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Sampled-Data Stabilization of a Class of Parabolic Systems

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***Abstract*— A semilinear scalar heat equation with the control input in the right-hand side, coupled to the homogenous Dirichlet or Neumann boundary conditions, is considered. Such a system represents a class of reaction-diffusion equations that model many physical phenomena. It is well-known that this system is stabilizable by a linear infinite-dimensional state-feedback. For realistic design, finite-dimensional discrete version realizations may be applied leading to local results.**

In the present paper we suggest a sampled-data controller design, where the sampled-data (in time) measurements of the state are taken in the finite number of fixed (sampled) spatial variables. It is assumed that the sampling in time and the sampling in space (i.e. the distance between the consequently sampled spatial variables) are bounded. Our sampled-data feedback with a constant gain is piecewise-constant in time and in space. Sufficient conditions for the exponential stabilization are derived in terms of Linear Matrix Inequalities (LMIs) depending on the controller gain. By solving these LMIs, upper bounds on the sampling in time and on the sampling in space are found that preserve the exponential stability. The results are extended to the sampled-data in space and to delayed in time sampled-data measurements. A numerical example illustrates the efficiency of the method.

I. INTRODUCTION

In the present paper we consider the following semilinear scalar heat equation

$$z_{\mathbf{t}}(x,t) = az_{\mathbf{xx}}(x,t) + \beta(z(x,t),x,t)z(x,t) + u(x,t), \quad x \in [0,\pi], \tag{1}$$

coupled to the Dirichlet

$$z(0,t) = z(\pi,t) = 0, \tag{2}$$

or to the Neumann

$$z_x(0,t) = z_x(\pi,t) = 0, \tag{3}$$

boundary conditions, where subindexes denote the corresponding partial derivatives and where $a > 0$, β is a smooth and bounded function $|\beta| \leq \beta_0$, $u(x,t)$ is control input. System (1) represents a class of reaction-diffusion equations that model many physical phenomena. Examples are numerous and among others include the problem of compressor rotating stall [9] with air injection actuator, where $z(x,t)$ denotes the axial flow through the compressor.

It is well-known that an open-loop system (1), (2) is unstable if $\beta_0 > a$ and that a linear infinite-dimensional feedback $u(x,t) = -Kz(x,t)$ with big enough $K > 0$ exponentially stabilizes the system [2]. For realistic design, finite-dimensional realizations [1], [14] may be applied. However, finite-dimensional control, which employs e.g. Galerkin truncation, leads to local results [14].

Also mobile collocated sensors and actuators (see [3] and references therein) or adaptive controllers [12] can be used. The latter methods are not easy to implement. In [9] the control input has been designed to enter the heat equation through a finite number of shape functions (e.g. step functions) and their respective amplitude values. Sufficient conditions has been derived then for the global stabilization of the infinite-dimensional dynamics. Sampled-data control of general linear infinite-dimensional systems has been studied in [13]. A Linear Matrix Inequalities (LMI) approach has been recently introduced for some classes of distributed parameter systems [6], [7]. This approach allows to derive simple finite-dimensional sufficient conditions for different robust control problems.

In the present paper we suggest a sampled-data controller design, where the sampled-data (in time) measurements of the state are taken in the finite number of fixed (sampled) spatial variables. It is assumed that the sampling in time and the sampling in space (i.e. the distance between the consequent sampled spatial variables) are bounded. Our sampled-data controller with a constant gain is piecewise-constant in time and in space. It can be implemented by a finite number of stationary actuators in the form of step functions and by zero-order hold devices. Sufficient conditions for the exponential stabilization are derived in terms of LMIs depending on the controller gain. By solving these LMIs, upper bounds on the sampling in time and on the sampling in space are found that preserve the exponential stability. The results are extended to the sampled-data in space and to delayed in time sampled-data measurements.

II. PROBLEM FORMULATION

We will formulate the results for the Dirichlet and for the Neumann conditions, but the results will be the same also for the mixed Dirichlet-Neumann boundary conditions. Our objective is to design an exponentially stabilizing sampled-data in space and in time controller:

$$u(x,t) = -Kz(\bar{x}_j, t_k), \ x_j \leq x < x_{j+1}, \ \bar{x}_j = \tfrac{x_{j+1}+x_j}{2}, \quad j = 0, ..., N-1, \ t \in [t_k, t_{k+1}), \ k = 0,1,2... \tag{4}$$

where $0 = x_0 < x_1 < ... < x_N = \pi$, $0 = t_0 < t_1 < ... < t_k ...$ The sampling in time and in space may be variable but bounded

$$0 < h_0 \leq t_{k+1} - t_k \leq h, \ x_{j+1} - x_j \leq \Delta. \tag{5}$$

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The closed-loop system (1), (4) has the form:

$$\begin{array}{l} z_{\mathbf{t}}(x,t) = az_{\mathbf{xx}}(x,t) + \beta(z(x,t),x,t)z(x,t) - Kz(\bar{x}_j,t_k), \\ x_j \le x < x_{j+1},\ j=0,...,N-1, \\ t \in [t_k,t_{k+1}),\ k=0,1,2... \end{array} \tag{6}$$

Our results will be based on the following relation:

$$z(\bar{x}_j,t_k) = z(x,t_k) - \int_{\bar{x}_j}^{x} z_\zeta(\zeta,t_k)d\zeta, \tag{7}$$

Then (6) can be presented as

$$\begin{array}{l} z_t(x,t) = az_{xx}(x,t) + \beta(z(x,t),x,t)z(x,t) \\ -K[z(x,t_k) - \int_{\bar{x}_j}^{x} z_\zeta(\zeta,t_k)d\zeta], \\ x_j \le x < x_{j+1},\ j=0,...,N-1, \\ t\in[t_k,t_{k+1}),\ k=0,1,2... \end{array} \tag{8}$$

We will consider also more general controllers

$$\begin{array}{l} u(x,t) = -Kz(\bar{x}_j,t_k-\eta_k), \\ x_j \le x < x_{j+1},\ j=0,...,N-1, \\ t\in[t_k,t_{k+1}),\ k=0,1,2..., \\ u(x,t)=0, \quad t<0, \end{array} \tag{9}$$

where $\eta_k \in [\eta_m,\eta_M]$ is an additional (e.g. actuator) time-delay. Representing $t_k - \eta_k = t - \tau(t)$, where $\tau(t) = t - t_k + \eta_k$, we have $\tau(t)\in[\eta_m,\tau_M]$ with $\tau_M = h+\eta_M$. The closed-loop system (1), (9) can be represented as:

$$\begin{array}{l} z_t(x,t) = az_{\mathbf{xx}}(x,t) + \beta(z(x,t),x,t)z(x,t) \\ -K[z(x,t-\tau(t)) - \int_{\bar{x}_j}^{x} z_\zeta(\zeta,t-\tau(t))d\zeta], \\ x_j \le x < x_{j+1},\ j=0,...,N-1,\ t\ge 0,\ \tau(t)\in[\eta_m,\tau_M], \\ z(x,t)=0, \quad t<0. \end{array} \tag{10}$$

The following generalized Halanay's and Wirtinger's inequalities will be useful:

Lemma 1: [10] (Halanay's inequality) Let $0<\delta_1<2\delta$ and let $V:[t_0-h,\infty)\to[0,\infty)$ be an absolutely continuous function that satisfies

$$\dot{V}(t) \le -2\delta V(t) + \delta_1 \sup_{-h\le\theta\le 0} V(t+\theta),\ t\ge t_0. \tag{11}$$

Then

$$V(t) \le e^{-2\gamma(t-t_0)} \sup_{-h\le\theta\le 0} V(t_0+\theta), \quad t\ge t_0, \tag{12}$$

where $\gamma>0$ is a unique positive solution of

$$\gamma = \delta - \frac{\delta_1 e^{2\gamma h}}{2}. \tag{13}$$

Lemma 2: [11] (Wirtinger's inequality). Let $z \in W^{1,2}([a,b],R)$ be a scalar function with $z(a)=0$ or $z(b)=0$. Then

$$\int_a^b z^2(\xi)d\xi \le \frac{4(b-a)^2}{\pi^2}\int_a^b \dot{z}^2(\xi)d\xi. \tag{14}$$

Moreover, if $z(a)=z(b)=0$, then

$$\int_a^b z^2(\xi)d\xi \le \frac{(b-a)^2}{\pi^2}\int_a^b \dot{z}^2(\xi)d\xi. \tag{15}$$

If additionally $z\in W^{2,2}([a,b],R)$, and $z(a)=z(b)=0$ then

$$\int_a^b \dot{z}^2(\xi)d\xi \le \frac{(b-a)^2}{\pi^2}\int_a^b \ddot{z}^2(\xi)d\xi. \tag{16}$$

III. LMI Conditions for Exponential Stabilization

In [6] a Lyapunov functional of the form

$$\begin{array}{l} V(t) = (p_1 - ap_3)\int_0^\pi z^2(x,t)dx + p_3 a\int_0^\pi z_x^2(x,t)dx \\ +\int_0^\pi[\tau_M r\int_{-\tau_M}^0\int_{t+\theta}^t e^{2\delta(s-t)}z_s^2(x,s)dsd\theta \\ +s\int_{t-\tau_M}^t e^{2\delta(s-t)}z^2(x,s)ds]dx \end{array} \tag{17}$$

with some constants $p_3>0$, $p_1>0$, $r\ge 0$ and $s\ge 0$ was introduced for stability analysis of the heat equation with time-delay

$$\begin{array}{l} z_t(x,t) = az_{xx}(x,t) + \beta(z(x,t),x,t)z(x,t) - Kz(x,t-\tau(t)), \\ t\ge 0,\ \tau(t)\in[\eta_m,\tau_M]. \end{array} \tag{18}$$

The above Lyapunov functional cannot be applied to (10) because of the additional time-delayed term $K\int_{\bar{x}_j}^{x} z_\zeta(\zeta,t-\tau(t))d\zeta$. Our main result will be based on application of Halanay's inequality and on the modified version of (17).

A. Sampled in spatial variable controller

We will start with the preliminary result, where we consider stabilization via sampled-data in space and continuous in time controller

$$u(x,t) = -Kz(\bar{x}_j,t),\ x_j\le x<x_{j+1},\ j=0,...,N-1, \tag{19}$$

The closed-loop system (1), (19) can be represented as:

$$\begin{array}{l} z_t(x,t) = az_{xx}(x,t) + \beta(z(x,t),x,t)z(x,t) \\ -K[z(x,t) - \int_{\bar{x}_j}^{x} z_\zeta(\zeta,t)d\zeta], \\ x_j\le x<x_{j+1},\ j=0,...,N-1,\ t\ge 0, \tau\in[0,\tau_M], \\ z(x,t)=0, \quad t<0. \end{array} \tag{20}$$

For the exponential stability analysis we apply the following Lyapunov functional

$$V(t) = \int_0^\pi z_x^2(x,t)dx. \tag{21}$$

Differentiating V we find

$$\dot{V} + 2\delta V = 2\int_0^\pi z_x(x,t)z_{xt}(x,t)dx + 2\delta\int_0^\pi z_x^2(x,t)dx. \tag{22}$$

Integrating by parts and using the boundary conditions, we have

$$\begin{array}{l} 2\int_0^\pi z_x(x,t)z_{xt}(x,t)dx = -2\int_0^\pi z_t(x,t)z_{xx}(x,t)dx \\ = -2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} z_{xx}(x,t)[az_{xx}(x,t) + (\beta-K)z(x,t) \\ +K\int_{\bar{x}_j}^{x} z_\zeta(\zeta,t)d\zeta)]dx. \end{array} \tag{23}$$

Integrating by parts we find

$$\begin{array}{l} 2(K-\beta)\int_0^\pi z_{xx}(x,t)z(x,t)dx = \\ -2(K-\beta)\int_0^\pi z_x^2(x,t)dx. \end{array} \tag{24}$$

By Young's inequality, for any scalar $\bar{R} > 0$ the following holds:

$$-2K\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[z_{xx}(x,t)\int_{\bar{x}_j}^{x} z_\xi(\xi,t)d\xi]dx \le K\bar{R}\int_0^\pi z_{xx}^2(x,t)dx + K\bar{R}^{-1}\sum_{j=0}^{N-1}\int_{\bar{x}_j}^{x_{j+1}}(\int_{\bar{x}_j}^{x} z_\xi(\xi,t)d\xi)^2dx. \tag{25}$$

Since $\int_{\bar{x}_j}^{x} z_\zeta(\zeta,t)d\zeta_{|x=\bar{x}_j} = 0$, application of Wirtinger's inequality (14) yields

$$\int_{x_j}^{x_{j+1}}(\int_{\bar{x}_j}^{x} z_\xi(\xi,t)d\xi)^2dx = \int_{x_j}^{\bar{x}_j}(\int_{\bar{x}_j}^{x} z_\xi(\xi,t)d\xi)^2dx + \int_{\bar{x}_j}^{x_{j+1}}(\int_{\bar{x}_j}^{x} z_\xi(\xi,t)d\xi)^2dx \le \frac{\Delta^2}{\pi^2}\int_{x_j}^{x_{j+1}} z_x^2(x,t)dx. \tag{26}$$

Denoting $\bar{R} = \frac{\Delta}{\pi}R$, we conclude from (22)-(26) that

$$\dot{V} + 2\delta V \le (\tfrac{\Delta}{\pi}RK - 2a)\int_0^\pi z_{xx}^2(x,t)dx + (\tfrac{\Delta}{\pi}R^{-1}K + 2\delta - 2(K-\beta))\int_0^\pi z_x^2(x,t)dx \le 0 \tag{27}$$

if (by Wirtinger inequality (16))

$$\begin{array}{l}\frac{\Delta}{\pi}RK - 2a \le 0,\\ \frac{\Delta}{\pi}RK - 2a + \frac{\Delta}{\pi}R^{-1}K + 2\delta - 2(K-\beta_0) \le 0.\end{array} \tag{28}$$

Inequality (27) yields

$$V(t) = \int_0^\pi z_x^2(x,t)dx \le e^{-2\delta t}V(0) = e^{-2\delta t}\int_0^\pi z_x^2(x,0)dx, \tag{29}$$

and, thus, (20) under the homogenous Dirichlet or Neumann boundary conditions is exponentially stable with the decay rate δ. We proved the following

Proposition 1: Given Δ, β_0 and R, let there exist δ and K such that linear scalar inequalities (27) are feasible. Then solutions to (20) under the Dirichlet boundary conditions (2) (or under the Neumann boundary conditions (3)) satisfy the inequality (29). Moreover, if $K \le \frac{2a\pi}{\Delta}$, then the system is exponentially stable with a small enough $\delta < 0$ provided the second inequality (28) is satisfied with $R = 1$.

B. The sampled-data in time and in space controller (4)

Our result in this section will be based on the following modification of Halanay's Lemma:

Lemma 3: Let $0 < \delta_1 < 2\delta$ and let $V : [t_0,\infty) \to [0,\infty)$ be continuous from the right and absolutely continuous for $t \ne t_k$, where $t_{k+1} - t_k \le h$. Assume that V satisfies

$$\lim_{t\to t_k^-} V(t) \ge V(t_k). \tag{30}$$

Let

$$\dot{V}(t) \le -2\delta V(t) + \delta_1 V(t_k),\ t \in [t_k, t_{k+1}). \tag{31}$$

Then

$$V(t) \le e^{-2\gamma(t-t_0)}V(t_0), \quad \forall t \ge t_0, \tag{32}$$

where γ is a unique positive solution of (13).

Proof: Let $V : [t_k, t_{k+1}) \to [0,\infty)$ satisfy (31) on $[t_k, t_{k+1})$ and denote by $\bar{V}(t)$ its continuation on $[t_k - h, t_k]$ defined by $\bar{V}(t) = V(t_k),\ t \in [t_k - h, t_k]$ and $\bar{V}(t) = V(t),\ t \in [t_k, t_{k+1})$. We have $\bar{V}(t_k) \le \sup_{-h\le\theta\le0}\bar{V}(t+\theta),\ t \in [t_k, t_{k+1})$ and thus

$$\dot{\bar{V}}(t) \le -2\delta\bar{V}(t) + \delta_1\sup_{-h\le\theta\le0}\bar{V}(t+\theta),\ t \in [t_k, t_{k+1}). \tag{33}$$

Then, by Halanay's inequality (11), $\bar{V}(t) = V(t) \le e^{-2\gamma(t-t_k)}V(t_k)\ \forall t \in [t_k, t_{k+1})$, where $\gamma > 0$ satisfies (13). Taking into account (30) and using the same arguments for $t \in [t_i, t_{i-1}), i \le k$, we obtain further

$$\begin{array}{l} e^{-2\gamma(t-t_k)}V(t) \le V(t_k) \le e^{-2\gamma(t-t_k)}V(t_k^-)\\ \le e^{-2\gamma(t-t_{k-1})}V(t_{k-1})\\ \le \ldots \le e^{-2\gamma(t-t_0)}V(t_0),\ t \in [t_k, t_{k+1}).\end{array}$$

■

For the exponential stability analysis of (1)-(4) we suggest to apply the following Lyapunov functional

$$\begin{array}{l} V(t) = (p_1 - \alpha a p_3)\int_0^\pi z^2(x,t)dx\\ + \int_0^\pi[ap_3 z_x^2(x,t) + r(t_{k+1}-t)\int_{t_k}^t e^{2\delta(s-t)}z_s^2(x,s)ds]dx\\ 0 < \alpha < 1,\ p_3 > 0,\ p_1 > 0,\ r > 0.\end{array} \tag{34}$$

By Wirtinger's inequality (15) we have

$$\begin{array}{l}(1-\alpha)ap_3\int_0^\pi z_x^2(x,t_k)dx \le\\ -\alpha a p_3\int_0^\pi z^2(x,t_k)dx + \int_0^\pi ap_3 z_x^2(x,t_k)dx \le V(t_k).\end{array} \tag{35}$$

The latter inequality will allow us to apply the modified Halanay's inequality (31).

The above t_k, t_{k+1}-dependent Lyapunov functional extends to the heat equation the one introduced in [5] for sampled-data control of finite-dimensional systems. It is piecewise-continuous in time and it does not grow in the jumps, since

$$\begin{array}{l} V(t_k) = (p_1 - \alpha a p_3)\int_0^\pi z^2(x,t)dx + \int_0^\pi ap_3 z_x^2(x,t_k)dx,\\ V(t_k^-) = V(t_k) + r(t_k - t_{k-1})\int_0^\pi\int_{t_{k-1}}^{t_k} e^{2\delta(s-t_{k-1})}z_s^2(x,s)dsdx.\end{array} \tag{36}$$

Theorem 1: Given positive scalars δ, Δ, h, K and R, let the four LMIs

$$\Phi_0 = \begin{bmatrix}\Psi_{11} & p_1 + (\beta - K - \alpha a)p_3 - vp_2\\ * & rh - 2p_3 + \frac{\Delta}{\pi}KRp_3\end{bmatrix} < 0,\ \beta = \pm\beta_0,$$

$$\Phi_1 = \begin{bmatrix}\Psi_{11} & p_1 + (\beta - K - \alpha a)p_3 - p_2 & Kp_2 h\\ * & -2p_3 + \frac{\Delta}{\pi}KRp_3 & Kp_3 h\\ * & * & -re^{-2\delta h}h\end{bmatrix} < 0, \tag{37}$$

where

$$\Psi_{11} = 2\delta p_1 + 2\delta(1-\alpha)ap_3 - 2(a - \beta + K)p_2 + \frac{\Delta}{\pi}KRp_2, \tag{38}$$

and the two LMIs

$$p_2 > \delta p_3, \tag{39}$$

$$KR^{-1}\Delta(p_3 + p_2) < 2\pi\delta p_3 a(1-\alpha), \tag{40}$$

hold for some positive scalars p_1, p_2, p_3 and r. Then solutions to (8) under the Dirichlet boundary conditions (2) (or under the Neumann boundary conditions (3)) satisfy the inequality

$$\int_0^\pi z_x^2(x,t)dx \le e^{-2\gamma(t-t_0)}\int_0^\pi z_x^2(x,t_0)dx,\ t \ge t_0, \tag{41}$$

where $\gamma > 0$ is a unique positive solution of (13) with $\delta_1 = KR^{-1}\Delta(p_3 + p_2)$.

Proof: Differentiating V we find

$$\begin{array}{l}\dot V(t) = 2(p_1 - \alpha a p_3)\int_0^\pi z(x,t)z_t(x,t)dx\\ +2p_3 a\int_0^\pi z_x(x,t)z_{xt}(x,t)dx - r\int_0^\pi\int_{t_k}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx\\ +r\int_0^\pi (t_{k+1}-t)z_t^2(x,t)dx\\ -2\delta\int_0^\pi r(t_{k+1}-t)\int_{t_k}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx, t\in[t_k,t_{k+1}).\end{array} \tag{42}$$

Integrating by parts and using the boundary conditions, we have

$$2p_3 a\int_0^\pi z_x(x,t)z_{xt}(x,t)dx = -2p_3 a\int_0^\pi z_t(x,t)z_{xx}(x,t)dx. \tag{43}$$

Therefore,

$$\begin{array}{l}\dot V + 2\delta V = 2(p_1 - \alpha a p_3)\int_0^\pi z(x,t)z_t(x,t)dx\\ -2p_3 a\int_0^\pi z_t(x,t)z_{xx}(x,t)dx - r\int_0^\pi\int_{t_k}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx\\ +r\int_0^\pi (t_{k+1}-t)z_t^2(x,t)dx + 2\delta(p_1-\alpha a p_3)\int_0^\pi z^2(x,t)dx\\ +2\delta a p_3\int_0^\pi z_x^2(x,t)dx,\ t\in[t_k,t_{k+1}).\end{array} \tag{44}$$

Denote

$$v_1(x,t) \triangleq \frac{1}{t-t_k}\int_{t_k}^t z_s(x,s)ds, \tag{45}$$

where by $v_{1|t=t_k}$ we understand the following: $\lim_{t\to t_k^+} v_1 = z_t(x,t_k)$. By Jensen's inequality [8] we have

$$\begin{array}{l}-r\int_0^\pi\int_{t_k}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx\\ \le -r\frac{1}{t-t_k}\int_0^\pi e^{-2\delta h}\left[\int_{t_k}^t z_s(x,s)ds\right]^2 dx\\ = -re^{-2\delta h}(t-t_k)\int_0^\pi v_1^2(x,t)dx.\end{array} \tag{46}$$

We apply further the descriptor method [4], [6] to (8), where the left-hand side of

$$\begin{array}{l}2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2 z(x,t)+p_3 z_t(x,t)][-z_t(x,t)+az_{xx}(x,t)\\ +\beta(z(x,t),x,t)z(x,t)-Kz(x,t_k)+K\int_{\bar x_j}^x z_\zeta(\zeta,t_k)d\zeta]dx=0\end{array} \tag{47}$$

or

$$\begin{array}{l}2\int_0^\pi[p_2 z(x,t)+p_3 z_t(x,t)][-z_t(x,t)+az_{xx}(x,t)\\ +\beta(z(x,t),x,t)z(x,t)-Kz(x,t_k)]dx\\ +2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2 z(x,t)+p_3 z_t(x,t)]K\int_{\bar x_j}^x z_\zeta(\zeta,t_k)d\zeta]dx=0\end{array} \tag{48}$$

with some free scalar $p_2 > 0$ is added to $\dot V(t) + 2\delta V(t)$. Representing

$$-Kz(x,t_k) = -Kz(x,t) + (t-t_k)Kv_1,$$

we arrive to

$$\begin{array}{l}\dot V(t) + 2\delta V(t) \le -re^{-2\delta h}(t-t_k)\int_0^\pi v_1^2(x,t)dx\\ +r\int_0^\pi (t_{k+1}-t)z_t^2(x,t)dx + 2\delta p_3 a\int_0^\pi z_x^2(x,t)dx\\ +2(p_1-\alpha a p_3)\int_0^\pi z(x,t)z_t(x,t)dx\\ +2\delta(p_1-\alpha a p_3)\int_0^\pi z^2(x,t)dx + 2ap_2\int_0^\pi z(x,t)z_{xx}(x,t)dx\\ +2\int_0^\pi[p_2 z(x,t)+p_3 z_t(x,t)][-z_t(x,t)\\ +[\beta(z(x,t),x,t)-K]z(x,t)+(t-t_k)Kv_1]dx\\ +2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2 z(x,t)+p_3 z_t(x,t)]K\int_{\bar x_j}^x z_\zeta(\zeta,t_k)d\zeta]dx,\\ t\in[t_k,t_{k+1})\end{array} \tag{49}$$

Integrating by parts and applying further Wirtinger inequality (15), we find

$$\begin{array}{l}2\delta p_3 a\int_0^\pi z_x^2(x,t)dx + 2ap_2\int_0^\pi z(x,t)z_{xx}(x,t)dx\\ = [-2ap_2 + 2\delta p_3 a]\int_0^\pi z_x^2(x,t)dx\\ \le [-2ap_2 + 2\delta p_3 a]\int_0^\pi z^2(x,t)dx,\end{array} \tag{50}$$

where we assume that $p_2 > \delta p_3$. Inequalities (50) and (39) yield

$$\begin{array}{l}\dot V(t) + 2\delta V(t) \le \int_0^\pi \eta^T\bar\Phi\eta dx\\ +2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2 z(x,t)+p_3 z_t(x,t)]K\int_{\bar x_j}^x z_\zeta(\zeta,t_k)d\zeta dx\end{array} \tag{51}$$

for $t\in[t_k,t_{k+1})$, where $\eta = col\{z(x,t), z_t(x,t), v_1\}$ and where

$$\bar\Phi \triangleq \begin{bmatrix} 2\delta p_1+2\delta(1-\alpha)ap_3-2(a-\beta+K)p_2 & p_1+(\beta-K-\alpha a)p_3-p_2 & Kp_2(t-t_k)\\ * & r(t_{k+1}-t)-2p_3 & Kp_3(t-t_k)\\ * & * & -re^{-2\delta h}(t-t_k)\end{bmatrix}.$$

By Young's inequality, for any scalar $\bar R > 0$ we have

$$\begin{array}{l}2Kp_2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} z(x,t)\int_{x_j}^x z_\zeta(\zeta,t_k)d\zeta dx\\ \le K\bar R p_2\int_0^\pi z^2(x,t)dx\\ +K\bar R^{-1}p_2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\int_{\bar x_j}^x z_\zeta^2(\zeta,t_k)d\zeta dx.\end{array} \tag{52}$$

Since $\int_{\bar x_j}^x z_\zeta^2(\zeta,t_k)d\zeta_{|x=\bar x_j} = 0$, by (14) we have

$$\int_{x_j}^{x_{j+1}}\int_{\bar x_j}^x z_\zeta^2(\zeta,t_k)d\zeta dx \le \frac{\Delta^2}{\pi^2}\int_0^\pi z_x^2(x,t_k)dx.$$

Choosing next $\bar R = \frac{\Delta}{\pi}R$, we find

$$\begin{array}{l}2Kp_2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} z(x,t)\int_{x_j}^x z_\zeta(\zeta,t_k)d\zeta dx\\ \le \frac{\Delta}{\pi}KRp_2\int_0^\pi z^2(x,t) + \frac{\Delta}{\pi}KR^{-1}p_2\int_0^\pi z_x^2(x,t_k)dx.\end{array} \tag{53}$$

Similarly

$$\begin{array}{l}2Kp_3\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\int_{x_j}^x z_t(x,t)z_\zeta(\zeta,t_k)d\zeta dx\\ \le \frac{\Delta}{\pi}KRp_3\int_0^\pi z_t^2(x,t) + \frac{\Delta}{\pi}KR^{-1}p_3\int_0^\pi z_x^2(x,t_k)dx.\end{array} \tag{54}$$

Hence, from (51)-(53) it follows

$$\begin{array}{l}\dot V + 2\delta V \le \int_0^\pi \eta^t\Phi\eta + \frac{\Delta}{\pi}KR^{-1}(p_3+p_2)\int_0^\pi z_x^2(x,t_k)dx\\ \le \frac{\Delta}{\pi}KR^{-1}(p_3+p_2)\int_0^\pi z_x^2(x,t_k)dx, \quad t\in[t-k,t_{k+1}),\end{array} \tag{55}$$

if

$$\Phi \triangleq \begin{bmatrix}\Psi_{11} & p_1+(\beta-K-\alpha a)p_3-p_2 & Kp_2(t-t_k)\\ * & r(t_{k+1}-t)-2p_3+\frac{\Delta}{\pi}KRp_3 & Kp_3(t-t_k)\\ * & * & -re^{-2\delta h}(t-t_k)\end{bmatrix} < 0,$$
$$t\in[t_k,t_{k+1}), \tag{56}$$

where Ψ_{11} is given by (38). Therefore, by Lemma 3 inequalities (40), (55) and (35) imply $V(t) \le e^{-2\gamma(t-t_0)}V(t_0)$ for $\gamma > 0$, satisfying (13) with $\delta_1 = \frac{\Delta}{\pi}KR^{-1}(p_3+p_2)/p_3 a(1-\alpha)$.

We will prove next that four LMIs (37) imply $\Phi < 0$. We note that Φ_0 and Φ_1 given by (37) are affine in β. Therefore, $\Phi_j < 0$ for all $\beta\in[-\beta_0,\beta_0]$ if LMIs (37) are satisfied. For

$t - t_k \to 0$ and $t - t_k \to h$ matrix inequality $\Phi < 0$ leads to $\Phi_0 < 0$ and $\Phi_1 < 0$ with notations given in (37). Denote by $\eta_0 = col\{z(x,t), z_t(x,t)\}$. Then $\Phi_0 < 0$ and $\Phi_1 < 0$ imply

$$\frac{t_{k+1}-t}{t_{k+1}-t_k}\eta_0^T\Phi_0\eta_0 + \frac{t-t_k}{t_{k+1}-t_k}\eta^T\Phi_1\eta = \eta^T\Phi_h\eta < 0,\ \forall \eta \neq 0,\\ t \in [t_k, t_{k+1}),$$

where

$$\Phi_h \overset{\Delta}{=} \begin{bmatrix} \Psi_{11} & p_1 + (\beta - K - \alpha a)p_3 - p_2 & Kp_2\frac{h}{t_{k+1}-t_k}(t-t_k) \\ * & r\frac{h}{t_{k+1}-t_k}(t_{k+1}-t) - 2p_3 + \frac{\Delta}{\pi}KRp_3 & Kp_3\frac{h}{t_{k+1}-t_k}(t-t_k) \\ * & * & -re^{-2\delta h}\frac{h}{t_{k+1}-t_k}(t-t_k) \end{bmatrix} < 0. \tag{57}$$

Since $t_{k+1} - t_k \leq h$, the feasibility of $\Phi_h < 0$ (by Schur complements) implies $\Phi < 0$, which completes the proof. ■

C. The time-delayed sampled-data controller (9)

We choose V of the form

$$\begin{aligned} V(t) &= (p_1 - \alpha a p_3)\int_0^\pi z^2(x,t)dx + p_3 a\int_0^\pi z_x^2(x,t)dx \\ &+ \int_0^\pi[\tau_M r\int_{-\tau_M}^0\int_{t+\theta}^t e^{2\delta(s-t)}z_s^2(x,s)dsd\theta \\ &+ s\int_{t-\tau_M}^t e^{2\delta(s-t)}z^2(x,s)ds]dx \end{aligned} \tag{58}$$

with some constants $0 < \alpha < 1$, $p_3 > 0$, $p_1 > 0$, $r \geq 0$ and $s \geq 0$. Differentiating V we find

$$\begin{aligned} &\dot V(t) + 2\delta V(t) = 2(p_1 - \alpha a p_3)\int_0^\pi z(x,t)z_t(x,t)dx \\ &+2p_3 a\int_0^\pi z_x(x,t)z_{xt}(x,t)dx \\ &-\tau_M r\int_0^\pi\int_{t-\tau_M}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx \\ &+\int_0^\pi[\tau_M^2 r z_t^2(x,t) + sz^2(x,t) - se^{-2\delta\tau_M}z^2(x,t-\tau_M)]dx \\ &+2\delta(p_1 - \alpha a p_3)\int_0^\pi z^2(x,t)dx + 2\delta p_3 a\int_0^\pi z_x^2(x,t)dx, \\ &t \in [t_k, t_{k+1}). \end{aligned} \tag{59}$$

By Jensen's inequality [8] we have

$$\begin{aligned} &-\tau_M r\int_0^\pi\int_{t-\tau_M}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx \\ &= -\tau_M r\int_0^\pi\int_{t-\tau_M}^{t-\tau(t)} e^{2\delta(s-t)}z_s^2(x,s)dsdx \\ &-\tau_M r\int_0^\pi\int_{t-\tau(t)}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx \\ &\leq -r\int_0^\pi e^{-2\delta\tau_M}\left[\int_{t-\tau_M}^{t-\tau(t)} z_s(x,s)ds\right]^2dx \\ &-r\int_0^\pi e^{-2\delta\tau_M}\left[\int_{t-\tau(t)}^t z_s(x,s)ds\right]^2dx \\ &= -re^{-2\delta\tau_M}[z(x,t-\tau(t)) - z(x,t-\tau_M)]^2dx \\ &-re^{-2\delta\tau_M}[z(x,t) - z(x,t-\tau(t))]^2dx. \end{aligned} \tag{60}$$

We apply further the descriptor method [4], [6] to (8), where the left-hand side of

$$\begin{aligned} &2\int_0^\pi[p_2 z(x,t) + p_3 z_t(x,t)][-z_t(x,t) + az_{xx}(x,t) \\ &+\beta(z(x,t),x,t)z(x,t) - Kz(x,t-\tau(t))]dx \\ &+2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2 z(x,t) + p_3 z_t(x,t)]\cdot \\ &K\int_{\bar x_j}^x z_\zeta(\zeta, t-\tau(t))d\zeta dx = 0 \end{aligned} \tag{61}$$

with some free scalar $p_2 > 0$ is added to $\dot V(t) + 2\delta V(t)$. From (43) we arrive to

$$\begin{aligned} &\dot V(t) + 2\delta V(t) \leq \int_0^\pi[\tau_M^2 r z_t^2(x,t) \\ &-re^{-2\delta\tau_M}[z(x,t-\tau(t)) - z(x,t-\tau_M)]^2]dx \\ &-re^{-2\delta\tau_M}[z(x,t) - z(x,t-\tau(t))]^2dx + 2\delta p_3 a\int_0^\pi z_x^2(x,t)dx \\ &+2\delta(p_1 - \alpha a p_3)\int_0^\pi z^2(x,t)dx \\ &+2(p_1 - \alpha a p_3)\int_0^\pi z(x,t)z_t(x,t)dx \\ &+2ap_2\int_0^\pi z(x,t)z_{xx}(x,t)dx \\ &+\int_0^\pi[sz^2(x,t) - se^{-2\delta\tau_M}z^2(x,t-\tau_M)]dx \\ &+2\int_0^\pi[p_2 z(x,t) + p_3 z_t(x,t)][-z_t(x,t) \\ &+\beta(z(x,t),x,t)z(x,t) - Kz(x,t-\tau(t))]dx \\ &+2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2 z(x,t) + p_3 z_t(x,t)]K\int_{\bar x_j}^x z_\zeta(\zeta,t_k)d\zeta dx. \end{aligned} \tag{62}$$

We take into account (50) provided that $p_2 > \delta p_3$. Setting $\eta = col\{z(x,t), z_t(x,t), z(x,t-h), z(x,t-\tau)\}$, we find that

$$\begin{aligned} &\dot V + 2\delta V \leq \int_0^\pi \eta^T\Phi_{\tau_M}\eta dx \\ &+\frac{\Delta}{\pi}KR^{-1}(p_3 + p_2)\int_0^\pi z_x^2(x,t-\tau)dx \\ &\leq \frac{\Delta}{\pi}KR^{-1}(p_3 + p_2)\int_0^\pi z_x^2(x,t-\tau)dx, \end{aligned} \tag{63}$$

if

$$\Phi_{\tau_M} \overset{\Delta}{=} \begin{bmatrix} \Psi_{\tau_M} & p_1 - p_2 + p_3(\beta - a\alpha) & 0 & re^{-2\delta\tau_M} - Kp_2 \\ * & r\tau_M^2 - 2p_3 + \frac{\Delta}{\pi}KRp_3 & 0 & -Kp_3 \\ * & * & -re^{-2\delta\tau_M} - se^{-2\delta\tau_M} & re^{-2\delta\tau_M} \\ * & * & * & -2re^{-2\delta\tau_M} \end{bmatrix} < 0, \tag{64}$$

where

$$\Psi_{\tau_M} = 2\delta p_1 + 2\delta p_3(1-\alpha)a + s - 2p_2(a - \beta - \frac{\Delta}{2\pi}KR) - re^{-2\delta\tau_M}. \tag{65}$$

Assuming further that $p_2 > \delta p_3$ and applying the Halanay's inequality (31), we obtain the following:

Theorem 2: Given δ, K and R, let there exist positive constants p_2, p_3,p_1, r and s such that LMIs $(p_3+p_2)KR^{-1}\Delta < 2\pi\delta p_3 a(1-\alpha)$, $p_2 > \delta p_3$ and two LMIs (64), where $\beta = \pm\beta_0$, are feasible. Then solutions to (8) under the Dirichlet boundary conditions (2) (or under the Neumann boundary conditions (3)) satisfy the inequality (41), where $\gamma > 0$ is a unique positive solution of (13) with $\delta_1 = \frac{\Delta}{\pi}\frac{p_3+p_2}{p_3 a(1-\alpha)}KR^{-1}$.

D. Example

Consider the controlled heat equation

$$\begin{aligned} z_t(x,t) &= z_{xx}(x,t) + \beta z(x,t) + u(x,t) \\ z(0,t) &= z(\pi,t) = 0 \end{aligned} \tag{66}$$

where the sampled-data control law (4) is chosen with $K = 3$ and where β is uncertain parameter satisfying $|\beta| \leq 1.8$.

For the continuous in time controller $u(x,t) = -3z(\bar x_j, t)$ by applying Proposition 1, we find that the system remains exponentially stable till $\Delta \leq 2.09$. Therefore, the controller $u(x,t) = -3z(\bar x_j, t)$ exponentially stabilizes the system if the space domain is divided into two subdomains with $x_{j+1} - x_j \leq 2.09,\ j = 0, 1$. For the continuous in space controller $u(x,t) = -3z(x,t_k)$, by using LMI Toolbox of Matlab we

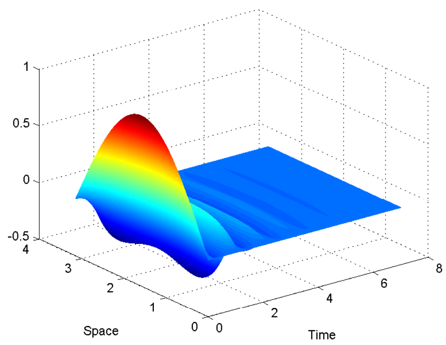

Fig. 1. Solution under the sampled in space and time control with $\Delta = \pi/2, h = 0.2$

verify LMI conditions of Theorem 1. This leads to the bound $h = 0.66 \geq t_{k+1} - t_k$, which preserves the stability.

We consider further $u(x,t) = -3z(\bar{x}_j, t_k - \eta_k)$ and apply Theorems 1 (for $\eta_k = 0$) and 2 with $R = 1$. Tables 1 and 2 show the maximum value of Δ as a function of h, which preserves the exponential stability of the system.

TABLE I

SAMPLED-DATA IN SPACE AND IN TIME CONTROLLER

α	0	0.1	0.15	0.45	0.9
δ	0.9	0.7	0.35	0.25	0.001
Δ_{max}	0.74	0.6	0.4	0.2	0.000001
h	0.00001	0.2	0.37	0.46	0.66

TABLE II

SAMPLED-DATA IN SPACE AND DELAYED IN TIME CONTROLLER

α	0	0.01	0.1	0.9
δ	0.9	0.8	0.7	0.001
Δ_{max}	0.74	0.69	0.6	0
τ_{max}	0	0.1	0.2	0.38

We proceed further with the numerical simulations, where we choose the initial function of the form $z(x,0) = \sin x$. We use a finite difference method. Our numerical simulations confirm the predicted upper bound on $t_{k+1} - t_k$ which preserves the stability. Hence, the conditions of Theorem 1 for the sampled-data in time controller are not conservative. Simulations of solution under the discretized in spatial variable controller with $x_{j+1} - x_j = \pi/2, \ j = 0, 1$, where the space domain is divided into two sub-domains, show that the closed-loop system is exponentially stable. This confirms the predicted by Proposition 1 behavior. Moreover, for $x_{j+1} - x_j = \pi/2, \ j = 0, 1$ the sampled-data in time and in space controller $u(x,t) = -3z(\bar{x}_j, t_k)$ preserves the stability for $t_{k+1} - t_k \leq 0.55$ (see Fig.1, where $t_{k+1} - t_k = 0.2$). The latter illustrates the conservativeness of our method.

IV. CONCLUSION

In the present paper we suggest a sampled-data controller design for a scalar semilinear heat equation under homogenous Dirichlet or Neumann boundary conditions. It is supposed that the sampling in time and the sampling in space (i.e. the distance between the consequently sampled spatial variables) are bounded. Our sampled-data feedback with a constant gain is piecewise-constant in time and in space. Sufficient conditions for the exponential stabilization are derived in terms of LMIs. By solving these LMIs, upper bounds on the sampling in time and on the sampling in space are found that preserve the exponential stability. A numerical example illustrates the efficiency of the method and its conservativeness. Thus, the results are analytical if the controller is sampled in time only (and it is continuous in space). This is consistent with the conclusions of [6]. The results are almost not conservative if the controller is sampled in space only. The conservativeness of our method for space-time sampled-data controller may stem from the application of Halanay's inequality.

Extension of the method to sampled-data control of some other classes of distributed parameter systems, as well as extension of convex optimization approach to robust control of distributed parameter systems may be topics for the future research.

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