Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

Monodromy Operator Approach to Time-Delay Systems: Fast-Lifting Based Treatment of Operator Lyapunov Inequalities

Tomomichi Hagiwara

***Abstract*— This paper establishes a new fundamental framework of an operator-theoretic approach to linear time-invariant (LTI) time-delay systems (TDSs). The state transition of TDSs is first viewed in discrete-time and described by a bounded operator called the monodromy operator. A stability condition in terms of the spectral radius of the monodromy operator is given, which in turn is related to an operator Lyapunov inequality about that operator. A special class of operators, parametrized by two finite-dimensional (FD) constant matrices and constructed via the fast-lifting technique, is then introduced, which is ensured to contain a nonempty subset of the solutions to the operator Lyapunov inequality whenever the TDS is stable and the fast-lifting parameter N is large enough. A fundamental framework for asymptotically exact stability analysis is thus established. An equivalent scaling treatment is also shown, and further generalization of the arguments is carried out with the use of Legendre polynomials in the construction of the above operator class. These arguments proceed in a rather "linear algebraic" way with many similarities to the stability analysis of FDLTI discrete-time systems. The presented framework could be said to be a "pseudo-discretization" technique; it allows one to essentially reduce the arguments on infinite-dimensional operators to those about two matrices, with an increasing degree of freedom as N gets larger, but without introducing any matrix approximation of infinite-dimensional operators.**

I. INTRODUCTION

Time-delay systems (TDS) are very commonly encountered in engineering and sciences. There hence exists a quite long and deep history of studies on this subject, e.g., [1]–[12]. This paper is stimulated by the recent study in [13] that has employed the lifting technique [14]–[17] developed and widely used in the area of sampled-data systems. The important idea in [13], related to the use of the lifting technique, is to focus on the states of the TDS Σ on the intervals $[(k-1)h, kh)$, $k \in \mathbb{N}$, where Σ consists of the finite-dimensional linear time-invariant (FDLTI) system F and the delay H with the delay length h (see Fig. 1). More precisely, the state transition between two such consecutive intervals is considered, and is described by what we call the monodromy operator, denoted by $\mathbf{T}$.

Unlike in an operator-theoretic approach as in [4], the monodromy operator discussed in the present paper is a bounded operator and thus is much easier to deal with. This

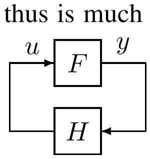

Fig. 1. Feedback system Σ with delay H.

T. Hagiwara is with the Department of Electrical Engineering, Kyoto University, Kyotodaigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan. hagiwara@kuee.kyoto-u.ac.jp

advantage is gained exactly by the discrete-time viewpoint we adopt for dealing with continuous-time TDSs. Such a viewpoint is also useful when we consider extending the framework to accommodate the case with discrete-time controllers. The fundamental theory about such treatment in discrete-time and the associated advantages mentioned above have already been discussed in our earlier paper [36], and the present paper corresponds to some improvement and extension of the arguments therein.

The contents of this paper are as follows. In Section II, we introduce the monodromy operator $\mathbf{T}$, which describes the state transition of Σ viewed in discrete-time. A stability condition of Σ is given in terms of this operator, and then will be further related to the associated operator Lyapunov inequality. The fast-lifting technique is then applied to introduce the fast-lifted monodromy operator $\mathbf{T}_N$, which plays a fundamental role in this study, where N is the fast-lifting parameter. The stability condition is also related to the operator Lyapunov inequality with respect to $\mathbf{T}_N$. Section III introduces the class $\mathcal{P}_N$ of operators that is ensured to contain a nonempty subset of the solutions to the operator Lyapunov inequality with respect to $\mathbf{T}_N$, whenever Σ is stable and the integer N is large enough. It should be stressed that the arguments proceed in a "linear algebraic" way with many similarities to the treatment of FDLTI discrete-time systems. The relationship of the present approach with the Lyapunov-Krasovskii functional techniques is also discussed. Section IV gives further extension of the class $\mathcal{P}_N$ by use of the Legendre polynomials, which is expected to help one conclude the stability of Σ with a smaller N.

The following notation is used in this paper. $(\cdot)^*$ and $\rho(\cdot)$ denote the adjoint and spectral radius of an operator, respectively. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of real numbers, complex numbers, integers and positive integers, respectively, while $\mathbb{N}_0$ means $\{0\} \cup \mathbb{N}$. $\mathcal{K}_\mu$ is a shorthand notation for the Hilbert space $(L_2([0,h);\mathbb{R}))^\mu$ with an underlying $h > 0$. $\otimes$ denotes the Kronecker product of matrices, and for a matrix $(\cdot)$, we use the shorthand notations $\mathbf{I}(\cdot)$, $\mathbf{O}(\cdot)$ and $\overline{(\cdot)}$ to mean diag$[I, (\cdot)]$, diag$[0, (\cdot)]$, and $I_N \otimes (\cdot)$, respectively; these notations are also used in a parallel fashion for an operator, too. The set of linear bounded operators on the Hilbert space $\mathcal{Z}$ is denoted by $\mathcal{L}(\mathcal{Z})$.

II. MONODROMY AND FAST-LIFTED MONODROMY OPERATORS OF TIME-DELAY SYSTEMS AND OPERATOR LYAPUNOV INEQUALITIES

Regarding the time-delay feedback system Σ in Fig. 1, we first remark that general linear time-invariant retarded/neutral

ISBN 978-963-311-370-7

delay differential equations with a single delay length $h > 0$ or commensurate delays can be formulated as such a feedback system. In this section, we introduce what we call the monodromy operator and fast-lifted monodromy operator for Σ. The stability of Σ is related to these operators and then to the operator Lyapunov inequalities with respect to these operators.

A. Monodromy Operator

We assume that F in Fig. 1 is represented by

$$\frac{dx}{dt} = Ax + Bu, \; y = Cx + Du \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times\mu}$, $C \in \mathbb{R}^{\mu\times n}$ and $D \in \mathbb{R}^{\mu\times\mu}$, and the input-output relation of H is given by $u(t) = y(t-h), \; h > 0$. The initial conditions of Σ are given by $x(0) = x_0$ and $y(\theta - h) = \widehat{u}_0(\theta), \; 0 \le \theta < h$ with some $\widehat{u}_0 \in \mathcal{K}_\mu$. This implies that $u(\theta) = \widehat{u}_0(\theta)$ $(0 \le \theta < h)$, and we can denote the lifted representation [14]–[17] of u by $\{\widehat{u}_k\}_{k=0}^{\infty}$, where $\widehat{u}_k(\theta) = u(kh + \theta)$. Let us denote $x(kh)$ simply by x_k. Then,

$$x(kh+\theta) = \exp(A\theta)x_k + \int_0^\theta \exp(A(\theta-\tau))B\widehat{u}_k(\tau)d\tau$$

$$\widehat{u}_{k+1}(\theta) = y(kh+\theta) = Cx(kh+\theta) + Du(kh+\theta) \tag{2}$$

Hence, Σ can be represented by

$$\begin{bmatrix} x_{k+1} \\ \widehat{u}_{k+1} \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_k \\ \widehat{u}_k \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} A_d & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{3}$$

where the matrix $A_d \in \mathbb{R}^{n\times n}$ and the operators $\mathbf{B} : \mathcal{K}_\mu \to \mathbb{R}^n$, $\mathbf{C} : \mathbb{R}^n \to \mathcal{K}_\mu$ and $\mathbf{D} : \mathcal{K}_\mu \to \mathcal{K}_\mu$ are defined as follows.

$$A_d = \exp(Ah), \quad \mathbf{B}f = \int_0^h \exp(A(h-\tau))Bf(\tau)d\tau$$

$$(\mathbf{C}v)(\theta) = C\exp(A\theta)v$$

$$(\mathbf{D}f)(\theta) = \int_0^\theta C\exp(A(\theta-\tau))Bf(\tau)d\tau + Df(\theta) \tag{4}$$

A similar representation to (3) has been given in [13] by taking the lifted representation $\widehat{x}_k$ of the state of F, in lieu of $\widehat{u}_k$. This implies that in our treatment, $\widehat{u}_k$ is regarded as independent of $x(t)$, and this will provide us with a chance for extending the treatment readily to the case with a discrete-time controller. That is, our treatment is well suited to immediate generalization to the setting of TDSs under sampled-data control. We call the operator $\mathbf{T}$ in (3) the monodromy operator of Σ (due to its resemblance with the monodromy matrix about a periodic differential equation), which is an operator on the product space $\mathcal{M} := \mathbb{R}^n \oplus \mathcal{K}_\mu$; $\oplus$ denotes the direct sum of Hilbert spaces, which generates a new Hilbert space (with a natural inner product) [23].

B. Fast-Lifting and Fast-Lifted Monodromy Operator

The monodromy operator introduced above plays a very important role, but from the viewpoint of the following arguments and the associated numerical computation to be studied in the sequel paper [18], it is not always convenient and easy to deal with. Hence we here introduce what might be called the "pseudo-discretization" of $\mathbf{T}$, even though no actual discretization (reduction to finite-dimensionality) is introduced at this stage. That is, we take $N \in \mathbb{N}$ and consider the fast-lifting $\mathbf{L}_N : \mathcal{K}_m \to (\mathcal{K}'_m)^N$ [19]–[22], where $\mathcal{K}'_m$ denotes $\mathcal{K}_m$ with h replaced by $h' := h/N$. By definition of $\mathbf{L}_N$,

$$(\mathbf{L}_N\widehat{u}_k)(\theta') = \check{u}_k(\theta') = \begin{bmatrix} \widehat{u}_k(\theta') \\ \widehat{u}_k(h'+\theta') \\ \vdots \\ \widehat{u}_k((N-1)h'+\theta') \end{bmatrix} \tag{5}$$

where $0 \le \theta' < h'$. Under the notation $\mathbf{I}(\cdot) = \mathrm{diag}[I, \; (\cdot)]$ for an operator $(\cdot)$, where I on the right hand side denotes the identity matrix on $\mathbb{R}^n$, let us define

$$\mathbf{T}_N = \mathbf{I}(\mathbf{L}_N)\mathbf{T}\,\mathbf{I}(\mathbf{L}_N)^{-1} =: \begin{bmatrix} A_d & \mathbf{B}_N \\ \mathbf{C}_N & \mathbf{D}_N \end{bmatrix}, \tag{6}$$

which is an operator on $\mathcal{M}'_N := \mathbb{R}^n \oplus (\mathcal{K}'_\mu)^N$. Then, it is easy to see that (3) can be rewritten as

$$\begin{bmatrix} x_{k+1} \\ \check{u}_{k+1} \end{bmatrix} = \mathbf{T}_N \begin{bmatrix} x_k \\ \check{u}_k \end{bmatrix} \tag{7}$$

and hence $\mathbf{T}_N$ given by (6) is called the fast-lifted monodromy operator of Σ. Roughly speaking, $\mathbf{T}_N$ is easier to deal with than $\mathbf{T}$ is, because it is associated with the smaller interval $[0, h')$ than the original interval $[0, h)$, which plays an important role in the following section.

C. Operator Classes $\mathfrak{B}_{\mathbf{F}N}$, $\mathcal{B}_{\mathbf{F}N}$, and Stability Conditions Based on Operator Lyapunov Inequalities

As could be expected, we have the following stability condition of Σ via the (fast-lifted) monodromy operator [36].

Theorem 1: Σ is exponentially stable if and only if $\rho(\mathbf{T}) < 1$, or equivalently, $\rho(\mathbf{T}_N) < 1$.

See [36] for the definition of exponential stability, and note that $\rho(\mathbf{T}) = \rho(\mathbf{T}_N), \; \forall N \in \mathbb{N}$ in the above theorem.

The purpose of the following part of this paper is to relate the stability of Σ with an operator Lyapunov inequality in terms of the fast-lifted monodromy operator $\mathbf{T}_N$, and then introduce a class of tractable operators containing a solution to the inequality. In such a study, the following definition plays an important role.

Definition 1: Given $N \in \mathbb{N}$, consider the linear bounded operator $\mathbf{F}$ on $\mathcal{M}'_N = \mathbb{R}^n \oplus (\mathcal{K}'_\mu)^N$ such that

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{00} & \mathbf{F}_{01} \\ \mathbf{F}_{10} & \mathbf{F}_{11} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{00} & \mathbf{F}_{01} \\ \mathbf{F}_{10} & \mathbf{F}_{110} + F_{11} \end{bmatrix} = \mathbf{F}_0 + F_1 \tag{8}$$

with $F_1 = \mathrm{diag}[0, F_{11}]$, where $\mathbf{F}_{110}$ (and thus $\mathbf{F}_0$) is a compact operator, and F_{11} is the operator of multiplication on $(\mathcal{K}'_\mu)^N$ defined by the constant matrix $F_{11} \in \mathbb{R}^{\mu N\times\mu N}$. The class of such operators $\mathbf{F}$ is denoted by $\mathcal{B}_{\mathbf{F}N} \subset \mathcal{L}(\mathcal{M}'_N)$. The class $\mathfrak{B}_{\mathbf{F}N}$ of operators on $\mathcal{M}$ is defined as $\mathfrak{B}_{\mathbf{F}N} := \{\mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{F}\,\mathbf{I}(\mathbf{L}_N) \,|\, \mathbf{F} \in \mathcal{B}_{\mathbf{F}N}\} \subset \mathcal{L}(\mathcal{M})$.

We can easily see that the fast-lifted monodromy operator $\mathbf{T}_N$ given by (6) belongs to the class $\mathcal{B}_{\mathbf{F}N}$ (in particular, by letting $N = 1$, we see $\mathbf{T} \in \mathcal{B}_{\mathbf{F}1} = \mathfrak{B}_{\mathbf{F}1}$).

We are in a position to state the following results [36].

Proposition 1: Let $\mathbf{G} \in \mathcal{L}(\mathcal{Z})$ for a Hilbert space $\mathcal{Z}$. Then, $\rho(\mathbf{G}) < 1$ if and only if there exists an operator $\mathbf{X} \succ 0$ on $\mathcal{Z}$ such that

$$\mathbf{G}^*\mathbf{X}\mathbf{G} - \mathbf{X} \prec 0 \tag{9}$$

Furthermore, if $\mathcal{Z} = \mathcal{M}'_N$, $\mathbf{G} \in \mathcal{B}_{\mathbf{F}N}$ and $\rho(\mathbf{G}) < 1$, then there exists $\mathbf{X} \in \mathcal{B}_{\mathbf{F}N}$ satisfying $\mathbf{X} \succ 0$ and (9).

Corollary 1: Let $\mathbf{G} \in \mathcal{L}(\mathcal{M})$. Then, $\rho(\mathbf{G}) < 1$ if and only if there exists an operator $\mathbf{X} \succ 0$ on $\mathcal{M}$ such that (9) holds. Furthermore, if $\mathbf{G} \in \mathfrak{B}_{\mathbf{F}N}$ and $\rho(\mathbf{G}) < 1$ for some $N \in \mathbb{N}$, then there exists $\mathbf{X} \in \mathfrak{B}_{\mathbf{F}N}$ satisfying $\mathbf{X} \succ 0$ and (9).

Note that Proposition 1 is related to the fast-lifted monodromy operator, while Corollary 1 is to the monodromy operator. We also remark that the strict positivity of operators in the above results is defined as follows.

Definition 2: Let $\mathcal{Z}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $\mathbf{G} \in \mathcal{L}(\mathcal{Z})$ be a self-adjoint bounded operator. We say that $\mathbf{G}$ is strictly positive definite, denoted by $\mathbf{G} \succ 0$, if there exists a scalar $\delta > 0$ such that $\langle(\mathbf{G} - \delta I)v, v\rangle > 0$ whenever $v \neq 0$. $\mathbf{G}_1 \succ \mathbf{G}_2$ (or $\mathbf{G}_2 \prec \mathbf{G}_1$) is a shorthand notation for $\mathbf{G}_1 - \mathbf{G}_2 \succ 0$.

III. A Class of Solutions to the Operator Lyapunov Inequalities and Scaling Treatment for Stability Analysis

By Theorem 1 and Proposition 1, the following arguments in this paper will center around the strictly positive solutions of the operator Lyapunov inequality

$$\mathbf{T}_N^* \mathbf{P} \mathbf{T}_N \prec \mathbf{P} \tag{10}$$

or equivalently (through the relation $\mathbf{X} = \mathbf{I}(\mathbf{L}_N)^* \mathbf{P} \mathbf{I}(\mathbf{L}_N)$)

$$\mathbf{T}^* \mathbf{X} \mathbf{T} \prec \mathbf{X} \tag{11}$$

In particular, we are interested in constructing an explicit representation of a solution so that we can develop a numerical method for stability analysis with that representation (which we indeed do in [18]). This section introduces a special class of operators $\mathcal{P}_N$ that is ensured to contain a nonempty subset of the solutions to (10) whenever Σ is stable. The class $\mathcal{P}_N$ is very simple and tractable, because every operator $\mathbf{P} \in \mathcal{P}_N$ is parametrized by two finite-dimensional constant matrices P and Π, and thus the following arguments (including those in [18]) proceed in a rather "linear algebraic" way, even though all the arguments indeed deal with infinite-dimensional operators. The generalized sampling and hold operators $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$ introduced in this section and used in the construction of the above class play an important role in establishing such a framework. For example, the necessary and sufficient condition for the (strict) positivity of $\mathbf{P}$ can be given as matrix inequality conditions in P and Π, thanks to the properties of $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$. The relevance of $\mathcal{P}_N$ with the existing methods via the Lyapunov-Krasovskii functionals [26]–[28] is also discussed. Further discussions about an equivalent scaling treatment, which uses the square root $\mathbf{S}$ of $\mathbf{P}$ as well as its inverse $\mathbf{S}^{-1}$, are also given; the operators $\mathbf{S}$ and $\mathbf{S}^{-1}$ can also be constructed quite easily in a "linear algebraic" way.

Similar arguments have been developed in our earlier study [36], but the arguments in the present paper are much more sophisticated in the following aspects. First, in the earlier study, $\mathbb{R}^n$ in the product space $\mathcal{M}'_N$ was dealt with by embedding it into the space $\widetilde{\mathcal{F}}'_n$ of constant functions defined on $[0, h')$, and thus was rather detouring. Even though we do introduce such a space of constant functions (which we denote $\mathcal{K}'^{(0)}_\mu$ instead) in the following, it is associated only with $\mathcal{K}_\mu$, and not[1] with $\mathbb{R}^n$. Furthermore, the expressions for the inverse and square root of an operator in the class $\mathcal{P}_N$ have not been developed in that earlier study, so that all the arguments were carried out in such a way that the references to the forms of the inverse and/or square root can be avoided. This forced us to develop rather complicated and less intuitive discussions, which would have prevented an easy understanding of the overall arguments. The following arguments are a much simpler alternative to deriving essentially the same results; such arguments easily allow further extension of operators $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$ to more general ones $\mathbf{J}_{Sk}$ and $\mathbf{J}_{Hk}$ in Section IV.

A. Generalized Sampling and Hold Operators

We first introduce the "generalized sampling and hold operators" that play a very important role for constructing the solutions to operator Lyapunov inequalities. To avoid making the arguments excessively complicated from the beginning, however, we first confine ourselves to the simplest case of the ("zero-order") generalized sampling and hold operators $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$; a more general case of sampling and hold operators will be discussed in Section IV.

The generalized sampling operator $\mathbf{J}_{S0} : \mathcal{K}'_\mu \to \mathbb{R}^\mu$ and the generalized hold operator $\mathbf{J}_{S0}^* =: \mathbf{J}_{H0} : \mathbb{R}^\mu \to \mathcal{K}'_\mu$ are defined by

$$\mathbf{J}_{S0} f = \frac{1}{\sqrt{h'}} \int_0^{h'} f(\theta')d\theta', \ (\mathbf{J}_{H0} v_0)(\theta') = \frac{1}{\sqrt{h'}} v_0 \tag{12}$$

where $0 \le \theta' < h'$. Here we define $\mathcal{K}'^{(0)}_\mu \subset \mathcal{K}'_\mu$ to be the space of constant functions $\{v\mathbf{1} \,|\, v \in \mathbb{R}^\mu\}$ with $\mathbf{1}(\theta') = 1$ $(0 \le \theta' < h')$. It follows readily that $\mathcal{K}'^{(0)}_\mu$ is a Hilbert space with the inner product on $\mathcal{K}'_\mu$ restricted to constant functions. Furthermore, we can consider the restriction of $\mathbf{J}_{S0}$ to $\mathcal{K}'^{(0)}_\mu$, which we also denote by the same symbol $\mathbf{J}_{S0}$ to keep the notation concise. Similarly, we can view $\mathbf{J}_{H0}$ also as a mapping from $\mathbb{R}^\mu$ to $\mathcal{K}'^{(0)}_\mu$, which is again denoted by the same symbol $\mathbf{J}_{H0}$. With respect to the inner product on $\mathcal{K}'^{(0)}_\mu$ mentioned above and that on $\mathbb{R}^\mu$, we again have $\mathbf{J}_{H0} = \mathbf{J}_{S0}^*$, and also $\mathbf{J}_{S0}\mathbf{J}_{H0} = I$ on $\mathbb{R}^\mu$ and $\mathbf{J}_{H0}\mathbf{J}_{S0} = I$ on $\mathcal{K}'^{(0)}_\mu$. It should be noted, however, that $\mathbf{J}_{H0}\mathbf{J}_{S0} \neq I$ on $\mathcal{K}'_\mu$ (in fact, $\mathbf{J}_{H0}\mathbf{J}_{S0}$ is an orthogonal projection from $\mathcal{K}'_\mu$ onto $\mathcal{K}'^{(0)}_\mu$).

B. A Class of Solutions to Operator Lyapunov Inequalities

In this subsection, we introduce a class of operators, denoted by $\mathcal{P}_N$, such that the class is ensured to contain a nonempty subset of the solutions to the operator Lyapunov inequality (10), provided that Σ is stable and N is large enough.

Let us consider the operator

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01}\mathbf{J}_{S0} & \cdots & P_{0N}\mathbf{J}_{S0} \\ \mathbf{J}_{H0}P_{01} & \mathbf{J}_{H0}P_{11}\mathbf{J}_{S0} + \Pi & \cdots & \mathbf{J}_{H0}P_{1N}\mathbf{J}_{S0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{J}_{H0}P_{0N} & \mathbf{J}_{H0}P_{N1}\mathbf{J}_{S0} & \cdots & \mathbf{J}_{H0}P_{NN}\mathbf{J}_{S0} + \Pi \end{bmatrix}$$
$$= \mathbf{I}(\overline{\mathbf{J}_{H0}}) P \, \mathbf{I}(\overline{\mathbf{J}_{H0}})^* + \mathbf{O}\left(\overline{\Pi}\right) \in \mathcal{B}_{\mathbf{F}N} \tag{13}$$

[1]The only exception is with the proof of Proposition 3 given later.

where $\overline{\Pi} := I_N \otimes \Pi = \mathrm{diag}[\Pi, \cdots, \Pi]$ (and similarly for $\overline{\mathbf{J}_{H0}}$), and $P := (P_{ij})_{i,j=0}^N \in \mathbb{R}^{(n+\mu N)\times(n+\mu N)}$ and $\Pi \in \mathbb{R}^{\mu\times\mu}$ are matrices such that

$$P + \mathbf{O}\left(\overline{\Pi}\right) > 0, \quad \Pi > 0 \tag{14}$$

Note that P corresponds to the "compact (in fact, finite-rank) part" of $\mathbf{P}$ while Π to the "noncompact (multiplication operator) part." We have the following result on $\mathbf{P}$, the proof of which is given in Appendix.

Proposition 2: The operator $\mathbf{P}$ given by (13) satisfies $\mathbf{P} \succ 0$ if and only if (14) holds.

The above proposition suggests that we could consider the class of operators of the form (13) satisfying (14) (which we denote by $\mathcal{P}_N \subset \mathcal{L}(\mathcal{M}'_N)$) as the candidates of the strictly positive solutions to the operator Lyapunov inequality (10). This can be restated that we could consider the class of operators $\mathfrak{Q}_N := \{\mathbf{Q} = \mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{P}\,\mathbf{I}(\mathbf{L}_N) \,|\, \mathbf{P} \in \mathcal{P}_N\} \subset \mathcal{L}(\mathcal{M})$ as the candidates to the solutions of (11). Note in fact that $\mathfrak{Q}_N \subset \mathfrak{B}_{\mathbf{F}1}$, $\forall N \in \mathbb{N}$ due to the special block-diagonal form of the second term on the right hand side of (13). In view of this fact, the following result [36] plays a crucial role, whose proof is given in Appendix.

Proposition 3: Suppose that $\mathbf{X} \in \mathfrak{B}_{\mathbf{F}1}$ is given. For any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, there exists $\mathbf{Q} = \mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{P}\mathbf{I}(\mathbf{L}_N) \in \mathfrak{B}_{\mathbf{F}1}$ with $\mathbf{P}$ given by (13) such that $\|\mathbf{X} - \mathbf{Q}\| < \varepsilon$. In particular, if $\mathbf{X} \succ 0$, there exists such $\mathbf{Q} \succ 0$ (i.e., $\mathbf{Q} \in \mathfrak{Q}_N$).

Note that the assumption $\mathbf{X} \in \mathfrak{B}_{\mathbf{F}1}$ in the above proposition does not cause any problem in our arguments since the operator Lyapunov inequality (11) is ensured to have such a strictly positive solution by Corollary 1 due to the fact that $\mathbf{T} \in \mathfrak{B}_{\mathbf{F}1}$. Combining the above arguments, we see that, as far as a theoretical side is concerned, we can have an asymptotically exact (i.e., non-conservative) stability analysis method by letting $N \to \infty$, even when we confine ourselves to $\mathbf{P} \in \mathcal{P}_N$. More precisely, if Σ is stable, then there exists $\mathbf{P} \in \mathcal{P}_N$ satisfying (10) for some finite N that is large enough. A numerical method for stability analysis based on this fact will be discussed in details in the sequel paper [18].

C. Comparison with the Lyapunov-Krasovskii Approach

We now discuss the significance of the above construction of the classes $\mathcal{P}_N$ and $\mathfrak{Q}_N$ in connection with existing studies.

We first remark that the class $\mathfrak{Q}_N$ has a very close relationship with the Lyapunov-Krasovskii functionals (LKF) of those type used in such studies as [26]–[28], even though the discrete-time viewpoint introduced in the present study might make it hard to have a completely rigorous comparison. It has been shown in these studies that whenever Σ is stable, its stability can indeed be ensured by using an LKF within the class of such a type of LKFs studied therein. Some important points here are that

(i) these studies correspond to discussions before discretization of the functionals, and
(ii) it has also been shown in [27],[8] that the term corresponding to Π in (13) is in fact unnecessary (i.e., an LKF could also be found even with "$\Pi = 0$," if desired).

Regarding the first issue, it should also be noted that

(iii) a series of studies have been carried out in [8],[30]–[33] to discretize the LKFs and derive finite-dimensional *continuous-time* LMI conditions.

Regarding the issue (i), we first remark that the studies of [26]–[28] involves some functions in the kernels of the LKFs and thus no infinite-dimensionality issue has been eliminated at the theoretical stage of the proof for the existence of the LKFs. In other words, it is not clear what sort of kernels may be considered tractable without leading to conservativeness of the analysis. In contrast, the significance of the construction of the class $\mathfrak{Q}_N$ (or $\mathcal{P}_N$) lies in that the theoretical consequence (i.e., the existence of a solution $\mathbf{X} \in \mathfrak{Q}_N$ to the Lyapunov inequality (11) for large N) has been established with the use of only two finite-dimensional constant matrices P and Π. Furthermore, determining if the corresponding $\mathbf{P}$ is strictly positive is simply a problem of matrix inequalities in (14), and it also turns out (see the following subsection) that the square root $\mathbf{S} := \mathbf{P}^{1/2}$ and its inverse $\mathbf{S}^{-1}$ can also be constructed explicitly with only matrix manipulations. These computations will be quite important not only in theoretical studies but also in the numerical computations studied in the sequel paper [18]. In this way, most arguments *even in the theoretical treatment* can be carried out in a fairly "linear algebraic" way with many similarities to the stability analysis of FDLTI discrete-time systems, and this is a significant advantage of applying the fast-lifting approach and the pseudo-discretization technique.

The above observation is related to the issue (ii) raised above. Unlike [27],[8], the present paper has ruled out the case of $\Pi = 0$ by (14). The use of $\Pi > 0$ is crucial in ensuring the *strict* positivity of $\mathbf{P}$ (in fact, it is a necessary condition). If $\Pi = 0$, then $\mathbf{P}$ reduces to a compact operator, so that $\mathbf{S}^{-1}$ does not exist as a bounded operator. This is not a convenient situation as we will see in the following subsection (see (19)) as well as in the sequel paper [18], and hence ensuring the strict positivity of $\mathbf{P}$ is very important in our arguments. We will also see that its strict positivity, easily dealt with by the representation of $\mathbf{P}$ in the form of (13), plays a key role in our arguments in allowing repeated "linear algebraic" (or "matrix-like") treatment of operators.

The aforementioned observation about the issue (i) is also related to the issue (iii) raised above. Before we touch on the relevance, we first remark that we can further the study of the present paper so that we can give a numerical method for computing the solution $\mathbf{P}$ to the operator Lyapunov inequality (10). More precisely, we can derive finite-dimensional LMI conditions about the matrices P and Π contained in the Lyapunov operator $\mathbf{P}$, as will be shown in the sequel paper [18]. Since the non-conservativeness (asymptotic exactness) of the pseudo-discretization approach on the theoretical side has been established through the very treatment of $\mathbf{P}$ with P and Π, the non-conservativeness with respect to the associated *numerical solution method* for the operator Lyapunov inequality can be established rather easily. Furthermore, in contrast to the discretized LKF techniques [8],[30]–[33] that lead to continuous-time LMI conditions, the resulting LMI conditions in [18] have an important feature that they are *discrete-time* LMI conditions, which are well suited to dealing with discrete-time controllers.

D. Scaling Treatment

For our later purposes, it is more convenient to relate the analysis with operator Lyapunov inequalities to the scaling treatment of operators (see [29] for related arguments). To this end, it is convenient to have an explicit representation for the square root of $\mathbf{P} \succ 0$, i.e., $\mathbf{S} = \mathbf{S}^* \succ 0$ such that $\mathbf{S}^*\mathbf{S} = \mathbf{P}$. We can indeed verify by a direct computation that such $\mathbf{S}$ is given by

$$\mathbf{S} = \mathbf{I}(\overline{\mathbf{J}_{H0}})S\,\mathbf{I}(\overline{\mathbf{J}_{H0}})^* + \mathbf{O}(\overline{\Pi_S}) \tag{15}$$

where the matrices S and Π_S are given by

$$S = \left(P + \mathbf{O}(\overline{\Pi})\right)^{1/2} - \mathbf{O}\left(\overline{\Pi_S}\right), \quad \Pi_S = \Pi^{1/2} \tag{16}$$

We readily have a similar rule

$$\mathbf{S}^{-1} = \mathbf{I}(\overline{\mathbf{J}_{H0}})S_{\rm inv}\,\mathbf{I}(\overline{\mathbf{J}_{H0}})^* + \mathbf{O}\left(\overline{\Pi_{S_{\rm inv}}}\right) \tag{17}$$

where the matrices $S_{\rm inv}$ and $\Pi_{S_{\rm inv}}$ are given by

$$S_{\rm inv} = \left(S + \mathbf{O}\left(\overline{\Pi_S}\right)\right)^{-1} - \mathbf{O}\left(\overline{\Pi_{S_{\rm inv}}}\right), \; \Pi_{S_{\rm inv}} = \Pi_S^{-1} \tag{18}$$

We henceforth denote the class of $\mathbf{S} = \mathbf{S}^* \succ 0$ such that $\mathbf{S}^*\mathbf{S} = \mathbf{P} \in \mathcal{P}_N$ by $\mathcal{S}_N$ (in fact, $\mathcal{S}_N$ is exactly the same class as $\mathcal{P}_N$, but having these two notations would be helpful for avoiding possible tangles in the arguments). It follows from Proposition 1 that when N is large enough, checking the stability of Σ amounts to finding $\mathbf{S} \in \mathcal{S}_N$ such that

$$\|\mathbf{S}\mathbf{T}_N\mathbf{S}^{-1}\| < 1 \tag{19}$$

Even though this scaling approach is essentially the same as solving an operator Lyapunov inequality, it leads to an advantage especially with respect to a numerical procedure for stability analysis. As will be studied in the sequel paper [18], we will develop, through what we call the quasi-finite-rank approximation of the fast-lifted monodromy operator $\mathbf{T}_N$, a numerically tractable method (by means of only finite-dimensional computations) that leads to an "approximate solution" to the operator Lyapunov inequality (10). We then give a method, through some perturbation analysis, to check if the approximate solution indeed solves the Lyapunov inequality rigorously. In such a context, scalar inequalities such as (19) turn out to be easier to deal with than the infinite-dimensional inequality condition (10), and with such a technique, the numerical search for an appropriate $\mathbf{S} \in \mathcal{S}_N$ (or equivalently, $\mathbf{P} = \mathbf{S}^*\mathbf{S} \in \mathcal{P}_N$) can be carried out in an asymptotically exact fashion. We will further discuss the role of N in [18] in connection with quasi-finite-rank approximation, where the parameter N will play an important role also in reducing the quasi-finite-rank approximation error to any degree.

IV. Further Generalization of Sampling and Hold Operators

This section discusses further generalizing the sampling and hold operators dealt with in Subsection III-A and considers generalizing the classes $\mathcal{P}_N$ and $\mathcal{S}_N$ introduced in Subsections III-B and III-D. This will be important because we will then have a better chance of finding a solution to the operator Lyapunov inequality (10) without increasing N.

The first step for the discussions is to generalize the sampling and hold operators $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$ via the use of the Legendre polynomials [34]. The Legendre polynomials $\psi_i^{[-1,1]}$, $i \in \mathbb{N}_0$ are defined by

$$\psi_i^{[-1,1]}(t) = \frac{1}{2^i i!}\frac{d^i}{dt^i}(x^2-1)^i \tag{20}$$

($t \in [-1,1]$), and satisfy

$$\langle \psi_i^{[-1,1]}, \psi_j^{[-1,1]} \rangle = \frac{2}{2i+1}\delta_{ij} \tag{21}$$

where $\langle \cdot,\cdot \rangle$ denotes the standard inner product on $L_2([-1,1];\mathbb{R})$, and δ_{ij} denotes the Kronecker delta. Let us define the orthonormal polynomials ψ_i of order i ($i \in \mathbb{N}_0$) by

$$\psi_i(\theta') = \sqrt{\frac{2i+1}{h'}}\psi_i^{[-1,1]}\left(\frac{2\theta'}{h'} - 1\right), \; \theta' \in [0,h'] \tag{22}$$

Let us denote by $\mathcal{K}_\mu'^{(k)}$ the set of μ-dimensional polynomial vector functions of order k at most, defined on $[0,h')$, which we regard to have the same inner product $\langle \cdot,\cdot \rangle$ as $\mathcal{K}_\mu'$. Note that the operator $v_0 \in \mathbb{R}^\mu \mapsto v_0\psi_0 \in \mathcal{K}_\mu'^{(0)} \subset \mathcal{K}_\mu'$ is nothing but $\mathbf{J}_{H0}$. Similarly, we define the generalized hold operator $\mathbf{J}_{Hk} : (\mathbb{R}^\mu)^{k+1} \to \mathcal{K}_\mu'^{(k)}$, $k \in \mathbb{N}_0$ by

$$(\mathbf{J}_{Hk}[v_0^T, \cdots, v_k^T]^T)(\theta') = \sum_{i=0}^{k} v_i\psi_i(\theta'), \; \theta' \in [0,h') \tag{23}$$

As in the case of $\mathbf{J}_{H0}$, we view $\mathbf{J}_{Hk}$ also as an operator from $(\mathbb{R}^\mu)^{k+1}$ to $\mathcal{K}_\mu'$ ($\supset \mathcal{K}_\mu'^{(k)}$) according to the context. By the orthonormality, $\mathbf{J}_{Hk}$ is norm-preserving (i.e., isometric), and thus by defining the generalized sampling operator $\mathbf{J}_{Sk} := \mathbf{J}_{Hk}^*$ (with respect to the inner product either on $\mathcal{K}_\mu'^{(k)}$ or $\mathcal{K}_\mu'$), we see that $\mathbf{J}_{Sk}\mathbf{J}_{Hk} = I$; this in turn immediately leads to that $\mathbf{J}_{Hk}\mathbf{J}_{Sk}$ is a (nonzero) orthogonal projection from $\mathcal{K}_\mu'$ to $\mathcal{K}_\mu'^{(k)}$.

It readily follows that for any matrix $X \in \mathbb{R}^{m\times m}$ and the associated multiplication operator, we have $X\mathbf{J}_{Hk} = \mathbf{J}_{Hk}[X]^k$, where $[X]^k$ is a shorthand notation for $I_{k+1} \otimes X$. From this property, we can show as in the proof of Proposition 2 that the modified form of $\mathbf{P}$ given by

$$\mathbf{P}^{(k)} = \mathbf{I}(\overline{\mathbf{J}_{Hk}})P^{(k)}\mathbf{I}(\overline{\mathbf{J}_{Hk}})^* + \mathbf{O}\left(\overline{\Pi}\right) \in \mathcal{B}_{\mathbf{F}N} \tag{24}$$

satisfies $\mathbf{P}^{(k)} \succ 0$ if and only if

$$P^{(k)} + \mathbf{O}\left(\overline{[\Pi]^k}\right) > 0, \quad \Pi > 0 \tag{25}$$

where $P^{(k)} \in \mathbb{R}^{(n+(k+1)\mu N)\times(n+(k+1)\mu N)}$. Let us partition $P^{(k)}$ as $P^{(k)} = (P_{ij}^{(k)})_{i,j=0}^N$ where $P_{00}^{(k)} \in \mathbb{R}^{n\times n}$, $P_{i0}^{(k)} \in \mathbb{R}^{(k+1)\mu\times n}$, $P_{0j}^{(k)} \in \mathbb{R}^{n\times(k+1)\mu}$, $P_{ij}^{(k)} \in \mathbb{R}^{(k+1)\mu\times(k+1)\mu}$ ($i,j = 1,\cdots,N$). If we consider the case

$$P_{00}^{(k+1)} = P_{00}^{(k)}, \quad P_{i0}^{(k+1)} = [(P_{i0}^{(k)})^T \; 0]^T,$$

$$P_{0j}^{(k+1)} = [P_{0j}^{(k)} \; 0], \quad P_{ij}^{(k+1)} = \mathrm{diag}[P_{ij}^{(k)}, 0] \tag{26}$$

and construct the corresponding $\mathbf{P}^{(k+1)}$, then we see that it is nothing but $\mathbf{P}^{(k)}$. Thus, if we define the class $\mathcal{P}_N^{(k)}$ as the set of $\mathbf{P}^{(k)}$ satisfying (25), we see that it is monotonically increasing with respect to $k \in \mathbb{N}_0$, i.e., $\mathcal{P}_N^{(k_1)} \subset \mathcal{P}_N^{(k_2)}$ in a strict sense for any fixed $N \in \mathbb{N}$ whenever $k_1 < k_2$. Hence by using $\mathbf{P}^{(k)}$ instead of $\mathbf{P}$, we can have a better chance of successfully determining the stability of Σ for each fixed N. Furthermore, since we obviously have $\mathcal{P}_N = \mathcal{P}_N^{(0)} \subset \mathcal{P}_N^{(k)},\ \forall k \in \mathbb{N}$, the asymptotic exactness property of our approach with respect to $N \to \infty$ remains the same under the use of $\mathbf{P}^{(k)}$.

We remark that $\mathfrak{Q}_N^{(k)} := \{\mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{P}^{(k)}\mathbf{I}(\mathbf{L}_N) \,|\, \mathbf{P}^{(k)} \in \mathcal{P}_N^{(k)}\} \subset \mathfrak{B}_{\mathbf{F}1} \subset \mathcal{L}(\mathcal{M}),\ \forall N \in \mathbb{N}$ even for $k > 0$. That is, the extension from $\mathfrak{Q}_N = \mathfrak{Q}_N^{(0)}$ to $\mathfrak{Q}_N^{(k)},\ k > 0$ is achieved only within the class $\mathfrak{B}_{\mathbf{F}1}$. Note that confining to such a class the solutions to the operator Lyapunov inequality (11) is not restrictive by Corollary 1 and the fact that $\mathbf{T} \in \mathfrak{B}_{\mathbf{F}1}$. We also remark that the formulas for the square root $\mathbf{S}^{(k)}$ of $\mathbf{P}^{(k)} \succ 0$ and the inverse $(\mathbf{S}^{(k)})^{-1}$ are given in parallel forms to those given in Subsection III-D. These observations ensure that the preceding arguments extend readily to the case when $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$ are replaced by $\mathbf{J}_{Sk}$ and $\mathbf{J}_{Hk}$, respectively.

Remark 1: It may be interesting to interpret the above technique in comparison with the discretized LKF technique [8],[30]–[33]; the latter uses piecewise linear kernels, but it does not seem straightforward, if not impossible, to generalize the method to incorporate higher order piecewise polynomials. On the other hand, by taking $A'_{Hk} \in \mathbb{R}^{(k+1)\mu\times(k+1)\mu}$ and $C'_{Hk} \in \mathbb{R}^{\mu\times(k+1)\mu}$ appropriately, we have $(\mathbf{J}_{Hk}v)(\theta') = C'_{Hk}\exp(A'_{Hk}\theta')v$ [18]; that is, $\mathbf{J}_{Hk}$ and $\mathbf{J}_{Sk} = \mathbf{J}_{Hk}^*$ have similar structures to $\mathbf{C}$ and $\mathbf{B}$, respectively, with the underlying horizon replaced by $[0, h')$. Hence the operators $\mathbf{J}_{Hk}$ and $\mathbf{J}_{Hk}$ are both easy to deal with within the state-space framework. This is quite important in developing the numerical method in [18].

V. Concluding Remarks

Stimulated by the study in [13], we have developed a novel approach to the stability analysis of the time-delay system Σ. The state transition of Σ was first viewed in discrete-time and described by the monodromy operator $\mathbf{T}$, and it was then transformed to define the fast-lifted monodromy operator $\mathbf{T}_N$. A stability condition in terms of the spectral radius $\rho(\mathbf{T}) = \rho(\mathbf{T}_N)$ was given, and it was then related to operator Lyapunov inequality in terms of $\mathbf{T}_N$. By introducing the generalized sample and hold operators $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$, the class of operators $\mathcal{P}_N$ was introduced that is ensured to contain a nonempty subset of the solutions to the operator Lyapunov inequality if Σ is stable and if N is large enough. Thus, a fundamental framework for asymptotically exact stability analysis has been established. An equivalent scaling treatment was also shown (which will be very important in the numerical computations for finding the solutions dealt with in the sequel paper [18]). These arguments are carried out in a rather "linear algebraic" way with many similarities to the stability analysis of FDLTI discrete-time systems.

The class $\mathcal{P}_N$ is parametrized by the two finite-dimensional constant matrices P and Π, and it was shown that determining the positivity and taking the square root or the inverse of the operators in $\mathcal{P}_N$ can be carried out via only working on the matrices P and Π. In this sense, an intriguing feature of the theoretical framework could be said to be a "pseudo-discretization" technique; it allows one to essentially reduce the arguments on infinite-dimensional operators to those about finite-dimensional matrices. The relationship of the class $\mathfrak{Q}_N = \{\mathbf{Q} = \mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{P}\mathbf{I}(\mathbf{L}_N) \,|\, \mathbf{P} \in \mathcal{P}_N\}$ with the Lyapunov-Krasovskii functionals (LKFs) studied in [26]–[28] was discussed from a theoretical point of view. Comparison with the discretized LKF technique [8],[30]–[33] was also given from the viewpoint of the generalization with higher-order polynomials via the introduction of the further generalized sample and hold operators $\mathbf{J}_{Sk}$ and $\mathbf{J}_{Hk}$.

We finally remark that a numerical computation method can be developed based on the theoretical development in this paper (see the sequel paper [18]); there it is established that the asymptotic exactness about N in the stability analysis is retained also in the associated numerical computations, although we do need to apply some approximation technique in the reduction to finite-dimensional computations through LMI optimization. The resulting LMI conditions are "in discrete-time" due to the discrete-time viewpoint adopted in our approach. Such conditions can be used not only for stability analysis of TDSs with continuous-time (or even discrete-time) controllers, but are also suited, in principle, to the design of discrete-time controllers for continuous-time TDSs. This is a very important advantage of our approach, because almost all LMI conditions for stability of TDSs are "in continuous-time" and thus are not compatible with discrete-time controller design, in spite of general preference for discrete-time controllers for implementation reasons and nontrivial problems of discretizing continuous-time controllers while retaining closed-loop stability/performance.

We thus believe that the fast-lifting-based pseudo-discretization technique developed in this paper opens a promising horizon to the treatment of TDSs, providing a powerful but tractable framework with many similarities with the treatment of FDLTI discrete-time systems.

References

[1] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York (1963).
[2] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York (1977).
[3] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*, Academic Press (1986).
[4] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York (1995).
[5] L. Dugard and E. I. Verriest, Eds., *Stability and Control of Time-Delay Systems*, Springer, London (1998).
[6] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer (1999).
[7] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, Springer, London (2001).
[8] K. Gu, V. L. Kharitonov and J. Chen, *Stability of Time-Delay Systems*, Birkhäuser, Boston (2003).
[9] W. Michiels and S.-I. Niculescu, *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach*, SIAM (2007).
[10] V. L. Kharitonov, Robust Stability Analysis of Time Delay Systems: A Survey, *Annual Reviews in Control*, Vol. 23, pp. 185–196 (1999).
[11] J.-P. Richard, Time-Delay Systems: An Overview of Some Recent Advances and Open Problems, *Automatica*, Vol. 39, pp. 1667–1694 (2003).

[12] K. Gu and S.-I. Niculescu, Survey on Recent Results in the Stability and Control of Time-Delay Systems, *Transactions of the ASME*, Vol. 125, pp. 158–165 (2003).
[13] K. Hirata and H. Kokame, Stability analysis of retarded systems via lifting technique, Proc. Conf. Decision and Control, pp. 5595–5596 (2003).
[14] Y. Yamamoto, A function space approach to sampled-data systems and tracking problems, *IEEE Trans. Automat. Contr.*, Vol. AC-39, No. 4, pp. 703–713 (1994).
[15] B. A. Bamieh and J. B. Pearson, A general framework for linear periodic systems with applications to H_∞ sampled-data control, *IEEE Trans. Automat. Contr.*, Vol. 37, No. 4, pp. 418–435 (1992).
[16] H. T. Toivonen, Sampled-data control of continuous-time systems with an H_∞ optimality criterion, *Automatica*, Vol. 28, No. 1, pp. 45–54 (1992).
[17] G. Tadmor, H_∞ optimal sampled-data control in continuous time systems, *Int. J. Control*, Vol. 56, No. 1, pp. 99-141 (1992).
[18] T. Hagiwara and T. Inui, Monodromy operator approach to time-delay systems: Numerical method for solving operator Lyapunov inequalities, Proc. MTNS 2010 (2010).
[19] T. Hagiwara, Causal/noncausal linear periodically time-varying scaling for robust stability analysis and their properties, Proc. the 17th International Symposium on Mathematical Theory of Networks and Systems, pp. 742–752 (2006).
[20] T. Hagiwara and H. Umeda, Modified fast-sample/fast-hold approximation for sampled-data system analysis, *European Journal of Control*, Vol. 14, No. 4, pp. 286–296 (2008).
[21] T. Hagiwara and K. Okada, Modified fast-sample/fast-hold approximation and γ-independent H_∞-discretization for general sampled-data systems by fast-lifting, *Int. J. Control*, Vol. 82, No. 9, pp. 1762–1771 (2009).
[22] T. Hagiwara, Separator-type robust stability theorem of sampled-data systems allowing noncausal LPTV scaling, *Automatica*, Vol. 45, No. 8, pp. 1868–1872 (2009).
[23] A. W. Naylor and G. R. Sell, *Linear Operator Theory in Engineering and Science*, Springer (1982).
[24] J. B. Conway, *A Course in Functional Analysis*, Second Ed., Springer (1990).
[25] N. Dunford and J. T. Schwartz, *Linear Operators, Part II: Spectral Theory*, New York, Interscience (1963).
[26] E. F. Infante and W. B. Castelan, A Liapunov functional for a matrix difference-differential equation, *J. Differential Equations*, Vol. 29, pp. 439–451 (1978).
[27] Huang Wenzhang, Generalization of Liapunov's theorem in a linear delay system, *J. Mathematical Analysis and Applications*, Vol. 142, pp. 83–94 (1989).
[28] V. L. Kharitonov and A. P. Zhabko, *Lyapunov-Krasovskii* approach to the robust stability analysis of time-delay systems, *Automatica*, Vol. 39, No. 1, pp. 15–20 (2003).
[29] G.-C. Rota, On Models for Linear Operators, *Comm. Pure Appl. Math.*, Vol. XIII, pp. 469–472 (1960).
[30] K. Gu, Discretized LMI set in the stability problem of linear uncertain time-delay systems. *Int. J. Control*, Vol. 68, pp. 923–934 (1997).
[31] K. Gu, A further refinement of discretized Lyapunov functional method for the stability of time-delay systems, *Int. J. Control*, Vol. 74, No. 10, pp. 967–976 (2001).
[32] K. Gu, Discretization schemes for Lyapunov-Krasovskii functionals in time-delay systems, *Kybernetika*, Vol. 37, No. 4, pp. 479–504 (2001).
[33] K. Gu and Y. Liu, Lyapunov-Krasovskii functional for uniform stability of coupled differential-functional equations, *Automatica*, Vol. 45, No. 3, pp. 798–804 (2009).
[34] K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, Third Ed., Springer (2009).
[35] P. R. Halmos, *A Hilbert Space Problem Book*, Second Ed., Springer (1982).
[36] T. Hagiwara, Fast-lifting approach to time-delay systems: Fundamental framework, Proc. Conf. Decision and Control, pp. 5292–5299 (2008).

Appendix

Proof of Proposition 2

We derive the necessary and sufficient condition for $\gamma^2\mathbf{P}-I \succ 0$, since $\mathbf{P} \succ 0$ if and only if $\gamma^2\mathbf{P}-I \succ 0$ for some sufficiently large γ^2. It follows from (13) that

$$\gamma^2\mathbf{P}-I = \mathbf{I}\left(\overline{\gamma^2\Pi - I}\right) + \mathbf{I}(\overline{\mathbf{J}_{H0}})\left(\gamma^2 P - 2\,\mathbf{I}(0)\right)\mathbf{I}(\cdot)^* \quad (27)$$

where $\mathbf{I}(\cdot)^*$ denotes $\mathbf{I}(\overline{\mathbf{J}_{H0}})^*$, which is obvious from the self-adjointness; we use such convention here. We claim that the condition $\gamma^2\mathbf{P}-I \succ 0$ is equivalent to the matrix condition

$$\mathbf{I}\left(\overline{\gamma^2\Pi - I}\right) + \left(\gamma^2 P - 2\,\mathbf{I}(0)\right) > 0 \quad (28)$$

To see this, we first apply an appropriate congruence transformation on (27) and rewrite the condition $\gamma^2\mathbf{P}-I \succ 0$ as

$$I + \mathbf{I}\left(\overline{(\gamma^2\Pi - I)^{-1/2}\mathbf{J}_{H0}}\right)\left(\gamma^2 P - 2\,\mathbf{I}(0)\right)\mathbf{I}(\cdot)^* \succ 0 \quad (29)$$

so that we have the identity operator explicitly on the first term. Here we used the fact that $\gamma^2\mathbf{P}-I \succ 0$ only if $\gamma^2\Pi - I > 0$. Since the second term on the left hand side of (29) is a compact (in fact, finite-rank) operator, (29) is equivalent to the condition that all eigenvalues of the matrix

$$I + \mathbf{I}(\cdot)^*\,\mathbf{I}\left(\overline{(\gamma^2\Pi - I)^{-1/2}\mathbf{J}_{H0}}\right)\left(\gamma^2 P - 2\,\mathbf{I}(0)\right) \quad (30)$$

are positive. Noting that $(\cdot)\mathbf{J}_{H0} = \mathbf{J}_{H0}(\cdot)$ for an operator of multiplication by a matrix $(\cdot)$, and $\mathbf{J}_{H0}^*\mathbf{J}_{H0} = I$, the above matrix equals $I + \mathbf{I}\left(\overline{(\gamma^2\Pi - I)^{-1}}\right)\left(\gamma^2 P - 2\,\mathbf{I}(0)\right)$, and thus the condition is further equivalent to

$$I + \mathbf{I}(\overline{(\gamma^2\Pi - I)^{-1/2}})\left(\gamma^2 P - 2\,\mathbf{I}(0)\right)\mathbf{I}(\cdot)^* > 0 \quad (31)$$

This condition is equivalent to (28), or $\gamma^2\left(P + \mathbf{O}\left(\overline{\Pi}\right)\right) - I > 0$. Recalling the condition $\gamma^2\Pi - I > 0$ and considering the condition for the existence of large enough γ^2 satisfying these matrix inequalities complete the proof.

Proof of Proposition 3

A. Preliminary Results for the Proof

We first prepare some preliminary results for the proof of Proposition 3. Let $\widehat{x} = \{\widehat{x}_k\}_{k\in\mathbb{Z}} \in l_2 = l_2(\mathbb{Z};\mathbb{C}^\mu)$ and let $l_{2,r} = \{\widehat{x} \in l_2 \,|\, \widehat{x}_0 \in \mathbb{R}^\mu,\ \widehat{x}_{-k} = \bar{\widehat{x}}_k\ (\forall k \in \mathbb{N})\}$, where $\bar{\cdot}$ denotes the complex conjugate vector. Let $\mathcal{F} : \mathcal{K}_\mu \to l_{2,r}$ be the Fourier expansion operator. Furthermore, let the projections $\widehat{\mathcal{P}}_k : l_{2,r} \to l_{2,r}$ $(k \in \mathbb{N}_0)$ be $\widehat{y} = \widehat{\mathcal{P}}_k\widehat{x}$ with $\widehat{y}_i = 0$ $(|i| \le k)$, $\widehat{y}_i = x_i$ $(|i| > k)$. The first result corresponds to a high-frequency attenuation property of compact operators.

Proposition 4: If $\mathbf{A}$ is a compact operator on $\mathcal{K}_\mu$, then $\|\widehat{\mathbf{A}}\widehat{\mathcal{P}}_k\| \to 0$ $(k \to \infty)$ for $\widehat{\mathbf{A}} = \mathcal{F}\mathbf{A}\mathcal{F}^{-1}$.

Proof: $\|\widehat{\mathbf{A}}\widehat{\mathcal{P}}_k\|$ converges since it is non-increasing with respect to k. If the limit is not 0, there exists $\varepsilon > 0$ such that $\|\widehat{\mathbf{A}}\widehat{\mathcal{P}}_k\| > \varepsilon$, $\forall k \in \mathbb{N}$. Hence, if we note $\widehat{\mathcal{P}}_k^2 = \widehat{\mathcal{P}}_k$, we see that there exists a sequence $\{\widehat{x}_k\}$ such that $\widehat{x}_k \in l_{2,r}$, $\|\widehat{x}_k\| \le 1$, $\widehat{\mathcal{P}}_k\widehat{x}_k = \widehat{x}_k$ and $\|\widehat{\mathbf{A}}\widehat{x}_k\| > \varepsilon$, $\forall k \in \mathbb{N}$. For this $\{\widehat{x}_k\}$ and for any $\widehat{y} \in l_{2,r}$, we have $\langle \widehat{y}, \widehat{x}_k\rangle = \langle \widehat{y}, \widehat{\mathcal{P}}_k\widehat{x}_k\rangle = \langle \widehat{\mathcal{P}}_k^*\widehat{y}, \widehat{x}_k\rangle = \langle \widehat{\mathcal{P}}_k\widehat{y}, \widehat{x}_k\rangle$. Since $\|\widehat{\mathcal{P}}_k\widehat{y}\| \to 0$ $(k \to \infty)$, it follows that $\langle \widehat{y}, \widehat{x}_k\rangle \to 0$ $(k \to \infty)$ for each fixed $\widehat{y} \in l_{2,r}$. Hence $\widehat{x}_k$ converges weakly to 0. Since $\widehat{\mathbf{A}}$ is a compact operator, $\widehat{\mathbf{A}}\widehat{x}_k$ must converge (strongly) to 0 (Proposition VI.3.3 of [24]). This contradicts $\|\widehat{\mathbf{A}}\widehat{x}_k\| > \varepsilon$, $\forall k \in \mathbb{N}$. ■

The second result is straightforward due to the restriction to a finite number of sinusoidal components (since $\mathbf{J}_{H0}\mathbf{J}_{S0}f'$ is a constant function corresponding to the average of f').

Lemma 1: Suppose that h' is a directly adjustable parameter. For any $K \in \mathbb{N}$, $\omega > 0$ and $\varepsilon > 0$, there exists $h'_0 > 0$ such that, whenever $h' < h'_0$, $\|(\mathbf{J}_{H0}\mathbf{J}_{S0} - I)f'\|^2 < \varepsilon^2\|f'\|^2$, $\forall f' \in \{\sum_{k=-K}^{K} e^{jk\omega t}\hat{f}_k \mid \hat{f} \in l_{2,r},\ 0 \le t < h'\} \subset \mathcal{K}'_\mu$.

The above result can be restated as follows if we note that $\mathcal{F}$ and $\mathbf{L}_N$ are norm-preserving (i.e., isometric).

Lemma 2: Let $h > 0$ be given. Suppose that h' is given by $h' = h/N$, $N \in \mathbb{N}$, and let $\mathbf{Z}_N = \overline{\mathbf{J}_{H0}\mathbf{J}_{S0}}\ (= \mathbf{Z}_N^*) : (\mathcal{K}'_\mu)^N \to (\mathcal{K}'_\mu)^N$. For any $K \in \mathbb{N}$ and $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\|(\widehat{\mathbf{Z}}_N - I)(I - \widehat{\mathcal{P}}_K)\hat{f}\|^2 < \varepsilon^2\|(I - \widehat{\mathcal{P}}_K)\hat{f}\|^2\ (\le \varepsilon^2\|\hat{f}\|^2)$ $(\forall N \ge N_0)$ for all $\hat{f} \in l_{2,r}$, where $\widehat{\mathbf{Z}}_N = \mathcal{F}\mathbf{L}_N^{-1}\mathbf{Z}_N\mathbf{L}_N\mathcal{F}^{-1}$.

Proof: We first note that $g_N^2 := \|(\widehat{\mathbf{Z}}_N - I)(I - \widehat{\mathcal{P}}_K)\hat{f}\|^2 = \|(\mathbf{Z}_N - I)\mathbf{L}_N\mathcal{F}^{-1}(I - \widehat{\mathcal{P}}_K)\hat{f}\|^2$, where each of the N components of $\mathbf{L}_N\mathcal{F}^{-1}(I - \widehat{\mathcal{P}}_K)\hat{f}$ corresponds to f' in Lemma 1 under $\omega = 2\pi/h$. Hence by taking $N_0 > h/h'_0$ so that $h' < h'_0$ $(\forall N \ge N_0)$, it follows from Lemma 1 that $g_N^2 < \varepsilon^2\|\mathcal{F}^{-1}(I - \widehat{\mathcal{P}}_K)\hat{f}\|^2$. This completes the proof. ■

Based on Proposition 4 and Lemma 2, we are now in a position to state the following result that plays a key role.

Proposition 5: Let $h > 0$ be given and let $\mathbf{A}$ be a compact operator on $\mathcal{K}_\mu$. For any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\|\mathbf{Z}_N\mathbf{L}_N\mathbf{A}\mathbf{L}_N^{-1}\mathbf{Z}_N - \mathbf{L}_N\mathbf{A}\mathbf{L}_N^{-1}\| < \varepsilon$ for all $N \ge N_0$.

Proof: Since $\|\mathbf{Z}_N\mathbf{L}_N\mathbf{A}\mathbf{L}_N^{-1}\mathbf{Z}_N - \mathbf{L}_N\mathbf{A}\mathbf{L}_N^{-1}\| = \|\mathbf{L}_N\mathcal{F}^{-1}(\widehat{\mathbf{Z}}_N\widehat{\mathbf{A}}\widehat{\mathbf{Z}}_N - \widehat{\mathbf{A}})\mathcal{F}\mathbf{L}_N^{-1}\|$, it is enough to show that for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\|\widehat{\mathbf{Z}}_N\widehat{\mathbf{A}}\widehat{\mathbf{Z}}_N - \widehat{\mathbf{A}}\| < \varepsilon$ $(\forall N \ge N_0)$. We show this by establishing the existence of $N_0 \in \mathbb{N}$ such that $\|\widehat{\mathbf{Z}}_N\widehat{\mathbf{A}}(\widehat{\mathbf{Z}}_N - I)\| < \varepsilon/2$ and $\|(\widehat{\mathbf{Z}}_N - I)\widehat{\mathbf{A}}\| < \varepsilon/2$ whenever $N \ge N_0$.

We first claim the latter inequality. To show it, we first note that for any $l, k \in \mathbb{N}$,

$$\begin{aligned} &\|(\widehat{\mathbf{Z}}_N - I)\widehat{\mathbf{A}}\| \\ &\le \|(\widehat{\mathbf{Z}}_N - I)\widehat{\mathbf{A}}\widehat{\mathcal{P}}_k\| + \|(\widehat{\mathbf{Z}}_N - I)\widehat{\mathcal{P}}_l\widehat{\mathbf{A}}(I - \widehat{\mathcal{P}}_k)\| \\ &\quad + \|(\widehat{\mathbf{Z}}_N - I)(I - \widehat{\mathcal{P}}_l)\widehat{\mathbf{A}}(I - \widehat{\mathcal{P}}_k)\| \\ &\le \|\widehat{\mathbf{A}}\widehat{\mathcal{P}}_k\| + \|\widehat{\mathcal{P}}_l\widehat{\mathbf{A}}(I - \widehat{\mathcal{P}}_k)\| \\ &\quad + \|(\widehat{\mathbf{Z}}_N - I)(I - \widehat{\mathcal{P}}_l)\widehat{\mathbf{A}}(I - \widehat{\mathcal{P}}_k)\| \end{aligned} \tag{32}$$

since $\mathbf{Z}_N = \overline{\mathbf{J}_{H0}\mathbf{J}_{S0}} \ne I$ is an orthogonal (i.e., self-adjoint) projection on $(\mathcal{K}'_\mu)^N$, and thus so is $I - \mathbf{Z}_N$ $(\ne 0)$; hence we used $\|\widehat{\mathbf{Z}}_N - I\| = 1$. By Proposition 4, there exists $k \in \mathbb{N}$ such that $\|\widehat{\mathbf{A}}\widehat{\mathcal{P}}_k\| < \varepsilon/6$. For this fixed k, there exists $l \in \mathbb{N}$ such that $\|\widehat{\mathcal{P}}_l\widehat{\mathbf{A}}(I - \widehat{\mathcal{P}}_k)\| < \varepsilon/6$ again by Proposition 4, since $\widehat{\mathcal{P}}_l^* = \widehat{\mathcal{P}}_l$. Finally, we have $\|(\widehat{\mathbf{Z}}_N - I)(I - \widehat{\mathcal{P}}_l)\widehat{\mathbf{A}}(I - \widehat{\mathcal{P}}_k)\| \le \|(\widehat{\mathbf{Z}}_N - I)(I - \widehat{\mathcal{P}}_l)\| \cdot \|\widehat{\mathbf{A}}(I - \widehat{\mathcal{P}}_k)\|$ and thus by Lemma 2, for these fixed k and l there exists $N_0 \in \mathbb{N}$ such that $\|(\widehat{\mathbf{Z}}_N - I)(I - \widehat{\mathcal{P}}_l)\widehat{\mathbf{A}}(I - \widehat{\mathcal{P}}_k)\| < \varepsilon/6$ $(\forall N \ge N_0)$. Hence the first claim is established.

The former inequality can be established in a similar fashion by taking the adjoint and noting that $\|(\widehat{\mathbf{Z}}_N\widehat{\mathbf{A}}(\widehat{\mathbf{Z}}_N - I))^*\| = \|(\widehat{\mathbf{Z}}_N - I)\widehat{\mathbf{A}}^*\widehat{\mathbf{Z}}_N\| \le \|(\widehat{\mathbf{Z}}_N - I)\widehat{\mathbf{A}}^*\|$ since $\|\widehat{\mathbf{Z}}_N\| = 1$. This completes the proof. ■

B. Proof

To establish Proposition 3, we need to show that, given any $\mathbf{X} = \mathbf{X}_0 + \mathrm{diag}[0, X_{11}] \in \mathfrak{B}_{\mathbf{F}1}$ with $\mathbf{X}_0$ being a compact operator, $\mathbf{P}$ with P and Π chosen appropriately can approximate $\mathbf{I}(\mathbf{L}_N)\mathbf{X}\mathbf{I}(\mathbf{L}_N)^{-1} \in \mathcal{L}(\mathcal{M}'_N)$ with arbitrary accuracy for large enough N. It is easy to see that we have to take $\Pi = X_{11}$. Then, what remains is to show that $\mathbf{P}_0 = \mathbf{I}(\overline{\mathbf{J}_{H0}})P\,\mathbf{I}(\overline{\mathbf{J}_{H0}})^*$ can approximate $\mathbf{I}(\mathbf{L}_N)\mathbf{X}_0\mathbf{I}(\mathbf{L}_N)^{-1}$ with arbitrary accuracy. We claim that

$$P = \mathbf{I}(\overline{\mathbf{J}_{H0}})^*\,\mathbf{I}(\mathbf{L}_N)\mathbf{X}_0\,\mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{I}(\overline{\mathbf{J}_{H0}}) \tag{33}$$

indeed does the task. That is, we claim that

$$\begin{aligned} &\|\mathbf{I}(\mathbf{Z}_N)\mathbf{I}(\mathbf{L}_N)\mathbf{X}_0\mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{I}(\mathbf{Z}_N) \\ &\quad - \mathbf{I}(\mathbf{L}_N)\mathbf{X}_0\mathbf{I}(\mathbf{L}_N)^{-1}\| \to 0\ (N \to \infty) \end{aligned} \tag{34}$$

Note that the operators involved in (34) are on $\mathcal{M}'_N = \mathbb{R}^n \oplus (\mathcal{K}'_\mu)^N$, which is a mixture of an Euclidean space and functional spaces. To apply the preceding arguments, however, it is easier for us to work only on functional spaces. For compatibility with the functional part $(\mathcal{K}'_\mu)^N$ with N components, we consider embedding $\mathbb{R}^n$ into the space of N-tuple constant functions $(\mathcal{K}_n'^{(0)})^N$ as $v\ (\in \mathbb{R}^n) \mapsto [(v\mathbf{1})^T, \cdots, (v\mathbf{1})^T]^T$. We also consider the hold operator $\mathbf{J}_{H0}(h)$ (i.e., with the underlying interval given by h, rather than h'). Then, first by considering the relationship between $\mathbb{R}^n$ and $(\mathcal{K}_n'^{(0)})^N$, it is not hard to see that for any operator $\mathbf{G}$ on $\mathcal{M}'_N = \mathbb{R}^n \oplus (\mathcal{K}'_\mu)^N$,

$$\|\mathbf{G}\| \le \|\mathrm{diag}[\mathbf{L}_N\mathbf{J}_{H0}(h), I]\mathbf{G}\,\mathrm{diag}[\mathbf{J}_{H0}(h)^*\mathbf{L}_N^{-1}, I]\| \tag{35}$$

where the operator on the right hand side is on $(\mathcal{K}_n'^{(0)})^N \oplus (\mathcal{K}'_\mu)^N$. Hence, by taking $\mathbf{G}$ to be the operator in (34) and considering the right hand side of (35), it suffices to prove

$$\begin{aligned} &\|\mathbf{I}(\mathbf{Z}_N)(I_2 \otimes \mathbf{L}_N)\widetilde{\mathbf{X}}_0(I_2 \otimes \mathbf{L}_N^{-1})\mathbf{I}(\mathbf{Z}_N) \\ &\quad - (I_2 \otimes \mathbf{L}_N)\widetilde{\mathbf{X}}_0(I_2 \otimes \mathbf{L}_N^{-1})\| \to 0\ (N \to \infty) \end{aligned} \tag{36}$$

where $\widetilde{\mathbf{X}}_0 = \mathrm{diag}[\mathbf{J}_{H0}(h), I]\mathbf{X}_0\,\mathrm{diag}[\mathbf{J}_{H0}(h)^*, I]$, and $I_2 \otimes \mathbf{L}_N$ is a (slightly abused) shorthand notation for $\mathrm{diag}[\mathbf{L}_N, \mathbf{L}_N]$ (with the first $\mathbf{L}_N$ acting on $\mathcal{K}_n^{(0)}$ while the second on $\mathcal{K}_\mu$). Second, since $\mathbf{Z}_N = \mathbf{Z}_N^* = I$ on $(\mathcal{K}_n'^{(0)})^N$, we multiply the identity $\mathrm{diag}[\mathbf{Z}_N, I]$ and its adjoint from the left and right of the first operator in (36), respectively. Then, proving (36) is equivalent to showing the claim that

$$\begin{aligned} &\|(I_2 \otimes \mathbf{Z}_N)(I_2 \otimes \mathbf{L}_N)\widetilde{\mathbf{X}}_0(I_2 \otimes \mathbf{L}_N^{-1})(I_2 \otimes \mathbf{Z}_N) \\ &\quad - (I_2 \otimes \mathbf{L}_N)\widetilde{\mathbf{X}}_0(I_2 \otimes \mathbf{L}_N^{-1})\| \to 0\ (N \to \infty) \end{aligned} \tag{37}$$

where the norm is on $(\mathcal{K}_n'^{(0)})^N \oplus (\mathcal{K}'_\mu)^N$. Because this norm is no larger than that on $(\mathcal{K}'_n)^N \oplus (\mathcal{K}'_\mu)^N$, this claim is indeed a direct consequence[2] from Proposition 5. This completes the proof of the first assertion, while the second assertion is a direct consequence of the the following facts: (i) the spectrum of a self-adjoint operator is continuous with respect to self-adjoint perturbations [35, p. 243]; (ii) an operator is strictly positive-definite if and only if its spectrum is contained in a closed interval on the positive real axis [25].

[2]To be more rigorous, it should be noted that $I_2 \otimes \mathbf{L}_N$ (on $\mathcal{K}_n \oplus \mathcal{K}_\mu$) is different from $\mathbf{L}_N$ (on $\mathcal{K}_m$ with $m = n + \mu$) before we can apply Proposition 5. In fact, there exists a time-invariant permutation (and hence orthogonal) matrix E such that $I_2 \otimes \mathbf{L}_N = E\mathbf{L}_N$. On the other hand, since the action of $\mathbf{Z}_N$ on the ith component of $f' \in (\mathcal{K}'_\mu)^N$ is exactly the same as that on the jth component (i.e., $\mathbf{Z}_N$ is simply a collection of μN identical "scalar operators"), it follows that $I_2 \otimes \mathbf{Z}_N$ on $(\mathcal{K}'_n)^N \oplus (\mathcal{K}'_\mu)^N$ is equivalent to $\mathbf{Z}_N$ (with μ replaced by $m = n + \mu$ and $(\mathcal{K}'_n)^N \oplus (\mathcal{K}'_\mu)^N$ identified with $(\mathcal{K}'_m)^N$). Hence, by the linearity of $\mathbf{Z}_N$ and constancy of E, we see that $(I_2 \otimes \mathbf{Z}_N)E = E(I_2 \otimes \mathbf{Z}_N) = E\mathbf{Z}_N$, where the last $\mathbf{Z}_N$ is on $(\mathcal{K}'_m)^N$. These observations together with the orthogonality of E validate the application of Proposition 5.