such a way that the matrices P and Π involved in* **P** *are easily dealt with as optimization parameters*. It follows from (14) that

$$\Omega = \mathrm{diag}[I, \overline{(\mathbf{C}')^*\mathbf{J}_{H0}}]P\,\mathrm{diag}[I, \overline{(\mathbf{C}')^*\mathbf{J}_{H0}}]^* + \mathrm{diag}[0, \overline{(\mathbf{C}')^*\Pi\mathbf{C}'}] \quad (54)$$

What we do next depends on whether $D = 0$ or not, if we are concerned with the computational load about optimization of P and Π. When $D = 0$, we simply introduce the matrix $C_d' := (\mathbf{J}_{H0})^*\mathbf{C}'$, i.e.,

$$C_d' := \frac{1}{\sqrt{h'}}\int_0^{h'} C\exp(A\theta')d\theta' \quad (55)$$

As shown in [7], $(\mathbf{C}')^*\Pi\mathbf{C}'$ can be represented as

$$(\mathbf{C}')^*\Pi\mathbf{C}' = (V_d')^T(\Pi\otimes I_r)(V_d') \quad (56)$$

with some appropriately defined $r \in \mathbb{N}$ and matrix V_d'. Thus, when $D = 0$, the following representation of $\Omega = \mathbf{K}_1^*\mathbf{P}\mathbf{K}_1$ suffices, in which the matrices P and Π appear in an affine form.

$$\Omega = \mathbf{I}\left(\overline{(C_d')^T}\right)P\,\mathbf{I}\left(\overline{C_d'}\right) + \mathbf{O}\left(\overline{(V_d')^T(\Pi\otimes I_r)(V_d')}\right) \quad (57)$$

When $D \neq 0$, however, it is crucial to deal with the two matrices $(\mathbf{C}')^*\mathbf{J}_{H0}$ and $(\mathbf{C}')^*\Pi\mathbf{C}'$ at a time for our purpose. Moreover, we actually have to deal with simultaneously the left hand side of (38) and other similar operator compositions involving $\mathbf{C}'$ so that we can arrive at (44) and (45). More precisely, we introduce the matrix factorization

$$\begin{bmatrix} \mathbf{J}_{H0}^* \\ \mathbf{B}' \\ (\mathbf{C}')^* \end{bmatrix}(\cdot)\begin{bmatrix} \mathbf{J}_{H0} & (\mathbf{B}')^* & \mathbf{C}' \end{bmatrix} = \begin{bmatrix} (Z_d')^T \\ W_d' \\ (V_d')^T \end{bmatrix}((\cdot)\otimes I_q)\begin{bmatrix} Z_d' & (W_d')^T & V_d' \end{bmatrix} \quad (58)$$

Such factorization is indeed possible with appropriately defined $q \in \mathbb{N}$ and matrices W_d', V_d' and Z_d' (independent of the matrix $(\cdot)$) by slightly extending the arguments of [7]. It follows that $(\mathbf{C}')^*\mathbf{J}_{H0} = (V_d')^T Z_d'$ and $(\mathbf{C}')^*\Pi\mathbf{C}' = (V_d')^T(\Pi\otimes I_q)(V_d')$. To summarize, $\Omega = \mathbf{K}_1^*\mathbf{P}\mathbf{K}_1$ admits the following representation, in which the matrices P and Π appear in an affine form.

$$\Omega = \mathbf{I}\left(\overline{(V_d')^T}\right)\left\{\mathbf{I}\left(\overline{Z_d'}\right)P\,\mathbf{I}\left(\overline{(Z_d')^T}\right) + \mathbf{O}\left(\overline{\Pi}\otimes I_q\right)\right\}\mathbf{I}\left(\overline{V_d'}\right) \quad (59)$$

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