Stability Analysis of Aperiodic Sampled-Data Control Systems: An Improved Approach Using Matrix Uncertainty

Yasuaki Oishi and Hisaya Fujioka

***Abstract*— Stability analysis is considered on a sampled-data control system with an uncertain/time-varying sampling interval. A stability condition is given in a linear matrix inequality, readily tested with the interior-point method. Conservatism of the condition can be reduced to any degree by dividing the region of the possible sampling intervals. Reduction of the computational complexity as well as generalization to design of a state-feedback gain is considered. This approach is a simplification of the previous approach of the same authors and does not need the real Jordan canonical form, which is difficult to compute for a large-sized matrix.**

***Keywords*—sampled-data control, linear matrix inequalities, matrix uncertainty, conservatism, asymptotic exactness**

I. INTRODUCTION

There is an increasing need to consider a sampled-data control system with an uncertain and/or time-varying sampling interval due to the recent progress of networked and embedded control. Analysis and synthesis of such an aperiodic sampled-data control system were considered in [3], [4], [5], [7], [10], [11], [13], [15]. The present authors proposed in [12] an approach based on a robust linear matrix inequality (robust LMI). This approach is free from numerical difficulty even with a small sampling interval due to the delta-operator technique [9]. It is also asymptotically exact in the sense that conservatism of the approach can be reduced to any degree by the region-dividing technique. On the other hand, the approach requires the real Jordan canonical form of some matrix and may not be suitable to a high-order system.

In this paper, we propose a new approach not requiring the real Jordan canonical form. The idea is to use matrix uncertainty to model the effect of the uncertain sampling interval rather than parametric uncertainty. Although this approach can be more conservative than the previous approach, it is readily applicable to a high-order system. We also introduce the region-dividing technique into the approach and prove its asymptotic exactness.

This paper is organized as follows. In Section II, a sampled-data control system is presented and a problem is formulated. In Section III, stability analysis is considered. Section IV introduces a region-dividing technique for less conservative stability analysis. Section V discusses design of a state-feedback gain. Section VI presents an example and Section VII concludes the paper.

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Notation is standard. The symbols O and I denote the zero matrix and the identity matrix of appropriate size. For a matrix A, the symbol $\overline{\sigma}(A)$ expresses its maximum singular value. For a symmetric matrix A, the symbols $\overline{\lambda}(A)$ and $\underline{\lambda}(A)$ stand for its maximum and minimum eigenvalues, respectively. For symmetric matrices A and B, the inequalities $A \succ B$ and $A \succeq B$ mean $\underline{\lambda}(A-B) > 0$ and $\underline{\lambda}(A-B) \geq 0$, respectively.

II. PROBLEM

We consider a continuous-time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with the n-dimensional state $x(t)$. We are to stabilize it by sampled-data state-feedback control with a constant gain F. Specifically, the state is measured only at discrete time instants $0 = t_0 < t_1 < t_2 < \cdots$ such that $\lim_{k\to\infty} t_k = \infty$ and the input is a piecewise constant signal determined as

$$u(t) = Fx(t_k) \quad (t_k \leq t < t_{k+1})$$

for each $k = 0, 1, 2, \ldots$. We refer to this control system by S henceforth.

The control system S is different from a conventional sampled-data control system in that the sampling interval $t_{k+1} - t_k$ is not necessarily constant but may vary with k. We assume availability of its bounds $\underline{h}$ and $\overline{h}$ such that

$$\underline{h} \leq t_{k+1} - t_k \leq \overline{h} \quad (k = 0, 1, 2, \ldots).$$

Our problem is to verify stability of the control system S. We use the same stability condition as [12], which is based on the delta-operator technique and numerically stable even for a small sampling interval.

Proposition 1: The control system S is exponentially stable if there exists a symmetric matrix Q such that

$$\begin{pmatrix} -\Psi(h)Q - Q\Psi(h)^{\mathrm{T}} & \sqrt{h}\Psi(h)Q \\ \sqrt{h}Q\Psi(h)^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad (\underline{h} \leq h \leq \overline{h}), \quad (1)$$

where

$$\Psi(h) = \frac{1}{h}\int_0^h \mathrm{e}^{At}\,\mathrm{d}t(A + BF).$$

◇

Proof. Note that in S, the states at two adjacent sampling instants are related by

$$x(t_{k+1}) = \Phi(t_{k+1} - t_k)x(t_k) \quad (k = 0, 1, 2, \ldots),$$

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with

$$\Phi(h) = \mathrm{e}^{Ah} + \int_0^h \mathrm{e}^{At}\,\mathrm{d}t\,BF = I + \int_0^h \mathrm{e}^{At}\,\mathrm{d}t(A+BF).$$

This $\Phi(h)$ satisfies $\Psi(h) = (\Phi(h) - I)/h$. Substituting this relationship to (1) and taking the Schur complement, we have

$$Q \succ O, \quad Q - \Phi(h)Q\Phi(h)^{\mathrm{T}} \succ O \quad (\underline{h} \le h \le \overline{h}). \tag{2}$$

This is nothing but the condition for the quadratic stability of S. Hence its exponential stability follows. □

An advantage of our stability condition (1) over the standard quadratic stability condition (2) is that it can be used for a small h without numerical difficulty. Indeed, in the limit of $h \to 0$, the matrix $\Psi(h)$ converges to $A+BF$, which is the system matrix of the continuous-time control system where the continuous-time state-feedback control $u(t) = Fx(t)$ is applied to $\dot{x}(t) = Ax(t) + Bu(t)$. In the same limit, the stability condition (1) assures stability of this continuous-time control system. On the other hand, the standard condition (2) suffers from numerical instability because, for a small h, $\Phi(h)$ is close to identity and $Q - \Phi(h)Q\Phi(h)^{\mathrm{T}}$ is close to zero.

The condition (1) is difficult to test because we need to find Q that satisfies the LMI for infinitely many values of h. In [12], we gave its sufficient condition, which is described in finitely many LMIs and is tractable with the standard interior-point method. In order to reduce conservatism of the condition, we modeled the effect of an uncertain sampling interval as parametric uncertainty.

A drawback of this approach is that it requires the real Jordan canonical form of the matrix A, which is hard to compute for a large-sized A. In order to avoid the real Jordan canonical form, we will consider an approach using matrix uncertainty.

III. Stability Analysis

We give a sufficient condition for (1) using matrix uncertainty. With this aim, we replace $\Psi(h)$ in (1) by a function in h and Ω, where Ω is a matrix uncertainty with a bounded norm. After the replacement, the inequality is convex in h and, thus, has to be tested only at $\underline{h}$ and $\overline{h}$, the lower and the upper bounds on h. The dependence on Ω can be removed by the S-procedure. As a result, we obtain a sufficient condition expressed by two LMIs.

We consider the case of $\underline{h} > 0$ in Section III-A and the case of $\underline{h} = 0$ in Section III-B. The equality $\underline{h} = 0$ means that a positive lower bound is not available for the sampling interval. This situation is tractable because our stability condition (1) does not collapse even in the limit of $h \to 0$.

A. *The case of* $\underline{h} > 0$

In this case, the function for the replacement of $\Psi(h)$ is

$$\Psi_{\widehat{h}}(h, \Omega) = U_{\widehat{h}}(h) + \Omega V_{\widehat{h}}(h) \tag{3}$$

with $\widehat{h}$ being any fixed number such that $\underline{h} \le \widehat{h} \le \overline{h}$. This is basically the Taylor expansion of $h\Psi(h)$ around $h = \widehat{h}$. The function $hU_{\widehat{h}}(h)$ corresponds to the zeroth- and first-order terms and the product $h\Omega V_{\widehat{h}}(h)$ to the higher-order terms. Specifically,

$$U_{\widehat{h}}(h) = \frac{1}{h}\Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t + (h - \widehat{h})\mathrm{e}^{A\widehat{h}}\Big](A+BF), \tag{4}$$

$$V_{\widehat{h}}(h) = \frac{h - \widehat{h}}{h} A\mathrm{e}^{A\widehat{h}}(A+BF) \tag{5}$$

and Ω is a matrix such that $\overline{\sigma}(\Omega) \le \omega_{\underline{h},\overline{h},\widehat{h}}$, where

$$\omega_{\underline{h},\overline{h},\widehat{h}} = \max\Big\{ |\overline{h} - \widehat{h}| \max_{0 \le t \le \overline{h} - \widehat{h}} \exp\Big[\overline{\lambda}\Big(\frac{A + A^{\mathrm{T}}}{2}\Big)t\Big], \\ |\underline{h} - \widehat{h}| \max_{\underline{h} - \widehat{h} \le t \le 0} \exp\Big[\underline{\lambda}\Big(\frac{A + A^{\mathrm{T}}}{2}\Big)t\Big] \Big\}. \tag{6}$$

The subscripts of ω will be dropped when they are clear from the context. Here, we can prove the following.

Lemma 2: For $\Psi_{\widehat{h}}(h, \Omega)$, $U_{\widehat{h}}(h)$, $V_{\widehat{h}}(h)$, and ω defined above, the following properties hold: (i) For any $\underline{h} \le h \le \overline{h}$, there exists a matrix Ω such that $\Psi(h) = \Psi_{\widehat{h}}(h, \Omega)$ and $\overline{\sigma}(\Omega) \le \omega$; (ii) Both $hU_{\widehat{h}}(h)$ and $hV_{\widehat{h}}(h)$ are affine in h; (iii) $V_{\widehat{h}}(h) = O$ at $h = \widehat{h}$. ◇

Proof. We prove the property (i) only. The properties (ii) and (iii) are obvious from the definition.

The property (i) is obvious for $h = \widehat{h}$. We assume $h \ne \widehat{h}$ in the sequel. By the definition of $\Psi(h)$, we have

$$\begin{aligned} \Psi(h) &= \frac{1}{h}\int_0^h \mathrm{e}^{At}\,\mathrm{d}t(A+BF) \\ &= \frac{1}{h}\Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t + \int_0^{h-\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t\,\mathrm{e}^{A\widehat{h}}\Big](A+BF) \\ &= \frac{1}{h}\Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t + (h-\widehat{h})\mathrm{e}^{A\widehat{h}} \\ &\qquad + \int_0^{h-\widehat{h}} \mathrm{e}^{At} - I\,\mathrm{d}t\,\mathrm{e}^{A\widehat{h}}\Big](A+BF) \\ &= \frac{1}{h}\Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t + (h-\widehat{h})\mathrm{e}^{A\widehat{h}} \\ &\qquad + \int_0^{h-\widehat{h}} \int_0^t \mathrm{e}^{Au}\,\mathrm{d}u\,\mathrm{d}t\,A\mathrm{e}^{A\widehat{h}}\Big](A+BF). \end{aligned} \tag{7}$$

The proof is complete if the maximum singular value of

$$\frac{1}{h - \widehat{h}} \int_0^{h-\widehat{h}} \int_0^t \mathrm{e}^{Au}\,\mathrm{d}u\,\mathrm{d}t$$

is less than or equal to ω.

Suppose $h - \widehat{h} > 0$ first. Since $\overline{\sigma}(\mathrm{e}^{Au}) \le \exp[\overline{\lambda}((A + A^{\mathrm{T}})/2)u]$ for $u \ge 0$ [6, p. 577], we have

$$\overline{\sigma}\Big(\frac{1}{h - \widehat{h}} \int_0^{h-\widehat{h}} \int_0^t \mathrm{e}^{Au}\,\mathrm{d}u\,\mathrm{d}t\Big) \\ \le \frac{1}{h - \widehat{h}} \int_0^{h-\widehat{h}} \int_0^t \exp\Big[\overline{\lambda}\Big(\frac{A + A^{\mathrm{T}}}{2}\Big)u\Big]\,\mathrm{d}u\,\mathrm{d}t.$$

Using the mean value theorem twice, we see that the right-hand side is equal to

$$t' \exp\Big[\overline{\lambda}\Big(\frac{A+A^{\mathrm{T}}}{2}\Big)u'\Big]$$

for some $0 \le u' \le t' \le h - \widehat{h} \le \overline{h} - \widehat{h}$. It is less than or equal to ω. The proof in the case of $h - \widehat{h} < 0$ is similar. □

The idea to use the mean value theorem to a parameter-dependent LMI is due to [2]. The function to bound the maximum singular value of the matrix exponential function has appeared in [4], [5]. Similarly to those papers, different bounds can be used instead.

We replace $\Psi(h)$ with $\Psi_{\widehat{h}}(h, \Omega)$ in the condition (1) to obtain the desired sufficient condition, which can be easily tested.

Theorem 3: Suppose $\underline{h} > 0$. Let $\widehat{h}$ be any number such that $\underline{h} \le \widehat{h} \le \overline{h}$. Define $U_{\widehat{h}}(h)$, $V_{\widehat{h}}(h)$, and ω as (4), (5), and (6), respectively. Then, the control system S is exponentially stable if there exist a symmetric matrix Q and two nonnegative numbers $s_{\underline{h}}$ and $s_{\overline{h}}$ such that

$$\begin{pmatrix} -U_{\widehat{h}}(h)Q - QU_{\widehat{h}}(h)^{\mathrm{T}} - s_h I & \sqrt{h}U_{\widehat{h}}(h)Q & -\omega QV_{\widehat{h}}(h)^{\mathrm{T}} \\ \sqrt{h}QU_{\widehat{h}}(h)^{\mathrm{T}} & Q & \omega\sqrt{h}QV_{\widehat{h}}(h)^{\mathrm{T}} \\ -\omega V_{\widehat{h}}(h)Q & \omega\sqrt{h}V_{\widehat{h}}(h)Q & s_h I \end{pmatrix} \succ O \quad (h = \underline{h}, \overline{h}). \tag{8}$$

◇

Proof. As an application of the S-procedure, it is shown in general [1, p. 24] that, for real matrices M_1, M_2, and M_3 of appropriate size, the inequality $M_1 + M_2\Omega M_3 + M_3^{\mathrm{T}}\Omega^{\mathrm{T}}M_2^{\mathrm{T}} \succ O$ holds for any matrix Ω such that $\overline{\sigma}(\Omega) \le \omega$ if and only if there exists a nonnegative number s such that

$$\begin{pmatrix} M_1 - sM_2M_2^{\mathrm{T}} & \omega M_3^{\mathrm{T}} \\ \omega M_3 & sI \end{pmatrix} \succ O.$$

We use this result with

$$M_1 = \begin{pmatrix} -U_{\widehat{h}}(h)Q - QU_{\widehat{h}}(h)^{\mathrm{T}} & \sqrt{h}U_{\widehat{h}}(h)Q \\ \sqrt{h}QU_{\widehat{h}}(h)^{\mathrm{T}} & Q \end{pmatrix},$$

$$M_2 = \begin{pmatrix} I \\ O \end{pmatrix},$$

$$M_3 = \begin{pmatrix} -V_{\widehat{h}}(h)Q & \sqrt{h}V_{\widehat{h}}(h)Q \end{pmatrix}.$$

Then, the assumption of the theorem implies

$$M_1 + M_2\Omega M_3 + M_3^{\mathrm{T}}\Omega^{\mathrm{T}}M_2^{\mathrm{T}} = \begin{pmatrix} -\Psi_{\widehat{h}}(h,\Omega)Q - Q\Psi_{\widehat{h}}(h,\Omega)^{\mathrm{T}} & \sqrt{h}\Psi_{\widehat{h}}(h,\Omega)Q \\ \sqrt{h}Q\Psi_{\widehat{h}}(h,\Omega)^{\mathrm{T}} & Q \end{pmatrix} \succ O \tag{9}$$

for $h = \underline{h}, \overline{h}$ and for any matrix Ω with $\overline{\sigma}(\Omega) \le \omega$. Here, the inequality (9) actually holds for all $\underline{h} \le h \le \overline{h}$. To see this, multiply $\sqrt{h}$ to the first row and and to the first column of the matrix in (9). Then we have an equivalent inequality

$$\begin{pmatrix} -h\Psi_{\widehat{h}}(h,\Omega)Q - hQ\Psi_{\widehat{h}}(h,\Omega)^{\mathrm{T}} & h\Psi_{\widehat{h}}(h,\Omega)Q \\ hQ\Psi_{\widehat{h}}(h,\Omega)^{\mathrm{T}} & Q \end{pmatrix} \succ O.$$

This inequality is convex in h because $h\Psi_{\widehat{h}}(h,\Omega)$ is affine in h due to Lemma 2 (ii). Hence, this inequality holds not only for $h = \underline{h}, \overline{h}$ but also for all $\underline{h} \le h \le \overline{h}$. By Lemma 2 (i), the stability condition of Proposition 1 is satisfied, which implies the exponential stability of S. □

The choice of $\widehat{h}$ is up to the user. In particular, the choice $\widehat{h} = \underline{h}$ or $\widehat{h} = \overline{h}$ is computationally attractive. Indeed, by Lemma 2 (iii), $V_{\widehat{h}}(h) = O$ either at $h = \underline{h}$ or $h = \overline{h}$ with this choice, which allows us to neglect the last row and the last column in the LMI (8). Furthermore, when $\underline{h}$ is close to zero, the choice $\widehat{h} = \underline{h}$ is preferable because the choice $\widehat{h} = \overline{h}$ makes $V_{\widehat{h}}(h)$ large at $h = \underline{h}$, which results in large effect of Ω in $\Psi_{\widehat{h}}(\underline{h}, \Omega)$.

B. The case of $\underline{h} = 0$

We next consider the case of $\underline{h} = 0$. Here, the choice $\widehat{h} = \underline{h} = 0$ is necessary for boundedness of $U_{\widehat{h}}(h)$ and $V_{\widehat{h}}(h)$. In this case, we have

$$U_0(h) = A + BF, \quad V_0(h) = A(A+BF), \tag{10}$$

which are actually independent of h. The function for replacement of $\Psi(h)$ is

$$\Psi_0(h,\Omega) = U_0(h) + \Omega V_0(h),$$

where Ω is a matrix uncertainty such that $\overline{\sigma}(\Omega) \le \omega_{0,\overline{h},0}$ with

$$\omega_{0,\overline{h},0} = \overline{h} \max_{0 \le t \le \overline{h}} \exp\Big[\overline{\lambda}\Big(\frac{A+A^{\mathrm{T}}}{2}\Big)t\Big]. \tag{11}$$

The subscripts of ω are again dropped. With this setup, we can derive a result corresponding to Theorem 3 in a similar way.

Theorem 4: Suppose $\underline{h} = 0$. Let $\widehat{h} = \underline{h} = 0$. Define $U_0(h)$ and $V_0(h)$ as (10) and ω as (11). Then, the control system S is exponentially stable if there exist a symmetric matrix Q and two nonnegative numbers $s_{\underline{h}}$ and $s_{\overline{h}}$ such that the inequality (8) holds. ◇

IV. A Region-Dividing Technique

A. Division of the region of the sampling interval

In our previous approach based on parametric uncertainty [12], conservatism of the stability condition can be reduced to any degree by dividing the region of the sampling interval, $[\underline{h}, \overline{h}]$. The same technique is effective in the present approach, too.

Henceforth, we let a *division* mean a set of subregions $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \mid j = 1, 2, \ldots, J\}$ such that

$$\underline{h} = \underline{h}^{[1]} < \overline{h}^{[1]} = \underline{h}^{[2]} < \overline{h}^{[2]} = \underline{h}^{[3]} < \cdots < \overline{h}^{[J]} = \overline{h}.$$

For a given division Δ, we consider the following stability condition.

Theorem 5: Let $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \mid j = 1, 2, \ldots, J\}$ be a division of $[\underline{h}, \overline{h}]$. For each j with $\underline{h}^{[j]} > 0$, let $\widehat{h}^{[j]}$ be any number in the subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$, let $U_{\widehat{h}^{[j]}}(h)$ and $V_{\widehat{h}^{[j]}}(h)$ be as (4) and (5), respectively, and let $\omega^{[j]}$ be $\omega_{\underline{h}^{[j]},\overline{h}^{[j]},\widehat{h}^{[j]}}$

in (6). For j with $\underline{h}^{[j]} = 0$, if any, let $\widehat{h}^{[j]} = 0$, let $U_0(h)$ and $V_0(h)$ be as (10), and let $\omega^{[j]}$ be $\omega_{0,\overline{h}^{[j]},0}$ in (11). Then, the control system S is exponentially stable if there exists a symmetric matrix Q and $2J$ nonnegative numbers $s^{[j]}_{\underline{h}^{[j]}}$ and $s^{[j]}_{\overline{h}^{[j]}}$ $(j = 1, 2, \dots, J)$ such that

(the formula at the bottom of this page).

◇

Proof. For each subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$, the discussion in the proof of Theorem 3 is applicable. Consequently, the inequality (1) holds for any h in $\cup_{j=1,2,\dots,J}[\underline{h}^{[j]}, \overline{h}^{[j]}] = [\underline{h}, \overline{h}]$. Proposition 1 then implies the exponential stability of S. □

B. Asymptotic exactness

Let us choose $\widehat{h}^{[j]} = \underline{h}^{[j]}$ for any subregion in the stability condition of Theorem 5. Then, we can reduce conservatism of this condition to any degree by using a sufficiently fine division. A corresponding result is known in the approach with parametric uncertainty [12]. Let us refer to $\overline{h}^{[j]} - \underline{h}^{[j]}$ as the *width* of the subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$ and $\max_{j=1,2,\dots,J}(\overline{h}^{[j]} - \underline{h}^{[j]})$ as the *maximum width* of the division $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \mid j = 1, 2, \dots, J\}$, which is denoted by $\overline{\mathrm{wid}}\,\Delta$.

Theorem 6: Suppose that there exists Q satisfying the original stability condition (1). Then, the same Q satisfies the condition of Theorem 5 for a division Δ having sufficiently small $\overline{\mathrm{wid}}\,\Delta$, when the choice $\widehat{h}^{[j]} = \underline{h}^{[j]}$ is adopted for any j. ◇

Proof. Consider the stability condition of Theorem 5 for some division $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \mid j = 1, 2, \dots, J\}$. This condition is equivalent to the existence of Q satisfying

$$\begin{pmatrix} -\Psi_{\widehat{h}^{[j]}}(h,\Omega)Q - Q\Psi_{\widehat{h}^{[j]}}(h,\Omega)^{\mathrm{T}} & \sqrt{h}\Psi_{\widehat{h}^{[j]}}(h,\Omega)Q \\ \sqrt{h}Q\Psi_{\widehat{h}^{[j]}}(h,\Omega)^{\mathrm{T}} & Q \end{pmatrix} \succ O$$

for any $\underline{h}^{[j]} \le h \le \overline{h}^{[j]}$, any Ω such that $\overline{\sigma}(\Omega) \le \omega_{\underline{h}^{[j]},\overline{h}^{[j]},\widehat{h}^{[j]}}$, and any $j = 1, 2, \dots, J$.

Pick up any j and subtract the left-hand side of (1) from the left-hand side of the above inequality to have

$$\begin{pmatrix} \Psi_{\widehat{h}^{[j]}}(h,\Omega) - \Psi(h) \\ O \end{pmatrix} \begin{pmatrix} -Q & \sqrt{h}Q \end{pmatrix} + \begin{pmatrix} -Q \\ \sqrt{h}Q \end{pmatrix} \begin{pmatrix} \Psi_{\widehat{h}^{[j]}}(h,\Omega)^{\mathrm{T}} - \Psi(h)^{\mathrm{T}} & O \end{pmatrix}. \quad (12)$$

By the equations (3) and (7), we have

$$\Psi_{\widehat{h}^{[j]}}(h,\Omega) - \Psi(h) = \left(\Omega - \frac{1}{h-\widehat{h}^{[j]}}\int_0^{h-\widehat{h}^{[j]}}\int_0^t \mathrm{e}^{Au}\,\mathrm{d}u\,\mathrm{d}t\right)V_{\widehat{h}^{[j]}}(h).$$

Here, both matrices in the parentheses have the maximum singular values less than or equal to $\omega_{\underline{h}^{[j]},\overline{h}^{[j]},\widehat{h}^{[j]}}$; The maximum singular value of $V_{\widehat{h}^{[j]}}(h)$ is bounded by some number independent of $\widehat{h}^{[j]}$ and h since $0 \le \underline{h}^{[j]} = \widehat{h}^{[j]} \le h \le \overline{h}$. Hence, the maximum singular value of $\Psi_{\widehat{h}^{[j]}}(h,\Omega) - \Psi(h)$ can be bounded from above by a number proportional to $\omega_{\underline{h}^{[j]},\overline{h}^{[j]},\widehat{h}^{[j]}}$. This number is further proportional to $\overline{h}^{[j]} - \underline{h}^{[j]}$. Consequently, if the maximum width $\overline{\mathrm{wid}}\,\Delta$ is small enough, the discrepancy (12) is small for any subregion in Δ, which establishes the theorem. □

C. Adaptive division

The technique of adaptive division in [12] is applicable to the present approach. This is a technique to refine a division only in a subregion necessary to do. With this technique, we can expect reduction of the computational cost. Although the discussion is parallel to that of [12], we reproduce it here for convenience of the readers.

We first restate our stability condition in a semidefinite programming (SDP) problem, which is an optimization problem constrained by LMIs. This is used to find a subregion to subdivide.

Theorem 7: Let $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \mid j = 1, 2, \dots, J\}$ be a division of $[\underline{h}, \overline{h}]$. For each $j = 1, 2, \dots, J$, let $\widehat{h}^{[j]}$, $U_{\widehat{h}^{[j]}}(h)$, $V_{\widehat{h}^{[j]}}(h)$, and $\omega^{[j]}$ be as in Theorem 5. Then, the control system S is exponentially stable if the following SDP problem has a maximum value unbounded from above:

maximize x
subject to $Q \succeq I$,
(the formula at the bottom of next page).

◇

Proof. If the SDP problem has a maximum value unbounded from above, it is feasible with a positive x. This means the existence of Q, $s^{[j]}_{\underline{h}^{[j]}}$, and $s^{[j]}_{\overline{h}^{[j]}}$ satisfying the condition of Theorem 5. Hence, the exponential stability of S follows. □

Suppose that we solve the SDP problem of Theorem 7 for some division Δ and obtain the (bounded) nonpositive maximum value. Since the stability of S is not assured in this case, we are to refine the division. Here, we notice a subregion having an active constraint at the optimal point. Since an active constraint prevents the optimal value from being improved, a subregion having an active constraint needs to be subdivided.

$$\begin{pmatrix} -U_{\widehat{h}^{[j]}}(h)Q - QU_{\widehat{h}^{[j]}}(h)^{\mathrm{T}} - s^{[j]}_h I & \sqrt{h}U_{\widehat{h}^{[j]}}(h)Q & -\omega^{[j]}QV_{\widehat{h}^{[j]}}(h)^{\mathrm{T}} \\ \sqrt{h}QU_{\widehat{h}^{[j]}}(h)^{\mathrm{T}} & Q & \omega^{[j]}\sqrt{h}QV_{\widehat{h}^{[j]}}(h)^{\mathrm{T}} \\ -\omega^{[j]}V_{\widehat{h}^{[j]}}(h)Q & \omega^{[j]}\sqrt{h}V_{\widehat{h}^{[j]}}(h)Q & s^{[j]}_h I \end{pmatrix} \succ O \qquad (h = \underline{h}^{[j]}, \overline{h}^{[j]}; j = 1, 2, \dots, J)$$

TABLE I

STABILITY ANALYSIS WITH ADAPTIVE DIVISION

dividing points	comp. time (s)	max. value
$0,\ 1.7294$	0.35	-5.95
$0,\ 0.8647,\ 1.7294$	0.37	-0.793
$0,\ 0.4324,\ 0.8647,\ 1.7294$	0.42	-0.362
$0,\ 0.4324,\ 0.8647,\ 1.2971,\ 1.7294$	0.46	-0.0658
$0,\ 0.4324,\ 0.8647,\ 1.2971,\ 1.5133,\ 1.7294$	0.51	-0.0141
$0,\ 0.4324,\ 0.8647,\ 1.2971,\ 1.5133,\ 1.6214,\ 1.7294$	0.57	-0.00328
$0,\ 0.4324,\ 0.8647,\ 1.2971,\ 1.5133,\ 1.6214,\ 1.6754,\ 1.7294$	0.63	-7.88×10^{-4}
$0,\ 0.4324,\ 0.8647,\ 1.2971,\ 1.5133,\ 1.6214,\ 1.6754,\ 1.7024,\ 1.7294$	0.68	-1.89×10^{-4}
$0,\ 0.4324,\ 0.8647,\ 1.2971,\ 1.5133,\ 1.6214,\ 1.6754,\ 1.7024,\ 1.7159,\ 1.7294$	0.76	-4.28×10^{-5}
$0,\ 0.4324,\ 0.8647,\ 1.2971,\ 1.5133,\ 1.6214,\ 1.6754,\ 1.7024,\ 1.7159,\ 1.7227,\ 1.7294$	0.89	-6.28×10^{-6}
$0,\ 0.4324,\ 0.8647,\ 1.2971,\ 1.5133,\ 1.6214,\ 1.6754,\ 1.7024,\ 1.7159,\ 1.7227,\ 1.7261,\ 1.7294$	0.65	$+\infty$

Based on this idea, we have the following algorithm that is expected to generate an efficient division, that is, a division that gives a less conservative result with little computational cost. Here, we mean by an *active subregion* a subregion having an active constraint. When the maximum value is not attained, an active constraint or an active subregion is not defined.

Algorithm 8:

0. Prepare a coarse division.
1. Solve the SDP problem of Theorem 7 corresponding to the current division.
2. Stop if the problem has a maximum value unbounded from above.
3. If the maximum value is attained, find and subdivide an active subregion. Otherwise, find and subdivide a subregion of the maximum width.
4. Return to Step 1 unless the number of subregions exceeds the prescribed number. ◇

This technique seems contradictory with Theorem 6 because a generated division is non-uniform in general and is not efficient for minimization of the maximum width. This contradiction is resolved by detailed analysis. The discussion is again similar to that in the previous approach [12].

V. DESIGN OF A STATE-FEEDBACK GAIN

Our approach can be used for design of a state-feedback gain as in [12]. In the stability conditions of Theorems 3 and 4, the matrix products $U_{\widehat{h}}(h)Q$ and $V_{\widehat{h}}(h)Q$ contain the factor $(A+BF)Q$. If we replace this factor by $AQ+BG$ and obtain Q and G that satisfy the stability condition, we can compute a stabilizing feedback gain as $F=GQ^{-1}$. The region-dividing technique is effective also in this case.

VI. EXAMPLE

The proposed approach is applied to the sampled-data control system with

$$A=\begin{pmatrix}0 & 1\\ 0 & -0.1\end{pmatrix},\quad B=\begin{pmatrix}0\\ 0.1\end{pmatrix},\quad F=(-3.75 \quad -11.5).$$

Stability of this control system has been analyzed with various approaches for various regions of the sampling interval h. For example, the approach of Naghshtabrizi *et al.* [11] verified stability for the region of h being $(0,1.1137]$; the approach of Mirkin [10] for $h\in(0,1.3659]$; the approach of Suh [15] for $h\in[0.5,1.729]$; the approach of Fujioka [5] for $h\in[0.01,1.72]$. In particular, our previous approach [12] verified stability for $h\in(0,1.7294]$. This region of h is almost maximum because this system is unstable for the constant sampling interval $h=1.7295$.

With the present approach in Section IV, we successfully verified the stability for $h\in(0,1.7294]$. This shows that the present approach is as efficient as our previous approach. Specifically, we set $\underline{h}=0$ and $\overline{h}=1.7294$ and adaptively generated a division of $[\underline{h},\overline{h}]$ according to Algorithm 8. With a division consisting of 11 subregions, we assured the stability. The process is summarized in Table I, where the stability is assured at the bottom line. Here, we chose $\widehat{h}^{[j]}=\underline{h}^{[j]}$ for all the subregions. The SDP problems were solved with the SDP solver SeDuMi [14] and the modeling language YALMIP [8]. The used computer was equipped with Intel Core 2 Duo U7500 ($1.06\,\mathrm{GHz}$) and memory of $2\,\mathrm{GBytes}$.

In our previous approach [12], a division consisting of 9 subregions, in place of 11, was sufficient for guaranteeing the stability. This appears to be because the present approach, which is based on matrix uncertainty, is more conservative than the previous approach, which is based on parametric uncertainty; The present approach requires a finer division to

$$\begin{pmatrix}-U_{\widehat{h}^{[j]}}(h)Q-QU_{\widehat{h}^{[j]}}(h)^{\mathrm{T}}-s_h^{[j]}I & \sqrt{h}U_{\widehat{h}^{[j]}}(h)Q & -\omega^{[j]}QV_{\widehat{h}^{[j]}}(h)^{\mathrm{T}}\\ \sqrt{h}QU_{\widehat{h}^{[j]}}(h)^{\mathrm{T}} & Q & \omega^{[j]}\sqrt{h}QV_{\widehat{h}^{[j]}}(h)^{\mathrm{T}}\\ -\omega^{[j]}V_{\widehat{h}^{[j]}}(h)Q & \omega^{[j]}\sqrt{h}V_{\widehat{h}^{[j]}}(h)Q & s_h^{[j]}I\end{pmatrix}\succeq xI \qquad (h=\underline{h}^{[j]},\overline{h}^{[j]};\ j=1,2,\ldots,J)$$

compensate its conservatism. Recall however that the present approach does not require the real Jordan canonical form of A.

We next designed a state-feedback gain F for the A and B above. We took the approach in Section V with $\underline{h} = 0$ and $\overline{h} = 5$. We chose $\widehat{h}^{[j]} = \underline{h}^{[j]}$ for all the subregions. As a result, a stabilizing gain $F = (-0.139 \ \ -2.40)$ was obtained with a division consisting of 6 subregions. The computational time was $0.58\,\mathrm{s}$. On the other hand, our previous approach gave a stabilizing gain without subdividing $(0, 5]$. This again shows that the present approach may need a finer division than the previous approach.

VII. Conclusion

An LMI condition is presented for stability of an aperiodic sampled-data system. This condition is applicable to a high-order system because it does not require the real Jordan canonical form unlike the previously proposed approach. Conservatism of the proposed condition can be decreased to any degree by division of the region of the possible sampling intervals. Adaptive division is also discussed for reduction of the computational cost.

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