On the Consistency of L^2-Optimal Sampled Signal Reconstructors

Gjerrit Meinsma and Leonid Mirkin

***Abstract*— The problem of restoring an analog signal from its sampled measurements is called the signal reconstruction problem. A reconstructor is said to be *consistent* if the resampling of the reconstructed signal by the acquisition system would produce exactly the same measurements. The consistency requirement is frequently used in signal processing applications as the design criterion for signal reconstruction. System-theoretic reconstruction, in which the analog reconstruction error is minimized, is a promising alternative to consistency-based approaches. The primary objective of this paper is to investigate, what are conditions under which consistency might be a byproduct of the system-theoretic design that uses the L^2 criterion. By analyzing the L^2 reconstruction in the lifted frequency domain, we show that non-causal solutions are always consistent. When causality constraints are imposed, the situation is more complicated. We prove that optimal relaxedly causal reconstructors are consistent either if the acquisition device is a zero-order generalized sampler or if the measured signal is the ideally sampled state vector of the antialiasing filter. In other cases consistency can no longer be guaranteed as we demonstrate by a numerical example.**

I. Introduction

In this paper we address the problem of reconstructing an analog signal v from its sampled measurements $\bar{y}$. The setup we study is depicted in Fig. 1. Here the measurement

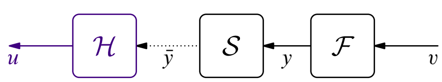

Fig. 1. Sampled signal reconstruction

channel consists of an analog (antialiasing) filter $\mathcal{F}$ and the ideal sampler $\mathcal{S}$ and the design parameter is the D/A device (hold / interpolator) $\mathcal{H}$, generating an analog reconstruction u of v according to the following law

$$u(t) = \sum_{i\in\mathbb{Z}} \phi(t-ih)\bar{y}[i], \quad t \in \mathbb{R}, \tag{1}$$

where $\phi(t)$ is the hold function (interpolation kernel) and $h > 0$ is is the sampling period. The goal is to design $\phi(t)$ so that u is in a sense close to v.

A widely used family of approaches to solve the reconstruction problem, especially in the signal processing literature, is based on the notion of *consistency*, introduced in [1], see also [2]. Loosely speaking, a reconstruction of an analog signal is said to be consistent if it would yield exactly the same measurements if it was reinjected into the measurement

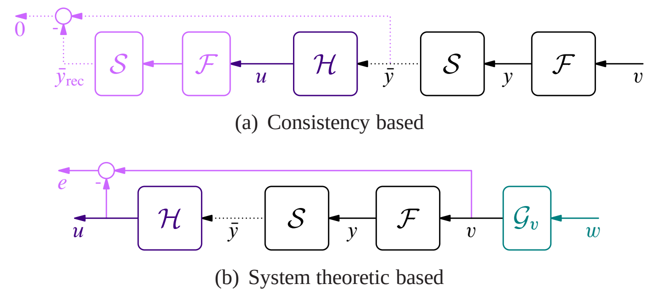

Fig. 2. Sampled signal reconstruction paradigms

system. For the scheme in Fig. 1, the consistency can be viewed through the block-diagram in Fig. 2(a), where we use lavender to represent (virtual rather than physical) signals and systems used in formulating the design criterion. The problem then is to design $\mathcal{H}$, or, more precisely, its discrete part, so that the A/D system from v to $\bar{y} - \bar{y}_{\rm rec}$ in Fig. 2(a) is zero for all admissible v. The simplicity and transparency of this criterion facilitates an efficient and meaningful design. The necessity to postulate (guess) the intersample waveform of the D/A conversion, however, is quite restrictive. For example, it complicates the incorporation of causality constraints into the design.

An alternative approach to sampled signal reconstruction is to directly minimize the analog reconstruction error $e := v - u$, see [3]–[5]. To render such an optimization meaningful, we have to account for properties of v. As accustomed in the control literature, these properties are accounted via modeling v as the output of a known system $\mathcal{G}_v$ (signal generator) driven by a normalized fictitious input w (denoted by teal blue in Fig. 2(b)). Reconstruction performance is then measured by a norm of the *error system*, which is the analog system from w to e in Fig. 2(b):

$$\mathcal{G}_{\rm e} := \mathcal{G}_v - \mathcal{HSFG}_v.$$

The L^2 formalism assumes that w is the standard white noise (or the Dirac delta in the deterministic case), in which case the minimization of the L^2-norm of the error system (in the causal case, it is the H^2 norm, see [6]), $\|\mathcal{G}_{\rm e}\|_2$, corresponds to the least mean square approach. We showed in [7], [8] that this problem can be solved analytically under causality constraints imposed upon $\mathcal{H}$ (finite preview).

Curiously, the consistency of the reconstruction does not necessarily interfere with the analog L^2 optimization criterion. We proved in [8], as a byproduct of our approach, that if $\mathcal{F} = 1$, the optimal reconstructors are always consistent, irrespective of the extent of the preview. This paper aims at extending this result to more general filters $\mathcal{F}$ (in other words,

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Gjerrit Meinsma is with the Dept. of Applied Math., University of Twente, 7500 AE Enschede, The Netherlands. E-mail: *g.meinsma@utwente.nl*.

Leonid Mirkin is with the Faculty of Mechanical Eng., Technion—IIT, Haifa 32000, Israel. E-mail: *mirkin@technion.ac.il*.

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to more general acquisition devices). Toward this end, we follow a different approach, as state-space arguments used to produce the result in [8] are not readily extendible to dynamical $\mathcal{F}$ and do not apply in the case when the transfer function of $\mathcal{F}$, $F(s)$, is irrational.

The paper is organized as follows. Section II reviews the L^2-optimal reconstruction problem and its solution in the lifted domain. The consistency of the resulting optimal reconstructors is then analyzed in Section III. In §III-A we consider non-causal solutions, while §III-B addresses solutions, obtained by imposing causality constraints.

Notation

Throughout the paper signals are represented by lowercase symbols such as $y(t) : \mathbb{R} \to \mathbb{C}$, overbars indicate discrete time signals, $\bar{y}[k] : \mathbb{Z} \to \mathbb{C}$, and the breve accent, $\breve{y}[k] : \mathbb{Z} \to \{[0,h) \to \mathbb{C}\}$, is used for lifted signals (see §II-B). Uppercase calligraphic symbols, like $\mathcal{G}$, denote continuous-time systems in time domains, whose impulse response / kernel is denoted with lowercase symbols, such as g, and the corresponding transfer function / frequency response is presented by uppercase symbols, like $G(s)$ and $G(\mathrm{j}\omega)$.

We use different accents to emphasize the dimensionality of the domain and range of lifted and semi-lifted systes, which helps us in keeping track of the signal space dimensions. The breve accent, such as in $\breve{\mathcal{G}}$, indicates that input and output space at each discrete time is infinite dimensional, like $\{[0,h) \to \mathbb{C}^n\}$. The acute accent indicates that $\acute{\mathcal{G}}$ maps an infinite-dimensional space, like $\{[0,h) \to \mathbb{C}^n\}$, to a finite-dimensional space, like $\mathbb{C}^n$. The grave accent says that $\grave{\mathcal{G}}$ maps a finite-dimensional space to an infinite-dimensional space.

II. L^2 Optimization

We start with formulating the L^2 optimization problem with causality constraints for the system in Fig. 2(b) and presenting its solution in the lifted domain from [7]. This solution will then be used for the consistency analysis of the L^2-optimal hold.

A. Problem formulation

Throughout we assume that

- $\boldsymbol{\mathcal{A}_1}$: $G_v(s)$ is rational and strictly proper, i.e., $G_v(\infty) = 0$;
- $\boldsymbol{\mathcal{A}_2}$: $F(s)$ is proper and is either rational or FIR with support in $[0,h]$;
- $\boldsymbol{\mathcal{A}_3}$: there is no unstable cancellations in $F(s)G_v(s)$;
- $\boldsymbol{\mathcal{A}_4}$: h is not pathological with respect to $\mathcal{F}\mathcal{G}_v$;
- $\boldsymbol{\mathcal{A}_5}$: the operator $\mathcal{S}\mathcal{F}\mathcal{G}_v$ is right invertible.

The rationality of G_v and $\boldsymbol{\mathcal{A}_2}$ are assumed for the sake of simplicity. If F is FIR, the A/D converter $\mathcal{S}\mathcal{F}$ corresponds to a zero-order generalized sampler, like the averaging sampler in the case of $F = \frac{1-\mathrm{e}^{-sh}}{hs}$. G_v must be strictly proper to guarantee the boundedness of the L^2 norm of the error system. Assumptions $\boldsymbol{\mathcal{A}_{3,4}}$ guarantee the stabilizability of the error system. $\boldsymbol{\mathcal{A}_5}$ just rules out the redundancy of the measurement channel.

We say that a hold $\mathcal{H}$ is *admissible* if it is of the form (1) and is stable, in the sense that it is a bounded operator $\ell^2(\mathbb{Z}) \to L^2(\mathbb{R})$. Also, $\mathcal{H}$ is said to be l*-causal* if its interpolation kernel $\phi(t)$ satisfies

$$\phi(t) = 0, \quad \text{whenever } t < -lh, \tag{2}$$

for some nonnegative integer l. The reconstruction problem is then cast as the following L^2 optimization for the setup in Fig. 2(b):

RP$_l$: Let $\mathcal{G}_v$ and $\mathcal{F}$ be causal and satisfy $\boldsymbol{\mathcal{A}_{1\text{–}5}}$ and $\mathcal{S}$ be the ideal sampler. For a given $l \in \mathbb{N}$, find an admissible and l-causal hold $\mathcal{H}$, which stabilizes $\mathcal{G}_\mathrm{e}$ and minimizes its L^2 system norm $\|\mathcal{G}_\mathrm{e}\|_2$.

Some explanatory remarks are in order:

Remark 2.1: The L^2-norm of h-time invariant (h-periodic) systems is defined through their lifted frequency responses, see (3) and [5, §V.D] for more details. In the causal ($l = 0$) case it is actually the H^2 norm of sampled-data systems, see [6, §12.2]. This norm has clear deterministic and stichastic interpretations. From a deterministic point of view, it might be convenient to think of it as the average energy of e, where the average is taken over all $w(t) = \delta(t - \sigma)$ in $\sigma \in [0,h)$:

$$\|\mathcal{G}_\mathrm{e}\|_2^2 = \frac{1}{h}\int_0^h \|\mathcal{G}_\mathrm{e}\delta(\cdot - \sigma)\|_{L^2(\mathbb{R})}^2 \mathrm{d}\sigma.$$

In the stochastic setting, $\|\mathcal{G}_\mathrm{e}\|_2^2$ equals the over time averaged sum of variances (power) of all n_e elements of e, provided w is a unit covariance analog white processes. ▽

Remark 2.2: By the stability of the error system we understand that it is a bounded operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. If the signal generator $\mathcal{G}_v$ is itself stable, the error system is stable whenever so is $\mathcal{H}$. In other words, in this case the stability requirement on $\mathcal{G}_\mathrm{e}$ is redundant. There are situations, however, when it might be required to include unstable dynamics into $\mathcal{G}_v$. This happens, for instance, when jω-axis poles are incorporated into $G_v(s)$ to impose steady-state requirements. In such situations the stability requirement imposes additional constraints on the reconstructor. ▽

Two main technical difficulties in dealing with the system in Fig. 2(b) are that it is a hybrid, continuous / discrete, system (this is especially accute regarding our design parameter, $\mathcal{H}$) and the continuous-time dynamics of the error system are not time invariant. These difficulties can be circumvented by using the *lifting technique* [6], which enables us to transform the problem to an equivalent pure discrete shift-invariant one.

B. Lifted-domain reformulation

The lifting transformation, or simply lifting, can be seen as a way of separating the behavior into a fully time *in*variant discrete-time behavior and a *finite-horizon* continuous-time (intersample) behavior. To be specific, given an analog signal $f : \mathbb{R} \to \mathbb{C}^{n_f}$, its *lifting* $\breve{f} : \mathbb{Z} \to \{[0,h) \to \mathbb{C}^{n_f}\}$ is the sequence of functions $\{\breve{f}[k]\}$ defined as

$$\breve{f}[k](\tau) = f(kh + \tau), \qquad k \in \mathbb{Z}, \tau \in [0,h).$$

In other words, with lifting we consider a function on $\mathbb{R}$ as a sequence of functions on $[0,h)$. The idea can be explained by

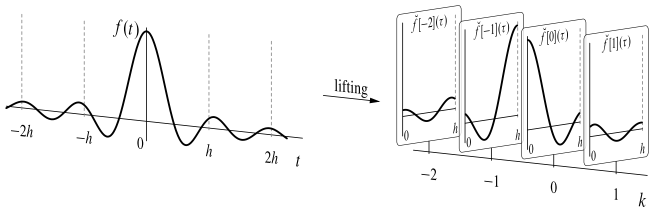


Fig. 3. Lifting analog signals

Fig. 3. Clearly, this incurs no loss of information, it is merely another representation of the signal.

The rationale behind the introduction of this representation is that it can losslessly convert the hybrid h-time invariant (h-periodic) system in Fig. 2(b) into a pure discrete shift-invariant one. Namely, by lifting all analog signals there we

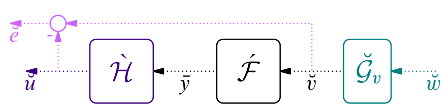


Fig. 4. System-theoretic reconstruction in the lifted domain

end up with the system depicted in Fig. 4 with the shift invariant signal generator $\breve{\mathcal{G}}_v$, acquisition system $\acute{\mathcal{F}}$, and the hold to be designed $\grave{\mathcal{H}}$. Here $\breve{\mathcal{G}}_v$ is the lifting of $\mathcal{G}_v$, i.e., the discrete system connecting the lifted sequences $\breve{w}$ with $\breve{v}$, and $\acute{\mathcal{F}}$ is the lifting of $\mathcal{SF}$, i.e., the discrete system connecting $\breve{v}$ with the discrete sequence $\bar{y}$. The lifted error system

$$\breve{\mathcal{G}}_{\mathrm{e}} = \breve{\mathcal{G}}_v - \grave{\mathcal{H}}\acute{\mathcal{F}}\breve{\mathcal{G}}_v$$

is then shift invariant for any $\phi(t)$ in (1).

With the regained shift invariance, we may analyze $\mathbf{RP}_l$ in the lifted frequency domain. To this end, we need some spaces. The Hilbert space L^2 consists of lifted frequency responses $\breve{G}(\mathrm{e}^{\mathrm{j}\theta})$, which are Hilbert-Schmidt operators for almost all $\theta \in [-\pi, \pi]$ and satisfy

$$\|\breve{G}\|_2 := \left(\frac{1}{2\pi h}\int_{-\pi}^{\pi}\|\breve{G}(\mathrm{e}^{\mathrm{j}\theta})\|_{\mathrm{HS}}^2 \mathrm{d}\theta\right)^{1/2} < \infty, \qquad (3)$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt operator norm. The space L^∞ consists of lifted frequency responses satisfying

$$\|\breve{G}\|_\infty := \operatorname*{ess\,sup}_{\theta\in[-\pi,\pi]} \sigma_{\max}[\breve{G}(\mathrm{e}^{\mathrm{j}\theta})] < \infty,$$

where $\sigma_{\max}$ stands for the operator maximal singular value. Another space we need is the Hardy space H^∞. It is defined as the set of transfer functions $\breve{G}(z)$, which are analytic for $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$ and satisfy

$$\|\breve{G}\|_{H^\infty} := \operatorname*{ess\,sup}_{z\in\mathbb{C}\setminus\mathbb{D}} \sigma_{\max}[\breve{G}(z)] < \infty.$$

H^∞ operators can be extended to $z \in \mathbb{T}$, resulting in a closed subspace of L^∞ with $\|\breve{G}\|_{H^\infty} = \|\breve{G}\|_\infty$. By $z^l H^\infty$ we then denote the subspace of L^∞ consisting of operators $\breve{G}(z)$ such that $z^{-l}\breve{G}(z) \in H^\infty$. Loosely speaking, H^∞ is the space of transfer functions, which are analytic and bounded in $\mathbb{C} \setminus \mathbb{D}$, whereas $z^l H^\infty$ is the space of analytic transfer functions with relaxed (if $l > 0$) or tightened (if $l < 0$) boundedness in $|z| \to \infty$. All definitions above extend straightforwardly to semi-lifted systems, like $\grave{\mathcal{H}}$ and $\acute{\mathcal{F}}$. Finally, it follows from the fact that $\grave{H}(z)$ is a finite-rank operator for almost all $z \in \mathbb{C}$ that $\grave{H} \in z^l H^\infty \Rightarrow \grave{H} \in L^2$.

Returning to $\mathbf{RP}_l$, it can be shown [5] that the hold as in (1) is admissible and l-causal iff its lifted transfer function $\grave{H} \in z^l H^\infty$ and the error system is stable iff its lifted transfer function $\breve{G}_{\mathrm{e}} \in L^\infty$. Thus, $\mathbf{RP}_l$ can be reformulated in the lifted frequency domain as follows:

$\breve{\mathbf{RP}}_l$: Given $\breve{G}_v$, $\acute{F}$, and $l \in \mathbb{N}$, find $\grave{H} \in z^l H^\infty$, which renders $\breve{G}_{\mathrm{e}} \in L^\infty \cap L^2$ and minimizes $\|\breve{G}_{\mathrm{e}}\|_2$.

Note that by $\boldsymbol{\mathcal{A}}_{1,2}$, the transfer functions $\breve{G}_v(z)$ and $\acute{F}(z)$ are rational, i.e., the lifted systems $\breve{\mathcal{G}}_v$ and $\acute{\mathcal{F}}$ admit finite-dimensional state space realizations.

C. Lifted-domain solution

In the solution of $\breve{\mathbf{RP}}_l$ we start with resolving the stability issue. To this end, note that the stabilizability of the error system is equivalent to the existence of the following coprime factorization:

$$\begin{bmatrix} \breve{G}_v \\ \acute{F}\breve{G}_v \end{bmatrix} = \begin{bmatrix} I & \grave{M}_v \\ 0 & \bar{M}_y \end{bmatrix}^{-1} \begin{bmatrix} \breve{N}_v \\ \acute{N}_y \end{bmatrix} \qquad (4)$$

for some $\grave{M}_v, \breve{N}_v \in H^\infty$ and coprime $\bar{M}_y, \acute{N}_y \in H^\infty$. In this case, the set of all stabilizing l-causal holds is given by

$$\grave{H} = -\grave{M}_v + \grave{Q}\bar{M}_y$$

and the set of all corresponding stable error systems is

$$\breve{G}_{\mathrm{e}} = \breve{N}_v - \grave{Q}\acute{N}_y,$$

where $\grave{Q} \in z^l H^\infty$ but otherwise arbitrary. Moreover, $\boldsymbol{\mathcal{A}}_1$ guarantees that $\breve{G}_{\mathrm{e}} \in L^2$ for every admissible $\grave{Q}$.

Having resolved the stability issue, we may use the standard Hilbert space optimization arguments to minimize $\|\breve{G}_{\mathrm{e}}\|_2$. To simplify the formulae, choose the factors in (4) that satisfy

$$\begin{bmatrix} \breve{N}_v \\ \acute{N}_y \end{bmatrix} \acute{N}_y^{\sim} = \begin{bmatrix} \acute{V}^{\sim} \\ I \end{bmatrix} \qquad (5)$$

for some $\acute{V} \in z^{-1}H^\infty$, where $\acute{V}^{\sim}$ denotes the conjugate transfer function $\acute{V}^{\sim}(z) = [\acute{V}(z^{-*})]^*$. In other words, we are looking for a numerator, for which $\acute{N}_y$ is co-inner and

$$\breve{N}_v\acute{N}_y^{\sim} =: \acute{V}^{\sim} = z\grave{V}_1 + z^2\grave{V}_2 + \cdots, \qquad (6)$$

where the sequence converges for almost every $z \in \mathbb{D}$. It can be shown [7] that such a numerator always exist if $\boldsymbol{\mathcal{A}}_5$ holds. Denote the optimal $\grave{Q}$ by $\grave{Q}_l$. By the Projection Theorem [9], it must satisfy

$$\langle \breve{N}_v - \grave{Q}_l\acute{N}_y, \grave{Q}\acute{N}_y\rangle_2 = 0$$

for all admissible $\grave{Q}$. Equivalently, using (5) we have:

$$\langle (\breve{N}_v - \grave{Q}_l\acute{N}_y)\acute{N}_y^{\sim}, \grave{Q}\rangle_2 = \langle \acute{V}^{\sim} - \grave{Q}_l, \grave{Q}\rangle_2 = 0.$$

This, in turn, leads to the following optimal choice of $\grave{Q}$:

$$\begin{aligned} \grave{Q} = \grave{Q}_l &:= \operatorname{proj}_{z^l H^\infty \cap L^2} \acute{V}^{\sim} \\ &= z\grave{V}_1 + z^2\grave{V}_2 + \cdots + z^l\grave{V}_l, \end{aligned} \qquad (7)$$

where $z^l H^\infty \cap L^2$ is the subspace of L^2, consisting of systems, whose lifted impulse response is zero for $k < -l$. Thus, the optimal hold is given by

$$\grave{H}_{\text{opt}} = -\grave{M}_v + \grave{Q}_l \bar{M}_y. \tag{8}$$

A state-space expression for this system can in principle be derived [7], [8]. For the consistency analysis, however, it is not essential.

III. Consistency Analysis

In the lifted frequency domain the consistency requirement, like that shown in Fig. 2(a) with $\mathcal{G}_v$ generating all admissible signals, reads

$$(I - \acute{F}\grave{H})\acute{F}\breve{G}_v = 0.$$

Moreover, by $\mathcal{A}_5$, $\acute{F}(z)\breve{G}_v(z)$ is right invertible for almost all $z \in \mathbb{C}$. Hence, the condition above reduces to

$$\acute{F}\grave{H} = I, \tag{9}$$

which is the condition that we shall check for the optimal reconstructor (8).

A. *Noncausal reconstruction ($l = \infty$)*

It follows from the second rows of (4) and (5) that

$$\bar{M}_y^\sim \bar{M}_y = (\acute{F}\breve{G}_v\breve{G}_v^\sim \acute{F}^\sim)^{-1}. \tag{10}$$

Then, by the first row of (5) and by (4) we have:

$$\begin{aligned}\acute{V}^\sim &= \breve{N}_v \acute{N}_y^\sim = (I + \grave{M}_v\acute{F})\breve{G}_v(\bar{M}_y\acute{F}\breve{G}_v)^\sim \\ &= (I + \grave{M}_v\acute{F})\breve{G}_v\breve{G}_v^\sim \acute{F}^\sim \bar{M}_y^\sim. \end{aligned} \tag{11}$$

Now, using the fact that in this case the optimal $\grave{Q}$ equals $\acute{V}^\sim$ and by (11) and (10), we have:

$$\begin{aligned}\grave{H}_{\text{opt}} &= -\grave{M}_v + \acute{V}^\sim \bar{M}_y \\ &= -\grave{M}_v + (I + \grave{M}_v\acute{F})\breve{G}_v\breve{G}_v^\sim \acute{F}^\sim \bar{M}_y^\sim \bar{M}_y \\ &= \breve{G}_v\breve{G}_v^\sim \acute{F}^\sim(\acute{F}\breve{G}_v\breve{G}_v^\sim \acute{F}^\sim)^{-1}.\end{aligned}$$

It is readily seen that this hold always satisfies (9). In other words,

- non-causal L^2-optimal reconstruction always produces consistent solutions,

no matter what are the signal generator $\mathcal{G}_v$ and the acquisition filter $\mathcal{F}$.

B. *l-causal reconstruction (finite l)*

Define

$$\grave{V}_{\text{tail}\,l} := \acute{V}^\sim - \grave{Q}_l \;= z^{l+1}\grave{V}_{l+1} + z^{l+2}\grave{V}_{l+2} + \cdots,$$

so that $\grave{Q}_l = \acute{V}^\sim - \grave{V}_{\text{tail}\,l}$. Using the result of the previous subsection we have that in this case

$$\acute{F}\grave{H}_{\text{opt}} = \acute{F}(-\grave{M}_v + \acute{V}^\sim\bar{M}_y - \grave{V}_{\text{tail}\,l}\bar{M}_y) = I - \acute{F}\grave{V}_{\text{tail}\,l}\bar{M}_y.$$

Thus, the optimal hold is consistent iff $\acute{F}\grave{V}_{\text{tail}\,l}\bar{M}_y = 0$, which reduces to

$$\acute{F}\grave{V}_{\text{tail}\,l} = 0 \tag{12}$$

because $\bar{M}_y$ is nonsingular.

A key observation, which we shall use in the analysis, is that while $\acute{V}^\sim$ is anti-causal,

$$\acute{F}\acute{V}^\sim = (I + \acute{F}\grave{M}_v)\bar{M}_y^{-1}$$

(follows from (11) and (10)) is *causal* as so are all its components. In other words, we have a causal system as the series interconnection of an anti-causal and a causal systems. Fig. 5 illustrates this situation in terms of its impulse response. This property will lead us to (12) in some situations as described below.

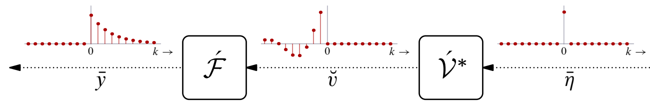


Fig. 5. Impulse response pattern of $\acute{\mathcal{F}}\acute{\mathcal{V}}^*$

1) FIR $\mathcal{F}$: Let $\mathcal{F}$ be an FIR system with the impulse response having support in $[0, h)$. This corresponds to the case when $\mathcal{SF}$ is a *zero-order generalized sampler*, acting as

$$\bar{y}[k] = \int_0^h f(\tau)v(kh - \tau)\mathrm{d}\tau, \tag{13}$$

where $f(\tau)$ is the impulse response of $\mathcal{F}$. In the lifted domain, this equation describes the following relation:

$$\bar{y}[k] = \acute{F}_1\breve{v}[k-1],$$

where $\acute{F}_1$ is an integral operator $L^2[0, h] \to \mathbb{R}^{n_{\bar{y}}}$ with the kernel f. This means that in this case

$$\acute{F}(z) = z^{-1}\acute{F}_1.$$

Now, using (6) we have:

$$\acute{F}\acute{V}^\sim = \acute{F}_1\grave{V}_1 + z\acute{F}_1\grave{V}_2 + z^2\acute{F}_1\grave{V}_3 + \cdots$$

Because this system must be causal, we have that

$$\acute{F}_1\grave{V}_i = 0 \quad \forall i = 2, 3, \ldots,$$

which implies that (12) holds for all $i \in \mathbb{N}$. Thus,

- l-causal L^2-optimal reconstruction always produces consistent solutions if $\mathcal{SF}$ is a zero-order generalized sampler of the form (13).

This result includes the result of [8] as a particular case for $f(\tau) = \delta(\tau)$.

2) y is the state of $\mathcal{F}$: Let now $\mathcal{F}$ be a finite-dimensional system having the following state-space realization:

$$F(s) = \left[\begin{array}{c|c} A_F & B_F \\ \hline I & 0 \end{array}\right]. \tag{14}$$

In this case, the lifting of $\mathcal{SF}$, $\acute{\mathcal{F}}$, describes the following input/output relation [6]:

$$\bar{y}[k+1] = \bar{A}_F\bar{y}[k] + \acute{B}_F\breve{v}[k],$$

where $\bar{A}_F := \mathrm{e}^{A_F h}$ and $\acute{B}_F : L^2[0, h] \to \mathbb{R}^n$ satisfies

$$\acute{B}_F\breve{v} = \int_0^h \mathrm{e}^{A_F(h-\tau)}B_F\breve{v}(\tau)\mathrm{d}\tau.$$

The impulse response as in Fig. 5, i.e., with $\bar{y}[k] = 0$ for all $k < 0$, can only be achieved if

$$\acute{B}_F \grave{V}_i = 0, \quad \forall i = 2, 3, \ldots \tag{15}$$

(as the impulse response of $\acute{V}^{\sim}$ is $\breve{v}[-k] = \grave{V}_k$ for all $k \in \mathbb{N}$). These equalities imply that

$$\acute{F}(z)\grave{V}_i = (zI - \bar{A}_F)^{-1}\acute{B}_F \grave{V}_i = 0$$

as well, which, in turn, leads to (12) for all $l \in \mathbb{N}$. Thus,

- l-causal L^2-optimal reconstruction always produces consistent solutions if $\mathcal{F}$ has a realization as in (14).

Obviously, this conclusion remains true if the realization of $\mathcal{F}$ has any square and nonsingular "C_F" matrix. If C_F is "fat," however, we can no longer guarantee (15) as the state vector of $\acute{F}$ needs not be zero (it must only belong to $\ker C_F$).

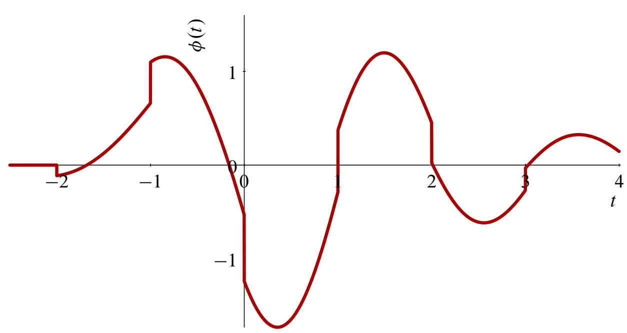
3) General finite-dimensional $\mathcal{F}$: In this case we can no longer guarantee the consistency of the reconstruction. To see this, consider an example with

$$G_v(s) = \frac{1}{s^2} \quad \text{and} \quad F(s) = \frac{8}{(2s+1)^2}. \tag{16}$$

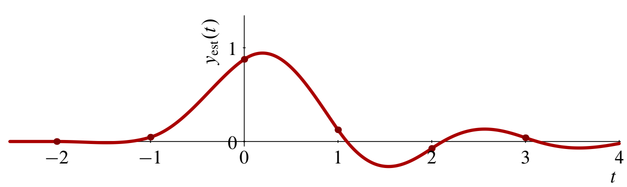
Fig. 6 presents simulation results for this example with the choices $h = 1$ and $l = 2$ (two steps preview). Fig. 6(a) depicts the impulse response $\phi(t)$ of the optimal reconstructor, $\mathcal{H}_{\text{opt}}$, and Fig. 6(b)—the impulse response $y_{\text{est}}(t)$ of the cascade of the optimal reconstructor and $\mathcal{F}$. Consistency in this case requires that sampling the latter signal by the ideal sampler (dark dots in Fig. 6(b)) produces the Kronecker delta, i.e., that

$$y_{\text{est}}(kh) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is clearly seen from the plot that this is not the case here. Thus, in this example we end up with a non-consistent reconstruction.



(a) Impulse response of $\mathcal{H}_{\text{opt}}$



(b) Impulse response of $\mathcal{F}\mathcal{H}_{\text{opt}}$

Fig. 6. Simulation results for system (16) with $h = 1$ and $l = 2$

IV. Concluding Remarks

In this paper we have analyzed the consistency of the L^2-optimal reconstruction of an analog signal from its sampled measurements. We have shown that if no causality constraints are imposed on the hold function, the optimal solution is always consistent. If the optimal hold is constrained to have some degree of causality, consistency can no longer be guaranteed in general. This was demonstrated by a counterexample. We have also determined two classes of the acquisition circuit for which consistency is guaranteed under any preview. Namely, this happens either if the acquisition device is a zero-order generalized sampler or if the measured signal is the ideally sampled state vector of the antialiasing filter.

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