A Spinor Approach to Port-Hamiltonian Systems

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***Abstract*— The key concept in the implicit definition of a port-Hamiltonian system is the geometric notion of a Dirac structure. A Dirac structure is a maximal isotropic subspace of the direct sum of the vector space of flows and its dual space of efforts. There exist several ways to represent a Dirac structure. One approach that appears to be unknown to the systems and control community is the pure spinor approach based on the work of É. Cartan. It is the purpose of this paper to explain the spinor formalism and to show how a port-Hamiltonian system is reformulated in the language of spinors.**

I. INTRODUCTION

An autonomous Hamiltonian system on a manifold $\mathcal{M}$ is determined by a function H on $\mathcal{M}$ together with a skew-symmetric mapping $J = J_x$ from $T_x^*\mathcal{M}$ to $T_x\mathcal{M}$ for each $x \in \mathcal{M}$. A curve $\gamma = \gamma(t)$ on $\mathcal{M}$ is a solution of the Hamiltonian system if the differential $dH \in T^*_{\gamma(t)}\mathcal{M}$ and the tangent vector $\dot{\gamma} \in T_{\gamma(t)}\mathcal{M}$ are related by the equation $\dot{\gamma} = J(dH)$. To simplify the notations, let V and V^* denote the coordinate spaces of $T_{\gamma(t)}\mathcal{M}$ and $T^*_{\gamma(t)}\mathcal{M}$ with respect to the standard bases of these spaces. Then the equation $\dot{\gamma} = J(dH)$ boils down to the more familiar equation $\dot{x} = J(\nabla H)^t$, where J is a skew-symmetric matrix and the pair $(\dot{x}, \nabla H) \in V \oplus V^*$ is the pair of coordinate vectors of $\dot{\gamma}$ and dH, respectively. Note that the transpose $(\nabla H)^t$ of the gradient is used here, because $\nabla H \in V^*$ is represented as a row vector.

Now the *explicit* system $\dot{x} = J(\nabla H)^t$ can be generalized to an *implicit* Hamiltonian system which can be written in the form $F\dot{x} = E(\nabla H)^t$ for some $(n \times n)$-matrices E and F satisfying the conditions (1) $EF^t + FE^t = 0$ and (2) rank $[F|E] = n$. The study of this class of systems is useful and relevant in the sense that both the so-called port-Hamiltonian systems (explicit Hamiltonian systems with power-conserving input and output connections) and the Hamiltonian systems with kinematic constraints can be written in the form of an implicit Hamiltonian system; see for example [9] and [4].

At some stage it has been recognized that for each implicit Hamiltonian system $F\dot{x} = E(\nabla H)^t$ the pairs $(\dot{x}(t), \nabla H(x(t)))$ are contained in a so-called *Dirac structure* on $V \oplus V^*$, the study of which has been initiated in [3] . There are several ways to represent a Dirac structure ($F\dot{x} = E(\nabla H)^t$ being one of them); see for example [4], [1]. In this paper we focus on another representation, namely the *spinor representation,* which makes use of a Clifford algebra structure which is naturally associated to the Dirac structure; see [5] for a modern account. The spinor concept has been discovered by É. Cartan in 1913 and the combined theory of spinors and Clifford algebras has played an important role in mathematical physics since Pauli and Dirac presented their equations of the electron in quantum mechanics in 1927 and 1928, respectively. It is the purpose of this paper to explain the spinor formalism and to show how a port-Hamiltonian system is reformulated in the language of spinors.

The condition $(\dot{x}, \nabla H) \in \mathcal{D}$ for some Dirac structure $\mathcal{D}$ translates in the language of exterior algebra into the algebraic identity

$$i_{\dot{x}}s + \nabla H \wedge s = 0, \tag{1}$$

where s denotes the so-called *pure spinor* associated to $\mathcal{D}$, $i_{\dot{x}}s$ the *interior* product of s by $\dot{x}$ and $\nabla H \wedge s$ the *exterior* product of s by ∇H. The beauty of this identity is that it contains all the relevant information of the (implicit) Hamiltonian system in a single algebraic identity, namely the flow $\dot{x}$, the effort ∇H and the spinor s encoding the interconnection structure of the system. If we write the exterior product $\nabla H \wedge s$ as $e_{\nabla H}s$ then the identity (1) can be rewritten as

$$(i_{\dot{x}} + e_{\nabla H})s = 0. \tag{2}$$

Note that (2) clearly exhibits the crucial feature of the spinor approach, namely that $\dot{x}$ and ∇H are represented as *operators* which act on the space of spinors.

The outline of this paper is as follows. In section 2 we give a brief summary of Dirac structures, in section 3 a very short introduction to spinors. In section 4 we describe the spinor representation of implicit autonomous Hamiltonian systems and finally in section 5 we apply this spinor approach to the port-Hamiltonian systems.

II. DIRAC STRUCTURES

Given a real n-dimensional vector space V, let the $2n$-dimensional space W be defined by $W = V \oplus V^*$. We shall denote the elements of W as (v, α), with $v \in V$ and $\alpha \in V^*$. Now, W is naturally equipped with the symmetric bilinear form $\langle\langle ., . \rangle\rangle$ defined by

$$\langle\langle (v, \alpha), (u, \beta) \rangle\rangle = \frac{1}{2}[\alpha(u) + \beta(v)]. \tag{3}$$

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Let $\{e_1, \ldots e_n\}$ be a basis of V and $\{e^1, \ldots, e^n\}$ the corresponding dual basis of V^*. Then the elements

$$f_i = (e_i, e^i),\ g_i = (e_i, -e^i),\ i = 1, \ldots, n,$$

are easily seen to form a basis of W, while satisfying the relations

$$\begin{cases} \langle\langle f_i, f_i \rangle\rangle = 1,\ i = 1, \ldots, n \\ \langle\langle g_i, g_i \rangle\rangle = -1,\ i = 1, \ldots, n \\ \langle\langle f_i, f_j \rangle\rangle = \langle\langle f_i, g_j \rangle\rangle = \langle\langle g_i, g_j \rangle\rangle = 0,\ i \neq j \end{cases}$$

Hence the bilinear form $\langle\langle ., . \rangle\rangle$ on W is non-degenerate with signature (n, n). A subspace U of W is called *isotropic* if $\langle\langle u_1, u_2 \rangle\rangle = 0$ for all $u_1, u_2 \in U$. The maximal dimension of an isotropic subspace of W is equal to n.

Definition 1: A Dirac structure on W is an isotropic subspace of W of maximal dimension n.

It is readily verified that the following three subspaces are examples of a Dirac structure on W :

1) $\{(v, 0) \in W \mid v \in V\} \cong V$.
2) $\{(0, \alpha) \in W \mid \alpha \in V^*\} \cong V^*$.
3) $\{(J(\alpha), \alpha) \in W \mid \alpha \in V^*\}$ for a skew-symmetric mapping $J : V^* \to V$; this is the graph of J.

III. Spinor spaces

Let W be a real m-dimensional vector space equipped with a non-degenerate quadratic form Q. There exists a well-established theory of the so-called *Clifford algebra* $Cl(W, Q)$ associated to (W, Q), which is closely connected to the existence of a *spinor space* for (W, Q). The idea of the latter is to represent W as a linear space of operators that act on a so-called spinor space S subject to a certain condition which involves the quadratic form Q. The set of linear operators on S is usually denoted by $End(S)$ (endomorphisms of S). We give the definition of a spinor space S associated to (W, Q) without going into the details of the Clifford algebra $Cl(W, Q)$. For more details and background the reader is referred to a textbook on this subject, such as [8] or [6].

Definition 2: A linear space S is called a spinor space associated to (W, Q) if there exists a linear mapping $\rho : W \to End(S)$ subject to the condition

$$\rho^2(w) = Q(w)1 \tag{4}$$

for each $w \in W$, i.e., such that the square of the operator $\rho(w)$ is equal to $Q(w)$ times the identity operator. The elements of a spinor space S are called spinors.

Remark 3: The theory of Clifford algebras guarantees the existence of a spinor space S for each quadratic vector space (W, Q). The Clifford algebra $Cl(W, Q)$ is in fact the algebra which is generated by the operators $\rho(w)$, $w \in W$.

Remark 4: Often equation (4) is replaced by the less formal one

$$w^2 = Q(w), \tag{5}$$

where $\rho(w)$ is simply identified with w and $Q(w)$ is understood to mean $Q(w)1$.

Remark 5: The polarization of equation (5) is given by

$$\frac{1}{2}(w_1 w_2 + w_2 w_1) = B(w_1, w_2),$$

for all $w_1, w_2 \in W$, where B is the symmetric bilinear form associated to the quadratic form Q.

Example 6: Let $W = \mathbb{R}^3$ with the quadratic form Q given by

$$Q(w) = w_1^2 + w_2^2 + w_3^2.$$

A spinor space for (W, Q) can be constructed by means of the well-known Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\rho : W \to End(\mathbb{C}^2)$ be defined by

$$\rho(w) = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3 = \begin{pmatrix} w_3 & w_1 - iw_2 \\ w_1 + iw_2 & -w_3 \end{pmatrix}.$$

One easily verifies that for each $w \in W$ we have

$$\rho^2(w) = (w_1^2 + w_2^2 + w_3^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Q(w)I,$$

hence $\mathbb{C}^2$ is a spinor space for (W, Q).

Now let us focus on a vector space W with the particular structure $W = V \oplus V^*$. The quadratic form Q on W which is associated to the bilinear form $\langle\langle ., . \rangle\rangle$ defined in (3) is given by

$$Q((v, \alpha)) = \alpha(v).$$

A well-known model for a spinor space S of this pair (W, Q) is given by $S \cong \wedge V^*$, the *exterior algebra* of the dual of V. For details about exterior algebras the reader is referred to any textbook covering the subject, such as [7]. In the following definition the spinor representation of $V \oplus V^*$ in the algebra $End(\wedge V^*)$ is simply denoted by (v, α), rather than by $\rho((v, \alpha))$; see Remark 4.

Definition 7: The (linear) action of $W = V \oplus V^*$ on the spinor space $\wedge V^*$ is defined by

$$(v, \alpha)s = i_v s + \alpha \wedge s, \tag{6}$$

for each $s \in \wedge V^*$, i.e., by the sum of the *interior product* $i_v s$ and the *exterior product* $\alpha \wedge s$ of the form s by v and α, respectively.

It follows from the theory of interior and exterior products that this embedding of W in the algebra $End(\wedge V^*)$ does indeed satisfy the required equation (5), i.e.,

$$(v, \alpha)^2 = \alpha(v)$$

for all $(v, \alpha) \in W$.

Given a non-zero spinor $s \in \wedge V^*$, let L_s be the linear subspace of W defined by

$$L_s = \{w \in W \mid ws = 0\}, \tag{7}$$

i.e., the set of all annihilators of s (obviously being a linear subspace). Since for each $w_1, w_2 \in L_s$

$$2\,\langle\langle w_1, w_2 \rangle\rangle\, s = (w_1 w_2 + w_2 w_1)s = 0,$$

the subspace L_s is isotropic.

Definition 8: $s \in \wedge V^*$ is called a *pure* spinor if L_s is a maximal isotropic subspace of $W = V \oplus V^*$, i.e., if $\dim L_s = \frac{1}{2} \dim W = n$.

Hence for each pure spinor s the linear subspace L_s is actually a Dirac structure, see Definition 1. It can be shown that conversely each maximal isotropic subspace of W can be represented in this way. It follows from (7) that $L_s = L_{\lambda s}$ for any non-zero $\lambda \in \mathbb{R}$, so a pure spinor belonging to a particular Dirac structure is not unique. Not every spinor is a pure spinor. A necessary condition for a spinor to be pure is stated in the following proposition; see [2].

Proposition 9: Any pure spinor $s \in \wedge V^*$ is necessarily contained in either the even part $\wedge^{\text{even}} V^*$ or the odd part $\wedge^{\text{odd}} V^*$ of the spinor space $\wedge V^*$.

Remark 10: Apart from the few cases $\dim V \leq 3$ this is not a sufficient condition for a spinor to be pure.

We conclude this section with the following three examples of (non-zero) pure spinors and their corresponding Dirac structures:

1) $s = 1 \in \wedge^0 V^*$ (0-form). Equation (6) becomes $(v, \alpha)1 = i_v 1 + \alpha \wedge 1 = \alpha$, so L_1 is the subspace of W of all (v, α) with $\alpha = 0$, i.e., $L_1 = V$.
2) $s = \theta \in \wedge^1 V^*$ (1-form). Equation (6) becomes $(v, \alpha)\theta = i_v \theta + \alpha \wedge \theta = \theta(v) + \alpha \wedge \theta$, so $(v, \alpha) \in L_\theta$ if and only if $\theta(v) = 0$ and $\alpha \wedge \theta = 0$, i.e.,
$$L_\theta = \ker \theta \oplus im\theta.$$
3) Given a 2-form $\omega \in \wedge^2 V^*$ let s be defined by
$$s = e^\omega = 1 + \omega + \frac{1}{2}\omega \wedge \omega + \cdots .$$
It follows that s is pure and $L_s = \{(v, -i_v \omega) \mid v \in V\}$, which is the graph of a skew-symmetric mapping from V to V^*.

IV. Implicit Hamiltonian systems

Let us consider an autonomous Hamiltonian system with state space X, Hamiltonian function H on X and a skew-symmetric matrix J. The dynamics of such a system is given by the equation
$$\dot{x} = J(\nabla H)^t, \tag{8}$$
where $\dot{x}$ is in the space of flows V and ∇H in the space of efforts V^*. Now let us focus on $(\dot{x}, \nabla H) \in V \oplus V^*$, rather than on the flow $\dot{x} \in V$ alone.

The pair $(\dot{x}, \nabla H) = (J(\nabla H), \nabla H)$ is contained in the graph $(J(\alpha), \alpha)$ of the mapping J, which is a Dirac structure on $V \oplus V^*$, recall Definition 1, because
$$\langle\langle (J(\alpha), \alpha), (J(\beta), \beta) \rangle\rangle = \frac{1}{2}[\alpha(J(\beta)) + \beta(J(\alpha))] = 0$$
for all $\alpha, \beta \in V^*$ and $\dim\{(J(\alpha), \alpha) \mid \alpha \in V^*\} = n$ $(= \dim V)$. More generally, we define

Definition 11: An implicit autonomous Hamiltonian system associated to a function H on X and a Dirac structure $\mathcal{D}$ on $V \oplus V^*$ is defined by the requirement
$$(\dot{x}(t), \nabla H(x(t)) \in \mathcal{D},$$
for all $t \geq t_0$.

Remark 12: Note that in this definition the Dirac structure $\mathcal{D}$ is constant, as are the spaces V and V^*. In many applications, however, the state space is a manifold $\mathcal{M}$, in which case the constant Dirac structure $\mathcal{D}$ on $V \oplus V^*$ in the above definition needs te be replaced by an x-dependent Dirac structure $\mathcal{D} = \mathcal{D}_x$ on the x-dependent space $T_x\mathcal{M} \oplus T_x^*\mathcal{M}$. Furthermore, the requirement $(\dot{x}(t), \nabla H(x(t)) \in \mathcal{D}$ is replaced in this more abstract setting by the requirement $(\dot{\gamma}(t), dH(\gamma(t))) \in \mathcal{D}_{\gamma(t)}$, where $\gamma = \gamma(t)$ describes a solution curve of the Hamiltonian system on the manifold $\mathcal{M}$.

Returning to the less abstract setting, we now wish to use the spinor representation for the description of an implicit Hamiltonian system. For each Dirac structure $\mathcal{D}$ there exists a pure spinor $s \in \wedge V^*$ such that $\mathcal{D} = L_s$, that is,
$$(\dot{x}, \nabla H) \in \mathcal{D} \Leftrightarrow (\dot{x}, \nabla H)s = 0.$$
According to definition (6) the condition $(\dot{x}, \nabla H)s = 0$ is equivalent to
$$i_{\dot{x}} s + \nabla H \wedge s = 0. \tag{9}$$

Note that this identity contains all the relevant information in a single identity, namely the flow $\dot{x}$, the effort ∇H and the interconnection structure encoded by the spinor s.

Remark 13: In the spirit of Remark 12, a more abstract version of identity (9) is given by
$$i_{\dot{\gamma}} s + dH \wedge s = 0,$$
where $\dot{\gamma}$ is a *vector field*, dH a *covector field* and s a *spinor field* along the solution curve γ on a manifold $\mathcal{M}$.

Let us have a closer look at identity (9) for the 2-dimensional case. Let $\{e_1, e_2\}$ be a basis of V and $\{e^1, e^2\}$ the corresponding dual basis of V^*. Then a basis of the spinor space $\wedge V^*$ is given by $\{1, e^1, e^2, e^1 \wedge e^2\}$, so a general spinor s is given by a linear combination
$$s = s_0 + s_1 e^1 + s_2 e^2 + s_{12} e^1 \wedge e^2.$$
Let us examine the condition $i_v s + \alpha \wedge s = 0$ with
$$v = v_1 e_1 + v_2 e_2 \in V \text{ and } \alpha = \alpha_1 e^1 + \alpha_2 e^2 \in V^*.$$
The interior product $i_v s$ and exterior product $\alpha \wedge s$ are easily found to be
$$\begin{aligned} &i_v s = s_1 v_1 + s_2 v_2 + s_{12}(-v_2 e^1 + v_1 e^2) \\ &\alpha \wedge s = s_0(\alpha_1 e^1 + \alpha_2 e^2) + (\alpha_1 s_2 - \alpha_2 s_1) e^1 \wedge e^2 \end{aligned}$$
and so $i_v s + \alpha \wedge s$ is equal to
$$s_1 v_1 + s_2 v_2 + (s_0 \alpha_1 - s_{12} v_2) e^1 + (s_0 \alpha_2 + s_{12} v_1) e^2 + (\alpha_1 s_2 - \alpha_2 s_1) e^1 \wedge e^2.$$
Setting this equal to zero amounts to setting each of the 4 coefficients of the spinor equal to zero, hence
$$\begin{cases} s_1 v_1 & +s_2 v_2 & & & = 0 \\ & -s_{12} v_2 & +s_0 \alpha_1 & & = 0 \\ s_{12} v_1 & & & +s_0 \alpha_2 & = 0 \\ & & s_2 \alpha_1 & -s_1 \alpha_2 & = 0 \end{cases}$$

Now the spinor s is pure if and only if the solution set of this linear system is 2-dimensional, i.e., if and only if

$$rank \begin{pmatrix} s_1 & s_2 & 0 & 0 \\ 0 & -s_{12} & s_0 & 0 \\ s_{12} & 0 & 0 & s_0 \\ 0 & 0 & s_2 & -s_1 \end{pmatrix} = 2.$$

According to proposition 9 a pure spinor s is contained either in $\wedge^{\text{even}}V^*$or in $\wedge^{\text{odd}}V^*$, i.e., $s = s_0 + s_{12}e^1 \wedge e^2$ or $s = s_1e^1 + s_2e^2$. In this case every nonzero spinor in $\wedge^{\text{even}}V^*$ or $\wedge^{\text{odd}}V^*$ is pure, see remark 10, so we end up with the following two possibilities:

pure spinor	Dirac structure
$s_0 + s_{12}e^1 \wedge e^2$	$\begin{pmatrix} 0 & s_{12} \\ -s_{12} & 0 \end{pmatrix} v = \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \end{pmatrix} \alpha^t$
$s_1e^1 + s_2e^2$	$\begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix} v = \begin{pmatrix} 0 & 0 \\ -s_2 & s_1 \end{pmatrix} \alpha^t$

(10)

Substituting $(v, \alpha) = (\dot{x}, \nabla H)$ in (10) we obtain the following two types of 2-dimensional implicit Hamiltonian systems:

pure spinor	Hamiltonian system
$s_0 + s_{12}e^1 \wedge e^2$	$\begin{pmatrix} 0 & s_{12} \\ -s_{12} & 0 \end{pmatrix} \dot{x} = \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \end{pmatrix} (\nabla H)^t$
$s_1e^1 + s_2e^2$	$\begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix} \dot{x} = \begin{pmatrix} 0 & 0 \\ -s_2 & s_1 \end{pmatrix} (\nabla H)^t$

Note that in the even case with $s_{12} \neq 0$ division by s_{12} yields the familiar form of a 2-dimensional explicit Hamiltonian system $\dot{x} = J(\nabla H)^t$, for some skew-symmetrix matrix J.

In a similar way we can work out the possibilities for the 3-dimensional case. In this case the odd and even spinors are of the following form:

even spinor : $s_0 + s_{12}e^1 \wedge e^2 + s_{13}e^1 \wedge e^3 + s_{23}e^2 \wedge e^3$
odd spinor : $s_1e^1 + s_2e^2 + s_3e^3 + s_{123}e^1 \wedge e^2 \wedge e^3$

This value of n is the largest one for which any nonzero spinor in $\wedge^{\text{even}}V^*$ or in $\wedge^{\text{odd}}V^*$ is pure. Expansion of the identity $i_{\dot{x}}s + \nabla H \wedge s = 0$ along the basis of the spinor space yields the following two situations:

parity spinor	Hamiltonian system
even	$J_0\dot{x} = s_0(\nabla H)^t$
odd	$s_{123}\dot{x} = J_1(\nabla H)^t$

(11)

where J_0 and J_1 are the skew-symmetric matrices depending on the spinor coordinates in the following way:

$$J_0 = \begin{pmatrix} 0 & s_{12} & s_{13} \\ -s_{12} & 0 & s_{23} \\ -s_{13} & -s_{23} & 0 \end{pmatrix}, \; J_1 = \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix}.$$

V. Port-Hamiltonian systems

We mentioned in the introduction that a port-Hamiltonian system can be written in the form of an implicit Hamiltonian system. As a result of that, port-Hamiltonian systems also have a spinor representation. We shall see that the interconnection structure of the system and the influence of the external flow are somehow jointly incorporated in a single spinor equation. Before showing how this works, let us briefly recall the definition of a port-Hamiltonian system. For details and background we refer to the work of A.J. van der Schaft and others. In the following V denotes the coordinate space of the *internal flows*, U the coordinate space of the space of *external flows*, J a matrix representation of a skew-symmetric mapping from V^* to V and g a matrix representation of a mapping from U to V, where we assume rank $g = \dim U \leq \dim V$.

A port-Hamiltonian system with Hamiltonian H on the state space X is a system of the form

$$\begin{cases} \dot{x} = J(\nabla H)^t + gf \\ e^t = g^t(\nabla H)^t \end{cases} \tag{12}$$

with input $f \in U$ and output $e \in U^*$. The gradient ∇H (represented as a row-vector) is an element of the space V^* of *internal efforts* and the output e (also a row vector) is contained in the space U^* of *external efforts*. A crucial property of these systems is that they are *power-preserving*. Using the fact that J is skew-symmetric we find

$$\frac{dH}{dt} = ef,$$

which simply states indeed that the internal power of the system is equal to the externally supplied power.

Now the system (12) can be written as an *implicit* Hamiltonian system by putting together the internal and external flows on the one hand and the internal and external efforts on the other, i.e., by using the spaces $V \oplus U$ and $V^* \oplus U^*$, respectively. One easily verifies that (12) is equivalent to the implicit system

$$\underbrace{\begin{pmatrix} I & -g \\ 0 & 0 \end{pmatrix}}_{F} \begin{pmatrix} \dot{x} \\ f \end{pmatrix} = \underbrace{\begin{pmatrix} J & 0 \\ g^t & I \end{pmatrix}}_{E} \begin{pmatrix} (\nabla H)^t \\ -e^t \end{pmatrix}, \tag{13}$$

with rank $[F|E] = \dim V$ and $EF^t + FE^t = 0$. This provides a kernel representation of a Dirac structure on the total space of flows and efforts, which basically brings us back to the mathematics of section 4 (simply replace V by $V \oplus U$). An implicit definition of a port-Hamiltonian system in terms of an underlying Dirac structure is the following modification of Definition 11:

Definition 14: An implicit port-Hamiltonian system with external flow $f \in U$ and external effort $e \in U^*$ associated to a function H on X and a Dirac structure $\mathcal{D}$ on $V \oplus U \oplus V^* \oplus U^*$ is defined by the requirement

$$((\dot{x}(t), f), (\nabla H(x(t)), -e)) \in \mathcal{D},$$

for all $t \geq t_0$.

With this interpretation of a port-Hamiltonian system the spinor representation is obtained in the same way as in section 4. Given a Dirac structure $\mathcal{D}$ on $V \oplus U \oplus V^* \oplus U^*$ there exists a corresponding pure spinor s in $\wedge(V^* \oplus U^*)$ in the sense that $((\dot{x}, f), (\nabla H, -e)) \in \mathcal{D}$ if and only if

$$i_{(\dot{x},f)}s + (\nabla H, -e) \wedge s = 0. \tag{14}$$

Note that (14) simply extends identity (9) in the sense that external flows and efforts are being added to the system. Both J and g from the system description (12) are encoded in the single spinor s featuring in (14).

Let's see how this works for the case

$$(\dim V, \dim U) = (2, 1),$$

i.e. for a 2-dimensional system with one input variable. Suppose $\{e_1, e_2\}$ is a basis of V and $\{e_3\}$ a basis of U, with dual bases $\{e^1, e^2\}$ and $\{e^3\}$, respectively. A general even spinor in $\wedge(V^* \oplus U^*)$ is given by

$$s = s_0 + s_{12} e^1 \wedge e^2 + s_{13} e^1 \wedge e^3 + s_{23} e^2 \wedge e^3.$$

Using the even case result in (11), while replacing $\dot{x}_3$ by f and $\frac{\partial H}{\partial x_3}$ by $-e$, we obtain the following spinor representation of the port-Hamiltonian system:

$$\begin{pmatrix} 0 & s_{12} & s_{13} \\ -s_{12} & 0 & s_{23} \\ -s_{13} & -s_{23} & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ f \end{pmatrix} = s_0 \begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ -e \end{pmatrix}. \quad (15)$$

Note that (15) can be written in the form

$$\begin{cases} \dot{x} = s_0 S^{-1} (\nabla H)^t - S^{-1} \begin{pmatrix} s_{13} \\ s_{23} \end{pmatrix} f \\ e^t = \begin{pmatrix} s_{13} & s_{23} \end{pmatrix} S^{-1} (\nabla H)^t \end{cases}$$

where $S = \begin{pmatrix} 0 & s_{12} \\ -s_{12} & 0 \end{pmatrix}$, which is the standard form (12) of a port-Hamiltonian system with

$$J = s_0 S^{-1} \text{ and } g = -S^{-1} \begin{pmatrix} s_{13} \\ s_{23} \end{pmatrix}.$$

VI. CONCLUSION

We have shown that a port-Hamiltonian system (12), or more generally any implicit port-Hamiltonian system as defined in Definition 14, can be written in the form of a single algebraic identity (14). In this identity the entire interconnection structure of the system is encoded by the spinor s while the total flow $(\dot{x}, f)$ and effort $(\nabla H, -e)$ of the system are represented as operators acting on s, the first one via the interior product and the second via the exterior product. This spinor approach offers a new way of looking at port-Hamiltonian systems which might motivate or enhance further research in various directions.

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