Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

Variational symmetry of discrete-time Hamiltonian systems and learning optimal control

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Abstract— **Variational symmetry is a property of Hamiltonian control systems. It is the basis for learning optimal control. In order to apply a continuous-time control strategy to a real plant, we need to discretize the data with respect to time. This paper investigates the variational symmetry for the discretized Hamiltonian systems. It is proved that the discretization based on the midpoint rule preserves the variational symmetry of the original system. Furthermore, a learning optimal control method based on the midpoint rule is proposed.**

I. Introduction

Variational symmetry is a property of (continuous-time) Hamiltonian control systems. It is the basis for learning optimal control [1], [2], [3] and it provides several learning control methods [4], [5]. Learning optimal control can solve a class of optimal control problems for those systems by iteration of experiments without using the precise knowledge of the plant model, whereas conventional iterative learning control methods can only solve trajectory tracking control problems [6], [7] or need precise information of the plant model [8].

Although variational symmetry is a property for continuous-time Hamiltonian systems, we need to use discrete-time signals (converted by A/D converters) in executing learning control since continuous-time signals cannot be handled by computers directly. In order to simulate the behavior of continuous-time dynamical systems precisely, several discretization (numerical integration) algorithms have been proposed. See, e.g., [9], [10], [11]. However, in general, the discretized system does not preserve the property of the original continuous-time system. This paper proves that the implicit midpoint rule [9], which is one of symplectic integration methods for Hamiltonian systems, preserves the variational symmetry of the original continuous-time system.

This paper provides the following results. A discrete-time system converted by the implicit midpoint rule from a (continuous-time) controlled Hamiltonian system possesses the variational symmetry. This fact allows us to obtain a novel discrete-time optimal learning control method based on the implicit midpoint rule. Furthermore, numerical simulations will demonstrate the effectiveness of the proposed algorithm compared to the conventional one.

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II. Variational symmetry

This section refers to the preliminary results on variational symmetry [1], [2], [3]. First of all, let us consider a continuous-time Hamiltonian control system.

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x} & = & (J-R)\nabla_x H(x,u,t) \quad x(t^0) = x^0 \\ y & = & -\nabla_u H(x,u,t) \\ x^1 & = & x(t^1) \end{array} \right. \tag{1}$$

Here the signals $u(t), y(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$ denote the input, the output and the state respectively. We call the matrices $J,\ R \in \mathbb{R}^{n\times n}$ satisfying $J = -J^{\mathrm{T}}$ and $R = R^{\mathrm{T}} \geq 0$ the Poisson matrix and the dissipation matrix, respectively. The scalar function $H(x,u,t) \in \mathbb{R}$ is called the Hamiltonian function describing the total physical energy of the system. We consider the behavior of this system on the time interval $\Omega := [t^0, t^1] \subset \mathbb{R}$. Let us denote the map from the initial state $x^0 \in \mathbb{R}^n$ and the input $u \in L_2(\Omega)$ to the terminal state $x^1 \in \mathbb{R}^n$ and the output $y \in L_2(\Omega)$ by $\Sigma : (x^0, u) \mapsto (x^1, y)$ ($\Sigma : \mathbb{R}^n \times L_2^m(\Omega) \to \mathbb{R}^n \times L_2^m(\Omega)$). Furthermore, a simpler notation describing the map $u \mapsto y$ for a fixed initial state x^0 is denoted by $\Sigma^{x^0} : L_2^m(\Omega) \to L_2^m(\Omega)$.

The variational system [12] of Σ can be regarded as a Fréchet derivative of the map Σ. The Fréchet derivative of Σ is a map $\mathrm{d}\Sigma(u)(v)$ which is linear in v satisfying

$$\Sigma(u+v) - \Sigma(u) = \mathrm{d}\Sigma(u)(v) + o(\|v\|).$$

The variational system $\mathrm{d}\Sigma$ of the Hamiltonian control system Σ has the following property called *variational symmetry*.

Lemma 1: [1], [3] The derivative $\mathrm{d}\Sigma(x^0,u)(\cdot)$ of $\Sigma(x^0,u)$ is described by a Hamiltonian system for any (x^0,u). Further, if there exist a matrix $N \in \mathbb{R}^{n\times n}$ satisfying

$$NJ = -JN, \quad NR = RN \tag{2}$$

$$\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix} \nabla^2 H(x,u,t) = \nabla^2 H(x,u,t) \begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix}, \tag{3}$$

then the time reversal version of the state-space realization of $(\mathrm{d}\Sigma(\cdot))^*$ is similar to the state-space realization of $\mathrm{d}\Sigma(\cdot)$.

Their state-space realizations are given as follows.

$$\mathrm{d}\Sigma(x^0,u)\ :\ (x_v^0,u_v)\mapsto(x_v^1,y_v)$$
$$\left\{\begin{array}{rcll} \dot x &=& (J-R)\nabla_x H(x,u,t) & x(t^0)=x^0\\ \dot x_v &=& (J-R)\nabla_{x_v} H_v(x_v,u_v,x,u,t) & x_v(t^0)=x_v^0\\ y_v &=& -\nabla_{u_v} H_v(x_v,u_v,x,u,t) & \\ x_v^1 &=& x_v(t^1) & \end{array}\right. \quad (4)$$

$$(\mathrm{d}\Sigma(x^0,u))^*\ :\ (x_a^1,u_a)\mapsto(x_a^0,y_a)$$
$$\left\{\begin{array}{rcll} \dot x &=& (J-R)\nabla_x H(x,u,t) & x(t^0)=x^0\\ \dot x_v &=& -(J-R)\nabla_{x_v} H_v(x_v,u_a,x,u,t) & \\ y_a &=& -\nabla_{u_a} H_v(x_v,u_a,x,u,t) & \\ x_v(t^1) &=& -(J-R)Nx_a^1 & \\ x_a^0 &=& -N^{-1}(J-R)^{-1}x_v(t^0) & \end{array}\right. \quad (5)$$

Here the Hamiltonian function $H_v(x_v,u_v,x,u,t)$ of the variational system $\mathrm{d}\Sigma$ and the variational adjoint system $(\mathrm{d}\Sigma)^*$ is given by

$$H_v(x_v,u_v,x,u,t)=\frac{1}{2}\begin{pmatrix} x_v\\ u_v\end{pmatrix}^{\mathrm{T}}\nabla^2 H(x,u,t)\begin{pmatrix} x_v\\ u_v\end{pmatrix}. \quad (6)$$

In the lemma, the symbols u_v, y_v and x_v denote the input, the output and the state of the variational system $\mathrm{d}\Sigma$, and u_a, y_a and x_a denote those of the variational adjoint system $(\mathrm{d}\Sigma)^*$.

This lemma implies that the variational system $\mathrm{d}\Sigma$ and the variational adjoint system $(\mathrm{d}\Sigma)^*$ have a pair of state space realizations time-reversal to each other. We call this property *variational symmetry* since it is equivalent to the property that the transfer function matrix of $\mathrm{d}\Sigma$ is symmetric if it is time-invariant.

Now let us denote the time-reversal operator on the time interval Ω by

$$(\mathcal{R}(u))(t):=u(t^1-t),\quad t\in\Omega.$$

Then Lemma 1 implies the following input-output property of the variational system $\mathrm{d}\Sigma$ and the variational adjoint $(\mathrm{d}\Sigma)^*$.

Theorem 1: [2], [3] Consider the Hamiltonian system (1) and suppose that it satisfies the assumptions (2) *and* (3) *in Lemma 1. Let $\phi(t),\psi(t)\in\mathbb{R}^n$, $t\in\Omega$ denote the state trajectory with respect to the inputs $v,w\in L_2^m(\Omega)$ and suppose that they satisfy the following condition.*

$$\mathcal{R}\left(\nabla^2 H(x,u,t)\big|_{\substack{x=\phi\\ u=v}}\right)=\nabla^2 H(x,u,t)\big|_{\substack{x=\psi\\ u=w}} \quad (7)$$

Then the following relationship holds.

$$\mathcal{S}\ (\mathrm{d}\Sigma(\phi(t^0),v))^*=(\mathrm{d}\Sigma(\psi(t^0),w))\ \mathcal{S} \quad (8)$$

Here the map $\mathcal{S}:\mathbb{R}^n\times L_2^m(\Omega)\to\mathbb{R}^n\times L_2^m(\Omega)$ is defined by

$$\mathcal{S}(x^0,u):=(-(J-R)Nx^0,\mathcal{R}(u)).$$

In particular, we have a simpler relationship

$$(\mathrm{d}\Sigma^{\phi(t^0)}(v))^*=\mathcal{R}\ (\mathrm{d}\Sigma^{\psi(t^1)}(w))\ \mathcal{R}. \quad (9)$$

This theorem allows one to relate the variational adjoint system $(\mathrm{d}\Sigma)^*$ to the variational system $\mathrm{d}\Sigma$ which plays a central role in learning optimal control [2], [3].

Furthermore, we would like to show that a class of mechanical systems satisfying the assumptions (2) and (3) in the theorem. Le us consider a *simple* Hamiltonian system (1) with the state $x=(q,p)\in\mathbb{R}^{2m}$, the Hamiltonian function

$$H(q,p,u)=\frac{1}{2}p^{\mathrm{T}}M(q)^{-1}p+P(q)-u^{\mathrm{T}}q, \quad (10)$$

Poisson and dissipation matrices

$$J=\begin{pmatrix} 0 & I\\ -I & 0\end{pmatrix},\quad R=\begin{pmatrix} 0 & 0\\ 0 & D\end{pmatrix}. \quad (11)$$

Here the matrix $M(q)\in\mathbb{R}^{m\times m}$ is called an inertia matrix and D denotes the friction coefficient.

Proposition 1: [1] Suppose that the Hamiltonian system (1) is simple (as defined above). Assume moreover that the inertia matrix M is constant. Then this system satisfy the conditions (2) and (3).

Thus conventional mechanical systems have variational symmetry. If it does not have a constant inertia matrix, a PD feedback (a class of generalized canonical transformations) lets the systems to satisfy the variational symmetry condition approximately. See [1], [2], [3] for the detail.

III. Variational symmetry of discrete-time Hamiltonian systems

Variational symmetry recalled in the previous section can be used for learning control. In executing the control for the real plants, we need to use the sampled discrete-time data since the L_2 signals are impossible to be stored in the computer memory. There is a possibility that the discretization used in the sampling procedure can cause a serious problem to the performance of the control system. This section investigates how the variational symmetry is affected by the sampling discretization of the continuous data.

A. Discretization

There are several discretization methods to obtain the solution of dynamical systems numerically. Consider a continuous-time dynamical system

$$\dot x=f(x). \quad (12)$$

The most famous and standard method is the Euler method. This method is to compute the difference of the states between a small time interval h as

$$x_{k+1}=x_k+hf(x_k)$$

where k denotes the time index satisfying $t=t^0+h\ k$. This method is popular since it is simple and easy to execute. But from the accuracy point of view, there exist more advanced implicit methods which require to solve implicit functions.

Some of the implicit methods are suitable for discretization of mechanical systems. For instance, recently, implicit Lagrangian methods [10], [11] were proposed and investigated.

This method can preserve important properties of mechanical systems such as Noether's preservation law. Concerning the discretization of Hamiltonian dynamics, the symplectic Euler and the implicit midpoint rule [9] are known as powerful implicit discretization methods to preserve the symplectic property of the original system. The discretization error of them is known to be smaller than the Euler method and they preserves the volume of the solution family in the phase space. Here we concentrate on the implicit midpoint rule which yields the following implicit discrete-time system with respect to a sampling period h.

$$x_{k+1} = x_k + hf((x_k + x_{k+1})/2) \tag{13}$$

Let us apply this method to the Hamiltonian system (1). Then we obtain

$$\Sigma^d : \begin{cases} x_{k+1} &= x_k + h(J-R)\nabla_x H(\frac{x_{k+1}+x_k}{2}, u_k, kh+t^0) \\ y_k &= -\nabla_u H(\frac{x_{k+1}+x_k}{2}, u_k, kh+t^0) \\ x(t^0) &= x^0 \\ x^1 &= x(t^1) \end{cases} \tag{14}$$

where $t^1 = t^0 + k^1 h$. We can prove the variational symmetry of this system.

Lemma 2: Consider the discretized Hamiltonian system (14). *Suppose that there exist a matrix $N \in \mathbb{R}^{n\times n}$ satisfying the assumptions* (2) *and* (3) *in Lemma 1. Then the time reversal version of the state-space realization of* $(\mathrm{d}\Sigma^d(\cdot))^*$ *is similar to the state-space realization of* $\mathrm{d}\Sigma^d(\cdot)$. *Their state-space realizations are given as follows.*

$$\mathrm{d}\Sigma^d(x^0,u) \ : \ (x_v^0, u_v) \mapsto (x_v^1, y_v)$$

$$\begin{cases} x_{k+1} &= x_k + h(J-R)\nabla_x H(\frac{x_{k+1}+x_k}{2}, u_k, kh+t^0) \\ x_{v,k+1} &= x_{v,k} + h(J-R) \\ & \quad \times \nabla_{x_v} H_v(\frac{x_{v,k+1}+x_{v,k}}{2}, u_{v,k}, \frac{x_{k+1}+x_k}{2}, u_k, kh+t^0) \\ y_v &= -\nabla_{u_v} H_v(\frac{x_{v,k+1}+x_{v,k}}{2}, u_{v,k}, \frac{x_{k+1}+x_k}{2}, u_k, kh+t^0) \\ x(t^0) &= x^0 \\ x_v(t^0) &= x_v^0 \\ x_v^1 &= x_v(t^1) \end{cases} \tag{15}$$

$$(\mathrm{d}\Sigma^d(x^0,u))^* \ : \ (x_a^1, u_a) \mapsto (x_a^0, y_a)$$

$$\begin{cases} x_{k+1} &= x_k + h(J-R)\nabla_x H(x_k, u_k, kh+t^0) \\ x_{v,k+1} &= x_{v,k} - h(J-R) \\ & \quad \times \nabla_{x_v} H_v(\frac{x_{v,k+1}+x_{v,k}}{2}, u_{a,k}, \frac{x_{k+1}+x_k}{2}, u_k, kh+t^0) \\ y_{a,k} &= -\nabla_{u_a} H_v(\frac{x_{v,k+1}+x_{v,k}}{2}, u_{a,k}, \frac{x_{k+1}+x_k}{2}, u_k, kh+t^0) \\ x(t^0) &= x^0 \\ x_v(t^1) &= -\frac{1}{h}(J-R)Nx_a^1 \\ x_a^0 &= -hN^{-1}(J-R)^{-1}x_v(t^0) \end{cases} \tag{16}$$

Here the Hamiltonian function $H_v(x_v, u_v, x, u, t)$ is defined in (6).

Proof: Lemma can be proved by direct computation with the coordinate transformation

$$x_v = -\frac{1}{h}(J-R)Nx_a$$

while the continuous time case we use

$$x_v = -(J-R)Nx_a$$

in the proof of Lemma 1. ■

Noe that the assumptions (2) and (3) required in Lemma 2 are equivalent to those Lemma 1 showing the the continuous time case. This means that if the original continuous-time Hamiltonian system (1) possesses variational symmetry, then the discretized system (14) has variational symmetry as well. This fact also suggests that Proposition 1 is directly applicable to the discrete-time case, that is, the simple Hamiltonian system satisfying the assumptions in Proposition 1 possesses variational symmetry.

Furthermore, the variational symmetry with respect to the input-output map of the variational system of the discretized Hamiltonian system $\mathrm{d}\Sigma^d$ is described as follows. Here $\Omega^d = [0,1,\ldots,k^1] = [0,1,\ldots,(t^1-t^0)/h]$ denotes the interval on l_2 corresponding to Ω.

Theorem 2: Consider the discretized Hamiltonian system (14) and suppose that it satisfies the assumptions (2) *and* (3). *Let $\phi^d(k), \psi^d(k) \in \mathbb{R}^n$, $k \in \Omega^d$ denote the state trajectory with respect to the inputs $v, w \in l_2^m(\Omega^d)$ and suppose that they satisfy the following condition.*

$$\mathcal{R}^d \left(\nabla^2 H(x,u,t^0+kh)\big|_{\substack{x=\phi^d \\ u=v}} \right) = \nabla^2 H(x,u,t^0+kh)\big|_{\substack{x=\psi^d \\ u=w}} \tag{17}$$

Then the following relationship holds.

$$\mathcal{S}^d \ (\mathrm{d}\Sigma^d(\phi^d(t^0),v))^* = (\mathrm{d}\Sigma^d(\psi^d(t^0),w)) \ \mathcal{S}^d \tag{18}$$

Here the map $\mathcal{R}^d$ is the time-reversal operator on the interval Ω^d and $\mathcal{S}^d : \mathbb{R}^n \times l_2^m(\Omega^d) \to \mathbb{R}^n \times l_2^m(\Omega^d)$ is defined by

$$\mathcal{S}^d(x^0,u) := (-\frac{1}{h}(J-R)Nx^0, \mathcal{R}^d(u)).$$

In particular, we have a simpler relationship

$$(\mathrm{d}\Sigma^{d,\phi^d(t^0)}(v))^* = \mathcal{R}^d \ (\mathrm{d}\Sigma^{d,\psi^d(t^1)}(w)) \ \mathcal{R}^d. \tag{19}$$

Proof: This theorem can be proved in a similar way to the proof of Theorem 1 using Lemma 2. ■

This theorem can be directly used to derive optimal leaning control algorithm for the discretized system (14). This will be explained in the following section.

IV. Optimal Learning Control

Optimal learning control for Hamiltonian control systems is a procedure to solve a class of optimal control problems by iteration of experiments. Let $\Sigma : u \mapsto y$ denote the input-output map of the Hamiltonian system as in (1) and define a cost function (functional) $\tilde{\Gamma}(u,y)$ of the input u and output y such as

$$\tilde{\Gamma}(u,y) = \int_0^T \left(u(t)^{\mathrm{T}} Q(t) u(t) + y(t)^{\mathrm{T}} R(t) y(t) \right) \mathrm{d}t.$$

See [2], [3] for the detail of this method. Then the cost function can be reduced to $\Gamma(u) := \tilde{\Gamma}(u, \Sigma(u))$ by regarding it as a function of u. Let us calculate the gradient $\nabla\Gamma(u)$.

$$\begin{aligned} \langle \nabla\Gamma(u), \mathrm{d}u \rangle &= \langle \nabla_u \tilde{\Gamma}(u,y), \mathrm{d}u \rangle + \langle \nabla_y \tilde{\Gamma}(u,y), \mathrm{d}y \rangle \\ &= \langle \nabla_u \tilde{\Gamma}(u,y), \mathrm{d}u \rangle + \langle \nabla_y \tilde{\Gamma}(u,y), \mathrm{d}\Sigma(u)(\mathrm{u}) \rangle \\ &= \langle \nabla_u \tilde{\Gamma}(u,y), \mathrm{d}u \rangle + \langle (\mathrm{d}\Sigma(u))^* \nabla_y \tilde{\Gamma}(u,y), \mathrm{d}u \rangle \\ &= \langle \nabla_u \tilde{\Gamma}(u,y) + (\mathrm{d}\Sigma(u))^* \nabla_y \tilde{\Gamma}(u,y), \mathrm{d}u \rangle \end{aligned}$$

which reduces to

$$\nabla\Gamma(u) = \nabla_u \tilde{\Gamma}(u,y) + (\mathrm{d}\Sigma(u))^* \nabla_y \tilde{\Gamma}(u,y)$$

Once we obtain the gradient $\nabla\Gamma(u)$, the following update law will derive an optimal solution which locally minimizes the cost function $\Gamma(u)$.

$$\begin{aligned} u_{(i+1)} &= u_{(i)} + K\ \nabla\Gamma(u) \\ &= u_{(i)} + K\left\{\nabla_u\tilde{\Gamma}(u,\Sigma(u)) + (\mathrm{d}\Sigma(u))^*\nabla_y\tilde{\Gamma}(u,\Sigma(u))\right\} \end{aligned}$$

Here K is a positive scalar (or matrix) gain and i denotes the number of experiments.

Usually the computation of the variational adjoint $(\mathrm{d}\Sigma(u))^*$ requires the precise knowledge of the plant Σ. The variational symmetry of the plant system, however, enables one to obtain the data of the gradient $\nabla\Gamma(u)$ without knowing the parameters of the model by estimating the adjoint $(\mathrm{d}\Sigma(u))^*$ via the variational symmetry in Theorem 1. In the discrete-time case, the variational symmetry of the discretized system given in Theorem 2 directly yields the discrete-time version of optimal learning control.

Here we consider a simple cost function for trajectory tracking control

$$\tilde{\Gamma}(y) = \frac{1}{2}\int_0^T (y(t) - y^d(t))^2\mathrm{d}t$$

with a symmetric desired output trajectory y^d. In this case, the gradient $\nabla\Gamma(u)$ becomes

$$\nabla\Gamma(u) = (\mathrm{d}\Sigma(u))^*(y - y^d).$$

Using the variational symmetry (9) in Theorem 1, we can approximate it with

$$\begin{aligned} \nabla\Gamma(u) &= (\mathrm{d}\Sigma(u))^*(y - y^d) \\ &= \mathcal{R}(\mathrm{d}\Sigma(u))\mathcal{R}(y - y^d) \\ &\approx \frac{1}{\epsilon}\mathcal{R}(\Sigma(u + \epsilon\mathcal{R}(y - y^d)) - \Sigma(u)). \end{aligned}$$

The above approximation derives the iteration law

$$\begin{aligned} u_{(2i+1)} &= u_{(2i)} + \epsilon_{(i)}\mathcal{R}^d(y_{(2i)} - y^d) \\ u_{(2i+2)} &= u_{(2i)} - \frac{K_{(i)}}{\epsilon_{(i)}}\mathcal{R}^d(y_{(2i+1)} - y_{(2i)}). \end{aligned} \tag{20}$$

where the initial input is $u_{(0)} = 0$.

Let us consider the conventional case. Simply discretizing this control law, we obtain the following discrete-time version of the iteration law which corresponds to the Euler method.

$$\begin{aligned} u_{(2i+1)}(kh) =& u_{(2i)}(kh) + \epsilon_{(i)}\mathcal{R}^d\left(y_{(2i)}(kh) - y^d(kh)\right) \\ u_{(2i+2)}(kh) =& u_{(2i)}(kh) \\ & - \frac{K_{(i)}}{\epsilon_{(i)}}\mathcal{R}^d\left(y_{(2i+1)}(kh) - y_{(2i)}(kh)\right). \end{aligned} \tag{21}$$

Next let us consider a discretization using the implicit midpoint rule. Note that Equation (14) implies

$$y_n = -\nabla_u H((x_{n+1} + x_n)/2, u_n). \tag{22}$$

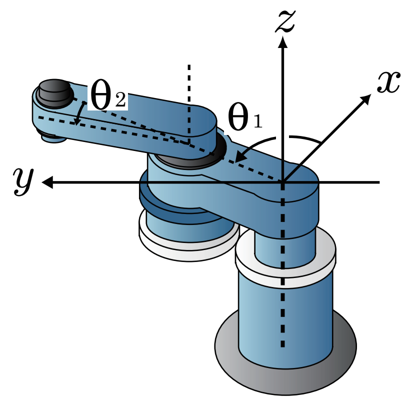

Fig. 1. 2-link manipulator

Suppose that the output y is linear in the state x. Then we obtain the following iteration rule.

$$\begin{aligned} u_{(2i+1)}(kh) =& u_{(2i)}(kh) \\ & + \epsilon_{(i)}\mathcal{R}^d\Bigg(\frac{y_{(2i)}(kh) + y_{(2i)}((k+1)h)}{2} \\ & - \frac{y^d(kh) + y^d((k+1)h)}{2} \\ u_{(2i+2)}(kh) =& u_{(2i)}(kh) \\ & - \frac{K_{(i)}}{\epsilon_{(i)}}\mathcal{R}^d\Bigg(\frac{y_{(2i+1)}(kh) + y_{(2i+1)}((k+1)h)}{2} \\ & - \frac{y_{(2i)}(kh) + y_{(2i)}((k+1)h)}{2}\Bigg). \end{aligned} \tag{23}$$

Furthermore, a simplified (approximated) version of the iteration law based on the implicit midpoint rule is proposed. This iteration law needs discretization with respect to the sampling period $h/2$.

$$\begin{aligned} u_{(2i+1)}(kh) =& u_{(2i)}(kh) \\ & + \epsilon_{(i)}\mathcal{R}^d\Bigg(y_{(2i)}\left(\left(k + \frac{1}{2}\right)h\right) \\ & - y^d\left(\left(k + \frac{1}{2}\right)h\right)\Bigg) \\ u_{(2i+2)}(kh) =& u_{(2i)}(kh) \\ & - \frac{K_{(i)}}{\epsilon_{(i)}}\mathcal{R}^d\Bigg(y_{(2i+1)}\left(\left(k + \frac{1}{2}\right)h\right) \\ & - y_{(2i)}\left(\left(k + \frac{1}{2}\right)h\right)\Bigg). \end{aligned} \tag{24}$$

A numerical simulation will demonstrate how those method work in the following section.

V. Numerical example

Consider a 2-link robot manipulator as depicted in Fig.1. The dynamics of this system can be described by a simple Hamiltonian system as in (1) with (10) and (11). The configuration state is $q = (\theta_1, \theta_2)$ denoting the joint angles of the manipulator. The inertia matrix $M(q)$ and the dissipation D are given by

$$M(q) = \begin{pmatrix} \rho_1 + \rho_2 + 2\rho_3 \cos\theta_2 & \rho_2 + \rho_3 \cos\theta 2 \\ \rho_2 + \rho_3 \cos\theta 2 & \rho_2 \end{pmatrix}$$
$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

with $\rho_1 = 2.550$, $\rho_2 = 0.7200$, $\rho_3 = 0.6500$, $d_1 = 0.2415$ and $d_2 = 0.2457$. We apply a local PD feedback with the P-gain 2400 and the D-gain 2300 to make the system asymptotically stable and to let it to satisfy the assumptions in Proposition 1 approximately. See [1] for the detail of this apparatus.

The control objective is a trajectory tracking control with respect to the joint angles of the manipulator as in the previous section. The desired trajectory $y^d(t)$, $t \in [0, 2]$ is selected as

$$y^d(t) = \begin{pmatrix} \theta_1^d(t) \\ \theta_2^d(t) \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{2}\cos(\pi t) \\ \frac{\pi}{2}\cos(\pi t) + 0.4 \end{pmatrix}. \qquad (25)$$

In order to evaluate the error with respect to the discretization, we define a cost function $\Pi(y)$ as

$$\Pi(y) := \|(\mathrm{d}\Sigma^{x^0}(u))^*(y_{(2i)} - y^d) - \mathcal{R}(\mathrm{d}\Sigma^{\xi^0}(w))\mathcal{R}(y_{(2i)} - y^d)\|_{l_2} \qquad (26)$$

The function $\Pi(y)$ becomes smaller as the behavior of the discretized Hamiltonian system becomes closer to that of the original. The design parameters for learning control are selected as $\epsilon_{(i)} = 1, K_{(i)} = \mathrm{diag}(2400, 2300)$ and $h = 0.2$.

Fig.2 depicts the history of the approximation error Π along the iteration of learning. The horizontal axis shows the number of iteration and the vertical axis shows the approximation error. The solid line (−), the circle (○) and the plus (+) denote the approximation errors with respect to the Euler method, the implicit midpoint rule, and the approximated version of the implicit midpoint rule as given in (21), (23) and (24), respectively. Fig.3 depicts the history of the corresponding cost functions along the iteration of learning.

Fig.2 shows that both the implicit midpoint rule and its approximated version have better accuracy in approximating the variational adjoint $(\mathrm{d}\Sigma)^*$ than the Euler method. Fig.3 shows that the implicit midpoint rule achieves the best convergence speed and that the conventional Euler method gives the worst performance for this control task.

Those results show that the proposed learning control method based on the implicit midpoint rule provides more accurate estimation of the gradient of the cost function and achieves better performance in learning control.

VI. Conclusion

This paper has investigated the variational symmetry of discretized Hamiltonian control systems. We have proved that the implicit midpoint rule preserves the variational symmetry of the original system. A learning control in discrete-time setting is proposed based on this fact. A numerical example demonstrates that the discretization based on the implicit midpoint rule actually has better accuracy than the conventional method and that the proposed method achieves better performance in learning control.

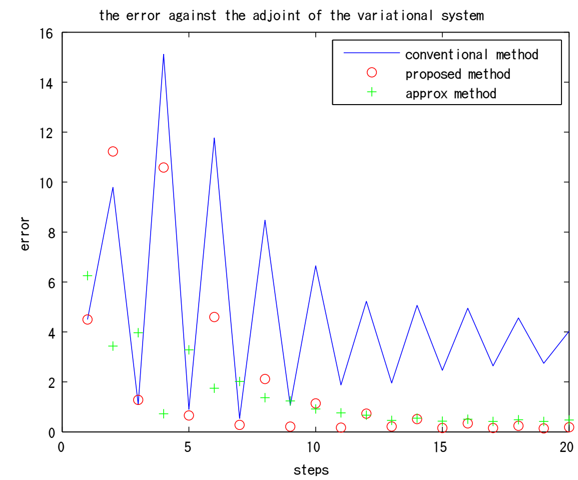

Fig. 2. Approximation error of the adjoint

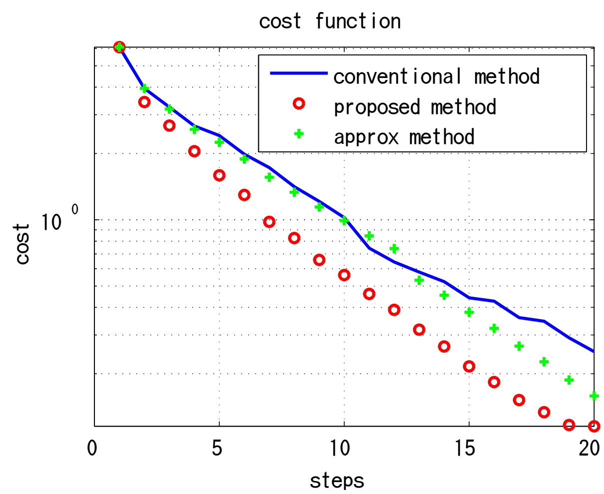

Fig. 3. History of the cost function

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