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Truncated norms and limitations on signal reconstruction

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***Abstract*— Design of optimal signal reconstructors over all samplers and holds boils down to canceling frequency bands from a given frequency response. This paper discusses limits of performance of such samplers and holds and develops methods to compute the optimal L^2-norm.**

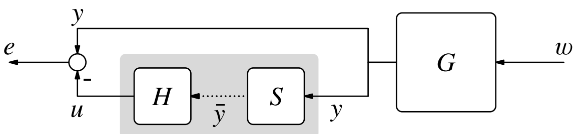

Fig. 1. Configuration

I. Introduction

The block diagram in Fig. 1 depicts a sampled data approach to optimal signal reconstruction. Here an analog signal y is given to a sampler S (with sampling period h) which produces some discrete signal $\bar{y}$ and this, in turn, is fed to a hold device H which produces an analog signal u. Ideally u equals y meaning that we reconstructed y error-free and we say that sampler and hold are optimal (with respect to some given norm) if they minimize the norm of the error mapping

$$G_{\text{e}} := (I - HS)G$$

from w to the reconstruction error $e := y - u$. The role of the system G is to weight certain frequency bands. The design of L^2-norm and L^∞-norm optimal samplers and holds among all h-periodic linear samplers and holds (possibly noncausal) has been solved [7], [6], [5] and the answer for LTI systems G in short is as shown in Fig. 2, that is: (a) fold the magnitude frequency response $|G(\mathrm{i}\omega)|$ back to the base-band $[0, \omega_{\text{N}}]$ where $\omega_{\text{N}} = \pi/h$ is the Nyquist frequency (Fig. 2(I-III)); (b) determine the upper-envelope of the folding and the corresponding frequency bands ($[0, \omega_{\text{C}}]$ and $[\omega_{\text{N}}, \omega_{\text{B}}]$ in Fig. 2(IV)); (c) then the optimal noncausal sampler-hold $H_{\text{opt}}S_{\text{opt}}$ among all linear h-periodic samplers and holds is the ideal band-pass filter that selects the frequency bands on which $|G(\mathrm{i}\omega)|$ contributes to the upper envelope (see Fig. 2(IV-V)). In particular the cascade of optimal sampler and hold is LTI, which is surprising considering that samplers and holds themselves are h-periodic. For the system of Fig. 2 the optimal sampler-hold has frequency response

$$(H_{\text{opt}}S_{\text{opt}})(\mathrm{i}\omega) = \begin{cases} 1 & |\omega| \in [0, \omega_{\text{C}}] \cup [\omega_{\text{N}}, \omega_{\text{B}}], \\ 0 & \text{elsewhere} \end{cases}$$

with $\omega_{\text{B}} = 2\omega_{\text{N}} - \omega_{\text{C}}$. Notice that the total length of the frequency band over which $H_{\text{opt}}S_{\text{opt}}$ is active, is exactly ω_{N}. This system can indeed be implemented as a cascade of sampler and hold, see [5] for details.

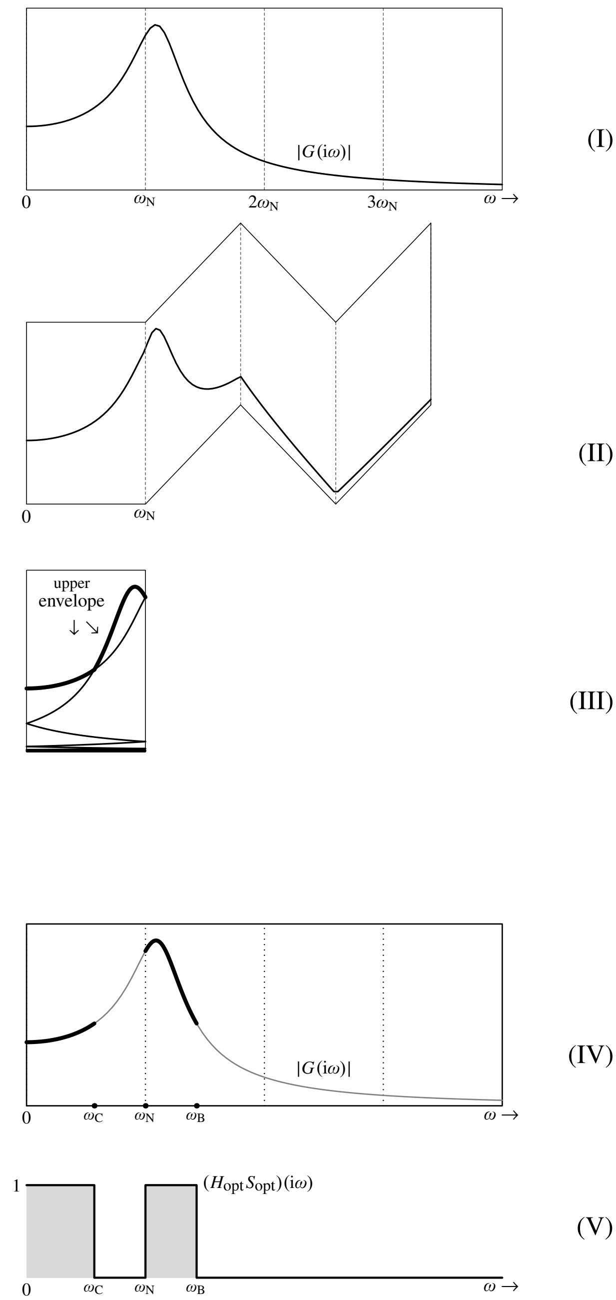

Fig. 2. Folding magnitude frequency response (upper 3 figures) and how to determine optimal sampler-hold (lower 2 figures)

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This result raises a couple of questions and in this paper we deal with two of them:

- How does the norm of the error mapping $(I - HS)G$ depend on the sampling period h?
  This part is largely based on [5, § VII.A].
- How to efficiently compute the L^2 norm of the optimal error mapping $(I - HS)G$?
  This boils down to computation of finite or semi-infinite integrals or rational functions. This problem is not new and has for instance been dealt with in the context of model reduction [3]. In our case the possible presence of imaginary poles makes matters somewhat more complicated.

On occasion we also consider L^∞-optimal samplers and holds. Considering that we optimize over non-causal samplers and holds it will be no surprise that the L^2-optimal sampler-hold are also L^∞-norm optimal.

A note on the L^2-norm and L^∞-norm is in order. Since we optimize over all linear h-periodic samplers and holds, the error system $G_\mathrm{e} := (I - HS)G$ whose norm we aim to minimize is linear but typically not time-invariant with respect to continuous-time shifts. It merely is time-invariant with respect to multiples of the sampling period. The L^2 system norm needs to be adjusted accordingly [1]. Fortunately the optimal cascade of sampler and hold $H_\mathrm{opt}S_\mathrm{opt}$ is LTI and since we only deal with optimal sampler-hold we can, once again, restrict ourselves to the familiar L^2 (and L^∞) of real LTI systems, which in terms of their frequency response reads

$$\|P\|_{L^2} = \sqrt{\frac{1}{\pi}\int_0^\infty |P(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega},$$
$$\|P\|_{L^\infty} = \operatorname{ess\,sup}_{\omega\in\mathbb{R}} |P(\mathrm{i}\omega)|.$$

Notation

Throughout h denotes the sampling period and $\omega_\mathrm{N} = \pi/h$ its Nyquist frequency. The conjugate $G^\sim$ of a real transfer matrix is defined as $G^\sim(s) = [G(-s)]^T$. A constant square matrix A is said to be *stable* if all its eigenvalues have strictly negative real part. The logarithm

$$\log(X)$$

in this paper always refers to the principal logarithm. The principal logarithm is defined for square matrices X without eigenvalues on the branch cut (the negative real axis, including zero). It is the unique matrix Q with $\mathrm{e}^Q = X$ and whose spectrum lies in the open vertical strip $\{z \mid -\pi < \mathrm{Im}(z) < \pi\}$ [2, Thm. 1.31].

II. Limits and bounds on reconstruction

This section is based on [5, § VII.A]. For the interpretation of the results in this section it is convenient to consider a white noise input w of unit intensity (zero mean and having constant spectral density 1). Then the squared norm $\|G\|_{L^2}^2$ is the power (variance) of $y = Gw$ and similarly the power of $u = H_\mathrm{opt}S_\mathrm{opt}Gw$ is $\|H_\mathrm{opt}S_\mathrm{opt}G\|_{L^2}^2$.

By Pythagoras the power of the reconstruction error $e = y - u$ then is the difference of the powers,

$$\|G_\mathrm{e}\|_{L^2}^2 = \|G\|_{L^2}^2 - \|H_\mathrm{opt}S_\mathrm{opt}G\|_{L^2}^2.$$

This also follows from $G_\mathrm{e} := (I - H_\mathrm{opt}S_\mathrm{opt})G$ and the fact that $(H_\mathrm{opt}S_\mathrm{opt})(\mathrm{i}\omega)$ at each ω is either zero or one.

The optimal sampler-hold $H_\mathrm{opt}S_\mathrm{opt}$ selects a series of non-overlapping frequency bands $B_i \subset [0, \infty)$ with a total length equal to the Nyquist frequency ω_N. It follows therefore that the power of u is bounded from above by[1]

$$\|H_\mathrm{opt}S_\mathrm{opt}G\|_{L^2}^2 = \frac{1}{\pi}\sum_i \int_{B_i} |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega \tag{1}$$
$$\leq \frac{\omega_\mathrm{N}}{\pi}\|G\|_{L^\infty}^2 = \frac{1}{h}\|G\|_{L^\infty}^2. \tag{2}$$

Consequently, the power of the reconstruction error $e = y - u$ is bounded from below as

$$\begin{aligned}\|G_\mathrm{e}\|_{L^2}^2 &= \|(I - H_\mathrm{opt}S_\mathrm{opt})G\|_{L^2}^2 \\ &= \|G\|_{L^2}^2 - \|H_\mathrm{opt}S_\mathrm{opt}G\|_{L^2}^2 \\ &\geq \|G\|_{L^2}^2 - \|G\|_\infty^2/h. \end{aligned}\tag{3}$$

Now let h_G be the sampling period at which (3) is zero

$$h_G := \frac{\|G\|_{L^\infty}^2}{\|G\|_{L^2}^2}.$$

This h_G is a fundamental limit in the sense that

Lemma II.1. *Error free reconstruction is impossible if the sampling period h exceeds h_G. Specifically the "signal-to-error ratio" (SER) satisfies*

$$SER := \frac{\|G\|_{L^2}^2}{\|G_e\|_{L^2}^2} \leq \frac{1}{1 - h_G/h} \qquad \forall h > h_G.$$

Proof. If $h > h_G$ then

$$\|G_\mathrm{e}\|_{L^2}^2 \geq \|G\|_{L^2}^2 - \|G\|_\infty^2/h = \|G\|_{L^2}^2(1 - h_G/h)$$

is positive. ■

The SER is the power of the signal y that we aim to reconstruct over the power of the reconstruction error e. This explains the term "SER". We see that the SER is at most 2 if $h \geq 2h_G$.

Also the L^∞ norm gives rise to limitations on perfect reconstruction. In fact, for certain values of h the L^∞ norm can not be reduced at all if $|G(\mathrm{i}\omega)|$ is not monotonically decaying. Indeed, suppose that the peak value of $|G(\mathrm{i}\omega)|$ is attained at some frequency, called resonance frequency,

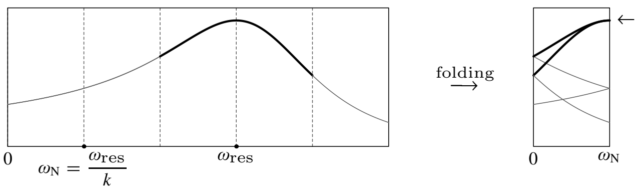
$$\omega_\mathrm{res} := \arg\max_{\omega\geq 0} |G(\mathrm{i}\omega)|$$

and that $\omega_\mathrm{res} > 0$. Suppose further that we sample at an integer fraction

$$\omega_\mathrm{N} = \frac{\omega_\mathrm{res}}{k}, \qquad \text{for some } k \in \mathbb{N}$$

[1] In fact since $H_\mathrm{opt}S_\mathrm{opt}$ selects those frequency bands B_i where $|G(\mathrm{i}\omega)|$ is maximal, it is not hard to show that the upperbound is tight in the sense that $\lim_{h\to\infty} h\|H_\mathrm{opt}S_\mathrm{opt}G\|_{L^2}^2 = \|G\|_{L^\infty}^2$ provided that $G(\mathrm{i}\omega)$ is continuous.

of this resonance frequency. Then folding of $|G(\mathrm{i}\omega)|$ reveals that the upper-envelope and the second envelope have the same peak value $\|G\|_\infty$ at either $\omega = 0$ or $\omega = \omega_\mathrm{N}$:

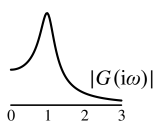


Since a single channel sampler-hold cancels only the upper envelope, the peak of $|G(\mathrm{i}\omega)|$ can not be reduced at all in this case and therefore:

Lemma II.2. *If $|G(\mathrm{i}\omega)|$ is continuous and $\omega_{res} > 0$ then sampling at rate $\omega_\mathrm{N} = \omega_{res}/k$ with $k \in \mathbb{N}$ is futile: $\|G_e\|_\infty = \|G\|_\infty$ is the best we can do and $HS = 0$ is one (of many) L^∞-optimal solutions.* △

Example II.3 (Resonance peaks)**.** Consider the second order LTI system G with resonance peak near $\omega = 1$,

$$G(\mathrm{i}\omega) = \frac{1}{(\mathrm{i}\omega + .2)^2 + 1}$$



Because of the peak, the reconstruction errors norms $\|G_\mathrm{e}\|_{L^2}$ and $\|G_\mathrm{e}\|_\infty$ need not be monotonous in the sampling period h, and indeed they are not: Fig. 3 shows the numerically computed $\|G_\mathrm{e}\|^2_{L^2}$ and $\|G\|_\infty$ as a function of h and ω_N. The reconstruction error norms converges to zero as $h \to 0$ and converge to $\|G\|_{L^2}$ and $\|G\|_\infty$ respectively as $h \to \infty$. In this example the fundamental time limit is

$$h_G = \frac{\|G\|^2_\infty}{\|G\|^2_{L^2}} = \frac{2.5^2}{125/104} = 5.2$$

exactly. As predicted, the L^∞ norm can not be reduced if $\omega_\mathrm{N} = \omega_\mathrm{res}/k \approx 1/k$, that is, if $h = k\pi/\omega_\mathrm{res} \approx k\pi$. As Fig. 3 suggests also the L^2 norm is close to a local maximum at these values. This can be interpreted as being close to pathological sampling, see [5]. △

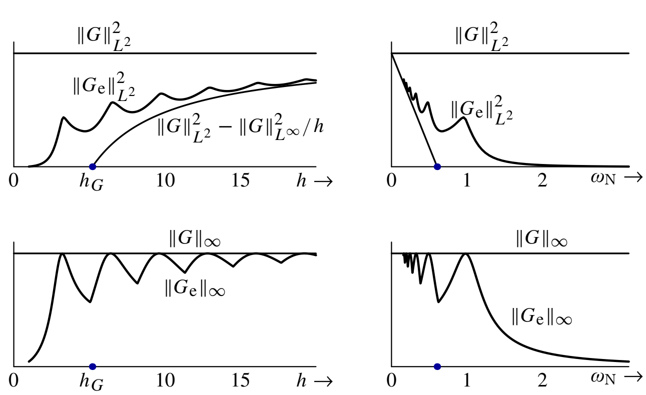


Fig. 3. Reconstruction error $\|G_\mathrm{e}\|^2_{L^2}$ (top) and $\|G_\mathrm{e}\|_\infty$ (bottom) as a function of h (left) and ω_N (right)

III. Computation of truncated L^2-norm

Finding the L^2-norm of the optimal $G_\mathrm{e} = (I - H_\mathrm{opt}S_\mathrm{opt})G$ involves computation of a finite or semi-infinite integral, $\frac{1}{\pi}\int_a^b |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega$ or $\frac{1}{\pi}\int_a^\infty |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega$. A complicating factor here is that our G may have imaginary poles. This rules out splitting of the spectrum of $G^\sim G$ using, for instance, Lyapunov equations. We have to work instead with the full $2n$-dimensional state representation of $K := G^\sim G$. To this end let $G(s) = C(sI - A)^{-1}B$ be a realization of G and define $\tilde{A}, \tilde{B}, \tilde{C}$ via

$$K := G^\sim G \overset{\mathrm{s}}{=} \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] := \left[\begin{array}{cc|c} A & 0 & B \\ -C^T C & -A^T & 0 \\ \hline 0 & B^T & 0 \end{array}\right] \tag{4}$$

and realize that $\mathrm{tr}(\tilde{C}\tilde{B}) = 0$ and that the imaginary poles of K are also imaginary poles of G.

Theorem III.1. *Suppose that $K(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$ with $\tilde{A}, \tilde{B}, \tilde{C}$ real and that $\mathrm{tr}(\tilde{C}\tilde{B}) = 0$. Then*

$$\int_{\omega_\mathrm{N}}^\infty \mathrm{tr}\, K(\mathrm{i}\omega)\,\mathrm{d}\omega = \mathrm{i}\,\mathrm{tr}(\tilde{C}\log(\omega_\mathrm{N} I - \tilde{A}/\mathrm{i})\tilde{B}) \tag{5}$$

provided that $\omega_\mathrm{N} > \omega_\mathrm{max} := \max|\omega_k|$ where the maximum is taken over all imaginary eigenvalues $\mathrm{i}\omega_k$ of $\tilde{A}$.

This theorem and other results in the section are proved in the appendix of this paper. The proof relies on elementary properties of the principal logarithm as documented in [2]. The result appears intuitive because

$$K(\mathrm{i}\omega) = -\mathrm{i}\tilde{C}(\omega I - \tilde{A}/\mathrm{i})^{-1}\tilde{B} \tag{6}$$

and its anti-derivative, motivated by the scalar case, is

$$\int K(\mathrm{i}\omega) = -\mathrm{i}\tilde{C}\log(\omega I - \tilde{A}/\mathrm{i})\tilde{B}. \tag{7}$$

There are however some points in the proof that are easily overlooked. In particular, the following: from a systems theoretic perspective one might prefer not to extract the factor i in $K(\mathrm{i}\omega)$ and use instead $K(\mathrm{i}\omega) = \tilde{C}(\mathrm{i}\omega I - \tilde{A})^{-1}\tilde{B}$. This wrongly suggests that $-\mathrm{i}\tilde{C}\log(\mathrm{i}\omega I - \tilde{A})\tilde{B}$ is a valid anti-derivative of $K(\mathrm{i}\omega)$ on $(\omega_\mathrm{max}, \infty)$. It is generally wrong because as ω varies in $(\omega_\mathrm{N}, \infty)$ some eigenvalues of $\mathrm{i}\omega I - \tilde{A}$ may cross the branch cut (the negative real axis) which makes the candidate antiderivative discontinuous (and wrong). Extracting i from the realization of $K(\mathrm{i}\omega)$ as done in (6) avoids this problem and in addition it has the advantage that the corresponding anti-derivative (7) is normalized to have zero trace at $\omega_\mathrm{N} = +\infty$ because $\mathrm{tr}(\tilde{C}\tilde{B}) = 0$, see the appendix.

The condition that $\mathrm{tr}(\tilde{C}\tilde{B}) = 0$ is necessary and sufficient for $\int_{\omega_\mathrm{N}}^\infty \mathrm{tr}\, K(\mathrm{i}\omega)\,\mathrm{d}\omega$ to exist. For SISO systems this is equivalent to $K(s)$ having relative degree 2 or more.

In the remaining subsections we summarize some minor extensions and special cases.

A. Proper $K(s)$

If the relative degree of $K(s)$ is not 2 or more, then the indefinite integral in (5) typically does not exist. A finite integral may still exist though. We formulate the result for proper K.

Lemma III.2. *Let $K(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$ be a realization with $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ real matrices. Then*

$$\begin{aligned}\int_{\omega_\mathrm{N}}^{\omega_\mathrm{F}} \operatorname{tr} K(\mathrm{i}\omega)\,\mathrm{d}\omega &= -\mathrm{i}\operatorname{tr}\tilde{C}[\log(\omega_\mathrm{F} I - \tilde{A}/\mathrm{i}) - \log(\omega_\mathrm{N} I - \tilde{A}/\mathrm{i})]\tilde{B} \\ &\quad + \operatorname{tr}(\tilde{D})(\omega_\mathrm{F} - \omega_\mathrm{N})\end{aligned}$$

as long as $\omega_\mathrm{N}, \omega_\mathrm{F} > \omega_{\max} := \max|\omega_k|$ where the maximum is taken over all imaginary eigenvalues $\mathrm{i}\omega_k$ of $\tilde{A}$.

In this finite case (still with $\omega_\mathrm{N}, \omega_\mathrm{F} > \omega_{\max}(A)$) the two logarithms can be combined into one $\log(\Omega)$ with

$$\begin{aligned}\Omega &:= (\omega_\mathrm{F} I - A/\mathrm{i})(\omega_\mathrm{N} I - A/\mathrm{i})^{-1} \\ &= (\mathrm{i}\omega_\mathrm{F} I - A)(\mathrm{i}\omega_\mathrm{N} I - A)^{-1}\end{aligned}$$

This is proved in Thm. IV.3. So

$$\int_{\omega_\mathrm{N}}^{\omega_\mathrm{F}} \operatorname{tr} K(\mathrm{i}\omega)\,\mathrm{d}\omega = -\mathrm{i}\operatorname{tr}(\tilde{C}\log(\Omega)\tilde{B} + \tilde{D}(\omega_\mathrm{F} - \omega_\mathrm{N}))$$

with Ω as defined above.

B. Stable A matrix

If the A matrix of G is stable then the computational effort can be further reduced and connections with Lyapunov and the classic L^2-norm can be established. It is a classic result that the squared L_2-norm

$$\|G\|_{L^2}^2 := \frac{1}{\pi}\operatorname{tr}\int_0^\infty G^\sim(\mathrm{i}\omega)G(\mathrm{i}\omega)\,\mathrm{d}\omega$$

of a stable finite dimensional system $G(s) = C(sI - A)^{-1}B$ can be computed via the solution of a linear equation. Specifically, if A is stable then

$$\|G\|_{L^2}^2 = \operatorname{tr}(B^T P B) \tag{8}$$

where P is the unique solution of the linear Lyapunov equation

$$A^T P + PA = -C^T C, \tag{9}$$

see e.g. [4, Lemma 2.1]. Now for given $\omega_\mathrm{N} \geq 0$, the squared truncated norm

$$\|G\|_{\omega_\mathrm{N}}^2 := \frac{1}{\pi}\operatorname{tr}\int_{\omega_\mathrm{N}}^\infty G^\sim(\mathrm{i}\omega)G(\mathrm{i}\omega)\,\mathrm{d}\omega \tag{10}$$

according to Theorem III.1 equals

$$\frac{\mathrm{i}}{\pi}\operatorname{tr}\tilde{C}\log(\omega_\mathrm{N} I - \tilde{A}/\mathrm{i})\tilde{B}.$$

This entails computation of a logarithm of a $2n \times 2n$ Hamiltonian matrix. Given the stability of A one can, if desired, reduce the computational burden somewhat.

Theorem III.3. *Suppose G is stable and strictly proper and let*

$$G(s) = C(sI - A)^{-1}B \tag{11}$$

be a realization with A, B, C are real matrices and A stable. Then (10) *equals*

$$\begin{aligned}\|G\|_{\omega_\mathrm{N}}^2 &= -\frac{2}{\pi}\operatorname{Im}\operatorname{tr}(B^T P\log(\omega_\mathrm{N} I - A/\mathrm{i})B) &(12)\\ &= \|G\|_{L^2}^2 - \frac{2}{\pi}\operatorname{Im}\operatorname{tr}(B^T P\log(\mathrm{i}\omega_\mathrm{N} I - A)B) &(13)\end{aligned}$$

where P is the unique solution of (9).

Stability of A in Theorem III.3 is exploited in two different ways: (a) then a solution of the Lyapunov equation (9) is guaranteed to exist, and (b) then the eigenvalues of the matrices $\omega I - A/\mathrm{i}$ and $\mathrm{i}\omega I - A$ whose logarithm we take are not on the branch cut (the negative real axis, including zero). This holds irrespective of the choice of $\omega_\mathrm{N} \in \mathbb{R}$.

For $\omega_\mathrm{N} = 0$ one recovers (8). Indeed for $\omega_\mathrm{N} = 0$, Equation (12) reduces to

$$\begin{aligned}\|G\|_{\omega_\mathrm{N}=0}^2 &= \frac{-2}{\pi}\operatorname{tr}\big(B^T P[\operatorname{Im}\log(-A/\mathrm{i})]B\big) \\ &= \frac{-2}{\pi}\operatorname{tr}\left((B^T P[\frac{-\pi}{2}I]B\right) \\ &= \operatorname{tr}\left(B^T PB\right).\end{aligned}$$

Here we used Thm. IV.2 of the appendix, which states that $\operatorname{Im}(\log(-A/\mathrm{i})) = -\frac{\pi}{2}I$ for every stable $A \in \mathbb{R}^{n\times n}$.

C. Stable A matrix, finite interval of integration

If $G(s)$ is proper but not strictly proper, $G(s) = C(sI - A)^{-1}B + D$ with $D \neq 0$, then the indefinite (10) is typically not defined, but the finite integral

$$\frac{1}{\pi}\operatorname{tr}\int_{\omega_\mathrm{N}}^{\omega_\mathrm{F}} G^\sim(\mathrm{i}\omega)G(\mathrm{i}\omega)\,\mathrm{d}\omega \tag{14}$$

does exist. In this case, (14) can be computed as

$$\begin{aligned}&\frac{2}{\pi}\operatorname{Im}\operatorname{tr}(R[\log(\omega_\mathrm{N} I - A/\mathrm{i}) - \log(\omega_\mathrm{F} I - A/\mathrm{i})]B) \\ &\quad + \frac{1}{\pi}\operatorname{tr}(D^T D)(\omega_\mathrm{F} - \omega_\mathrm{N}) &(15)\end{aligned}$$

where $R := B^T P + D^T C$. Once again the above two logarithms can be combined into one $\log(\Omega)$ with $\Omega := (\omega_\mathrm{N} I - A/\mathrm{i})(\omega_\mathrm{F} I - A/\mathrm{i})^{-1}$ Hence, (14) can be computed as

$$\begin{aligned}&\frac{2}{\pi}\operatorname{Im}\operatorname{tr}(R(\log((\omega_\mathrm{N} I - A/\mathrm{i})(\omega_\mathrm{F} I - A/\mathrm{i})^{-1}))B) \\ &\quad + \frac{1}{\pi}\operatorname{tr}(D^T D)(\omega_\mathrm{F} - \omega_\mathrm{N}). &(16)\end{aligned}$$

IV. Appendix: Principal logarithms and proofs

In this appendix we collect some basic properties of principal logarithms and proofs of the results of Section III. The logarithm log always refers to the principal logarithm [2, Thm. 1.31].

A. Basic properties of Principal Logarithm

All proofs are done by using matrix function definition via Jordan canonical form [2, Dn. 1.2]. Let the Jordan canonical form of the matrix $A/\mathrm{i} \in \mathbb{C}^{n\times n}$ be given by,

$$\frac{A}{\mathrm{i}} = Z\,\mathrm{diag}(J_1, J_2, \ldots, J_p)Z^{-1} \qquad (17)$$

where $Z \in \mathbb{C}^{n\times n}$, J_k is the kth Jordan block with eigenvalue λ_k [2, Eqn. (1.2)]. Also let $\sigma(A)$ denote the set of eigenvalues of A.

Theorem IV.1. *Suppose $A \in \mathbb{C}^{n\times n}$ and let $\omega_{max} := \max_k \omega_k$ where the maximum is taken over all imaginary eigenvalues $\mathrm{i}\omega_k$ of A. Then*

1) $\log(\omega I - A/\mathrm{i})$ *is analytic in* $\omega \in (\omega_{\max}, \infty)$ *and*
$$\frac{d}{d\omega}\log(\omega I - A/\mathrm{i}) = (\omega I - A/\mathrm{i})^{-1}$$
for all $\omega \in (\omega_{\max}, \infty)$
2) $\lim_{\omega\to\infty} \log(\omega I - A/\mathrm{i}) - \log(\omega)I = 0$
3) $\lim_{\omega\to\infty} \mathrm{Im}\,(\log(\omega I - A/\mathrm{i})) = 0$

Proof.

1) Let λ_k be eigenvalue of Jordan block $J_k \in \mathbb{C}^{m_k\times m_k}$. Using (17), [2, Dn 1.2], and [2, Eqn. (1.34)], we have
$$\log(\omega I - A/\mathrm{i}) = Z\,\mathrm{diag}(L_1(\omega), \ldots, L_p(\omega))Z^{-1},$$
where $L_k(\omega) \in \mathbb{C}^{m_k\times m_k}$ is given by,
$$\begin{bmatrix} \log(\omega-\lambda_k) & -(\omega-\lambda_k)^{-1} & \cdots & -\frac{(\omega-\lambda_k)^{-(m_k-1)}}{m_k-1} \\ & \log(\omega-\lambda_k) & \ddots & \vdots \\ & & \ddots & -(\omega-\lambda_k)^{-1} \\ & & & \log(\omega-\lambda_k) \end{bmatrix}.$$
Clearly $\log(\omega-\lambda_k)$ and $(\omega-\lambda_k)^j$ for any $j \in \{1, 2, \ldots\}$ is analytic for $\omega \in (\omega_{\max}, \infty)$. Therefore,
$$\frac{d}{d\omega}\log(\omega I - A/\mathrm{i}) = Z\,\mathrm{diag}(F_1(\omega), \ldots, F_p(\omega))Z^{-1}$$
where $F_k(\omega)$ is given by,
$$\begin{bmatrix} (\omega-\lambda_k)^{-1} & (\omega-\lambda_k)^{-2} & \cdots & (\omega-\lambda_k)^{-m_k} \\ & (\omega-\lambda_k)^{-1} & \ddots & \vdots \\ & & \ddots & (\omega-\lambda_k)^{-2} \\ & & & (\omega-\lambda_k)^{-1} \end{bmatrix}.$$
The result now follows because $F_k(\omega)(\omega I - J_k) = I_{m_k}$, where I_{m_k} is identity matrix of size m_k.
2) Since $\lim_{\omega\to\infty} \log(\omega - \lambda_k) - \log(\omega) = 0$ and $\lim_{\omega\to\infty}(\omega-\lambda_k)^{-j} = 0$ for $j \in \{1, 2, \ldots\}$, we have that $\lim_{\omega\to\infty} L_k(\omega) - \log(\omega)I_{m_k} = 0$.
3) As the imaginary part of $\log(\omega)I$ is zero, we have that $\lim_{\omega\to\infty}\mathrm{Im}(L_k(\omega))$ equals $\lim_{\omega\to\infty}\mathrm{Im}(L_k(\omega) - \log(\omega)I_{m_k}) = 0$.

■

Theorem IV.2. *Given a stable matrix $A \in \mathbb{C}^{n\times n}$, then*

1) $\mathrm{Im}(\log(-A/\mathrm{i})) = \mathrm{Im}(\log(-A)) - \frac{\pi}{2}I$
2) *If* $A \in \mathbb{R}^{n\times n}$, *then* $\mathrm{Im}(\log(-A/\mathrm{i})) = -\frac{\pi}{2}I$

Proof. Let $\omega \in \mathbb{C}$ and $\mathrm{Re}(\omega) < 0$. Then $-\omega$ has positive real part and $-\omega/\mathrm{i}$ negative imaginary part. Considering that the branch cut of the principal logarithm is the negative real axis, we get

$$\log(-\omega/\mathrm{i}) = \log(-\omega) - \mathrm{i}\frac{\pi}{2}.$$

Since A is a stable matrix, $\log(-A)$ exists. Therefore, using [2, Thm. 1.15a],

$$\begin{aligned}\mathrm{Im}(\log(-A/\mathrm{i})) &= \mathrm{Im}(\log(-A) - \mathrm{i}\frac{\pi}{2}I) \\ &= \mathrm{Im}(\log(-A)) - \frac{\pi}{2}I.\end{aligned}$$

If $A \in \mathbb{R}^{n\times n}$, then $\mathrm{Im}(\log(-A)) = 0$ [2, Thm 1.18c]. ■

Theorem IV.3. *Given a matrix $A \in \mathbb{C}^{n\times n}$ and $\omega_{\mathrm{N}}, \omega_{\mathrm{F}} \in (\omega_{\max}, \infty)$, where $\omega_{\max} := \max\{\sigma(A/\mathrm{i}) \cap \mathbb{R}\}$ then*

$$\begin{aligned}&\log\left((\omega_{\mathrm{F}} I - A/\mathrm{i})(\omega_{\mathrm{N}} I - A/\mathrm{i})^{-1}\right) \\ &= \log(\omega_{\mathrm{F}} I - A/\mathrm{i}) - \log(\omega_{\mathrm{N}} I - A/\mathrm{i}).\end{aligned}$$

Proof. Let λ denote an eigenvalue of A. Since product of the matrices $(\omega_{\mathrm{F}} I - A/\mathrm{i})$ and $(\omega_{\mathrm{N}} I - A/\mathrm{i})^{-1}$ commutes and $|\arg(\omega_{\mathrm{F}} - \lambda/\mathrm{i}) + \arg((\omega_{\mathrm{N}} - \lambda/\mathrm{i})^{-1})| < \pi$ if $\omega_{\mathrm{N}}, \omega_{\mathrm{F}} \in (\omega_{\max}, \infty)$, the result follows from [2, Thm 11.(2,3)]. ■

B. Proofs for Section III

Proof of Theorem III.1. For $\omega \in (\omega_{\mathrm{N}}, \infty)$ the matrix $\omega I - \tilde{A}/\mathrm{i}$ has no eigenvalues on the branch cut (negative real axis) because $\omega > \omega_{\max}$. So the principal logarithm $\log(\omega I - \tilde{A}/\mathrm{i})$ exists for all such ω [2, Thm 1.31] and it is the antiderivative of $(\omega I - \tilde{A}/\mathrm{i})^{-1}$ (Thm. IV.1.(1)). Using Thm. IV.1.(2) and the fact that $\mathrm{tr}(\tilde{C}\tilde{B}) = 0$ we now obtain

$$\begin{aligned}&\int_{\omega_{\mathrm{N}}}^{\infty} \mathrm{tr}\,K(\mathrm{i}\omega)\,\mathrm{d}\omega \\ &= -\mathrm{i}\int_{\omega_{\mathrm{N}}}^{\infty} \mathrm{tr}\,\tilde{C}(\omega I - \tilde{A}/\mathrm{i})^{-1}\tilde{B}\,\mathrm{d}\omega \\ &= \mathrm{i}\,\mathrm{tr}(\tilde{C}\log(\omega_{\mathrm{N}} I - \tilde{A}/\mathrm{i})\tilde{B}) - \mathrm{i}\lim_{\omega\to\infty}\log(\omega)\,\mathrm{tr}(\tilde{C}\tilde{B}) \\ &= \mathrm{i}\,\mathrm{tr}(\tilde{C}\log(\omega_{\mathrm{N}} I - \tilde{A}/\mathrm{i})\tilde{B}).\end{aligned}$$

■

Proof of Lemma III.2 and following statement. The proof of the lemma is entirely similar to that of Theorem III.1. The statement following the lemma is a consequence of Theorem IV.3. ■

Proof of Theorem III.3. With P the solution of (9) we can split $G^{\sim}G$ as

$$G^{\sim}G = H + H^{\sim} \qquad (18)$$

with $H(s) = B^T P(sI - A)^{-1}B$, see e.g. [4, proof of Lemma 12.8]. Now the antiderivative of $H(\mathrm{i}\omega)$ with respect to ω (see Theorem IV.1.1) is

$$\int H(\mathrm{i}\omega) = B^T P \int (\mathrm{i}\omega I - A)^{-1}B$$

$$= -\mathrm{i}B^T P \log(\omega I - A/\mathrm{i})B + \text{constant}.$$

Since $H^{\sim}(\mathrm{i}\omega)$ is the complex conjugate transpose of $H(\mathrm{i}\omega)$ we thus have (up to a constant)

$$\begin{aligned}\operatorname{tr}\int G^{\sim}(\mathrm{i}\omega)G(\mathrm{i}\omega) &= 2\operatorname{Re}\operatorname{tr}\int H(\mathrm{i}\omega)\\ &= 2\operatorname{Re}\operatorname{tr}(-\mathrm{i}B^T P\log(\omega I - A/\mathrm{i})B)\\ &= 2\operatorname{Im}\operatorname{tr}(B^T P\log(\omega I - A/\mathrm{i})B).\end{aligned}$$

Therefore, using Theorem IV.1.3

$$\begin{aligned}\pi\,||G||^2_{\omega_\mathrm{N}} &= 2\operatorname{Im}\operatorname{tr}(B^T P\log(\omega I - A/\mathrm{i})B)\Big|_{\omega_\mathrm{N}}^{\infty}\\ &= 2\operatorname{Im}\operatorname{tr}(-B^T P\log(\omega_\mathrm{N} I - A/\mathrm{i})B).\end{aligned} \tag{19}$$

Note that $-(\mathrm{i}\omega_\mathrm{N} I - A)$ is stable for every $\omega_\mathrm{N} \in \mathbb{R}$, therefore, using Theorem IV.2.1, (19) can also be written as,

$$\begin{aligned}\pi\,||G||^2_{\omega_\mathrm{N}} =& 2\operatorname{tr}(\frac{\pi}{2}B^T PB)\\ &- \operatorname{Im}\operatorname{tr}(B^T P\log(\mathrm{i}\omega_\mathrm{N} I - A)B).\end{aligned} \tag{20}$$

■

Proof statements Section III-C. Formula (15) follows form standard manipulation. The claim that the two logarithms can be combined into one again follows from Thm. IV.3. ■

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