Observability of partial states of invariant systems

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Abstract— **This paper considers the observability of partial states of an invariant control system on a Lie group. Specifically we consider in this paper left invariant systems where the outputs and the partial states are given by actions of the group on different manifolds. Depending of the type of these actions we give characterizations of partial state observability and for some cases additional sufficient observability criteria.**

I. INTRODUCTION

Observability criteria for invariant systems on Lie groups have been considered in several works starting in the seventies, e.g. [3], [4], [1]. In recent years, there has been some renewed interest in the study of observers and observability of invariant systems motivated by applications in robotics [2], [8]. However, the works in this area focus on observability and observers for the full state of the system. In applications it might not be relevant or even impossible to estimate the full state of the system. Instead only a partial state, i.e. the projection of the state onto a homogeneous space, might be of interest. This paper discusses the observability of such partial states. We focus on left invariant systems on Lie groups where the outputs and the partial states are given by actions of the group state on a reference point in an homogeneous space. We give, for the case that the action on the output space is a right action, a full characterization of observability of partial states. For the case that this action is a left action we provide besides characterizations of partial state observability additional sufficient observability criteria.

II. NOTATION AND DEFINITIONS

In this paper G will denote a connected Lie group with identity element e and Lie algebra $\mathfrak{g}$. Furthermore M, N will be smooth manifolds. We remind the reader that a smooth map $h\colon G \times M \to M$ is called a *left action* if $h(X, h(Y, y)) = h(XY, y)$ and $h(e, y) = y$ for all $X, Y \in G$, $y \in M$. It is called a *right action* if $h(X, h(Y, y)) = h(YX, y)$ and $h(e, y) = y$ for all $X, Y \in G$, $y \in M$. An action is called *transitive* if for all $x, y \in M$ exists an $X \in G$ with $h(X, x) = y$. The stabilizer subgroup $\{X \in G \mid h(X, y) = y\}$ of a point $y \in M$ with respect to an action $h\colon G \times M \to M$ is denoted by $\mathrm{stab}_h(y)$. We use $\mathcal{L}S$ for the Lie algebra of a subgroup $S \subset G$. Ad will denote the adjoint representation of G and $\mathrm{id}_{\mathfrak{g}}$ the identity map $\mathfrak{g} \to \mathfrak{g}$. Furthermore, for left invariant vector fields $F_0, \ldots, F_m$ on G we use the notation $\mathrm{Lie}\{F_0, \ldots, F_m\}$ for the smallest Lie algebra in $\mathfrak{g}$ containing $F_0, \ldots, F_m$. Here, the standard identification of $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on G is used.

Finally, we recall that the core $\mathrm{Core}_G(S)$ of a subset S of G is given by $\mathrm{Core}_G(S) = \bigcap_{X \in G}\{XYX^{-1} \mid Y \in S\}$ and is largest normal subgroup of G contained in S [5].

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III. PARTIAL STATE OBSERVABILITY

In this paper we consider left invariant systems on G, i.e.

$$\begin{aligned} \frac{d}{dt}X &= F_0(X) + \sum_{i=1}^{m} u_i F_i(X) \\ y &= h(X, y_0) \end{aligned} \tag{1}$$

with $F_0(X), \ldots, F_m(X)$ left invariant vector fields on G, $h\colon G \times M \to M$ a transitive left- or right action of G on M and y_0 a reference point on M. An input $u\colon \mathbb{R} \to \mathbb{R}^m$ for system (1) is called *admissible* if it is smooth and uniformly bounded.

The subject of interest in this paper is the observability of *partial states*, i.e. the observability of

$$z = k(X, z_0) \tag{2}$$

where $k\colon G \times N \to N$ is a smooth, transitive left- or right action of G on a manifold N and z_0 a reference point. Since in this setting G is a fiber bundle over N, one can view partial states as generalizations of the projections in $\mathbb{R}^n$ of the state onto linear subspaces. To avoid confusion, we will refer to the states on the group as *full states* and to the observability of these states as *full state observability*.

To define observability of partial states we need an appropriate notion of distinguishability of partial states.

Definition 1: Two partial states $z_1 \neq z_2 \in N$ are called *distinguishable* if for all X_1, $X_2 \in G$ with $k(X_1, z_0) = z_1$, $k(X_2, z_0) = z_2$ there exists and admissible input $u\colon \mathbb{R} \to \mathbb{R}^m$ with corresponding trajectories $X_1(t)$, $X_2(t)$ and a time $t \geq 0$ such that $h(X_1(t), y_0) \neq h(X_2(t), y_0)$.

The definition of partial state observability is now a straightforward extension of observability of the states in G.

Definition 2: System (1) is called *partial state observable* with respect to (2) if all partial states z_1, $z_2 \in N$ are distinguishable.

Obviously, for $N = G$ and $k(X, Y_0) := XY_0$, $Y_0 \in G$ we get the observability of full states on the group. To simplify our notation, we will use partial state observable for partial state observable with respect to (2).

From Definition 2 follows directly the following characterization of partial state observability.

Proposition 1: System (1) is called *partial state observable* with respect to (2) if and only if for all triples

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(X_1, X_2, u) of initial states $X_1, X_2 \in G$ and admissible inputs u we have for the corresponding trajectories $X_1(t), X_2(t), X_1(0) = X_1, X_2(0) = X_2$ with respect to the input u that

$$\big(\forall t \in \mathbb{R}_0^+ \ h(X_1(t), y_0) = h(X_2(t), y_0)\big) \quad \text{implies } k(X_1(0), z_0) = k(X_2(0), z_0).$$

To examine the observability of partial or full states, we need a suitable notion of an error of different full states on the group. There are two different canonical types of an error between full states on the group — the *left invariant error* and the *right invariant error* defined as follows: Given two solutions $X_1(t)$, $X_2(t)$ for the same input, but possibly different initial values, we define

$$E_r(t) := X_1(t)X_2(t)^{-1} \qquad \text{right invariant error,}$$
$$E_l(t) := X_2(t)^{-1}X_1(t) \qquad \text{left invariant error.}$$

Note that these two errors are not equivalent, and the correct choice usually depends on the invariance properties of the system on the group and the type of the actions [8], [7].

IV. Criteria for partial state observability

The goal of this paper is to give criteria for partial state observability, similar to the classical observability criteria for systems on Lie groups like [4]. However, the type of observability criteria depends strongly on the type of action on M and N, i.e. if the actions are left or right actions. Here, we discuss the two different cases of the action on the output space M being a left respectively a right action separately.

A. Action on output space: right action

We start with the case that the action on the output space h is a right action. In this case the system dynamics project to a control system on the output space, see [7], and the unobservable subgroup for *full states* is given by $\mathrm{stab}_h(y_0)$. Hence the system is full state observable if and only if $\mathrm{stab}_h(y_0)$ is trivial. However, for partial state observability the situation is more subtle and depends on the type of action k.

Let us first consider the case that k is a right action, too. Then we can use the right invariant error E_r to derive the following characterization of partial state observability from Proposition 1.

Proposition 2: Let h and k be right actions. System (1) is partial state observable if and only if for all triples (X_1, X_2, u) of initial states $X_1, X_2 \in G$ and admissible inputs u we have for the corresponding trajectories $X_1(t), X_2(t), X_1(0) = X_1, X_2(0) = X_2$ with respect to the input u that

$$\big(\forall t \in \mathbb{R}_0^+ E_r(t) \in \mathrm{stab}_h(y_0)\big) \text{ implies } E_r(0) \in \mathrm{stab}_k(z_0).$$

The partial state observability depends on the dynamics of the right invariant error E_r. However, it can be shown that for left invariant systems

$$\frac{d}{dt}E_r = 0,$$

i.e. the right invariant error is constant, see [8]. This yields the following observability criterion.

Theorem 1: If h and k are right actions, then the system is partial state observable if and only if $\mathrm{stab}_h(y_0) \subset \mathrm{stab}_k(z_0)$.

Note that Theorem 1 depends only on the output maps h,k and is completely independent of the control system on the group.

Example 1: As an example consider the attitude dynamics of a rigid-body in $\mathbb{R}^3$. The attitude is represented by a $X \in \mathrm{SO}(3)$ and obeys the dynamics

$$\frac{d}{dt}X = X\Omega, \quad \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

with $\omega_1, \omega_2, \omega_3$ the angular velocities [8]. The angular velocities are the inputs of this left invariant system.
Assume we measure an inertial direction in the body fixed frame. That means that we measure for a fixed $y_0 \in \mathbb{R}^3$ the values $y = X^T y_0$. To simplify our calculations we assume that $|y_0| = 1$. Therefore, we measure $y = h(X, y_0)$ with $h\colon \mathrm{SO}(3) \times S^2 \to S^2$ the right action $h(X, y) = X^T y$.
Consider a second fixed inertial direction in the body fixed frame, i.e. $z = h(X, z_0)$ with $z_0 \in S^2$. When is z observable from measurements of y ? By Theorem 1 this is only the case if and only if $\mathrm{stab}_h(z_0) \subset \mathrm{stab}_h(y_0)$, i.e. $z_0 = \pm y_0$. So, in general, a second inertial direction will not be observable from measurements of y.

We now discuss the case that k is a left action. Then we get a slightly different characterization of partial state observability.

Proposition 3: Let h be a right action and k be a left action. System (1) is partial state observable if and only if for all triples (X_1, X_2, u) of initial states $X_1, X_2 \in G$ and admissible inputs u we have for the corresponding trajectories $X_1(t), X_2(t), X_1(0) = X_1, X_2(0) = X_2$ with respect to the input u that

$$\big(\forall t \in \mathbb{R}_0^+ E_r(t) \in \mathrm{stab}_h(y_0)\big) \text{ implies } E_l(0) \in \mathrm{stab}_k(z_0).$$

While the difference between Propositions 2 and 3 appears to be minor, it results in a significant difference between the criteria for partial state observability.

Theorem 2: If h is a right action and k is a left action, then the system is partial state observable if and only if $\mathrm{stab}_h(y_0) \subset \mathrm{Core}_G(\mathrm{stab}_k(z_0))$.

Again the observability criterion is independent of the controllability properties of the system. This criterion of Theorem 2 is much more restrictive than the one of Theorem 1, since $\mathrm{Core}_G(\mathrm{stab}_k(z_0))$ will be smaller than $\mathrm{stab}_k(z_0)$ unless the stabilizer is a normal subgroup. In particular, we get the following corollary for groups without non-trivial normal subgroups.

Corollary 1: Let G be a Lie group which has only the trivial normal subgroups[1] and consider an arbitrary invariant

[1] Since the notion of a simple Lie group is different from the one for algebraic groups, we refrain from using ‘simple’ in this context.

system on G. Assume that h is a right action and k is a non-trivial left action. Then the system is partial state observable if and only if it is full state observable.

Example 2: Consider any left invariant system on $\mathrm{SO}(3)$ with h a right action and k a left action. Then partial state observability is equivalent to full state observability.

It might appear surprising that that Theorem 2 yields are more restrictive condition for partial state observability than Theorem 1. However, this can be explained by the fact that the system on G projects to a system on the output space. Since system (1) projects to a system on M, two full states on G are distinguishable if and only if they belong to different fibers of the map $X \mapsto h(X, y_0)$. Furthermore, if two full states on G are located in the same fiber, for any fixed input their corresponding solutions are located in a common fiber for all $t \geq 0$. Thus the question of partial state observability boils down to the question if states in the same fiber can give raise to different partial states. If k is a right action this holds if and only if $\mathrm{stab}_h(y_0) \not\subset \mathrm{stab}_k(z_0)$. However, if k is a left action this holds if and only if $\mathrm{stab}_h(y_0) \not\subset \mathrm{Core}_G(\mathrm{stab}_k(z_0))$, yielding the condition of Theorem 2.

B. Action on output space: left action

We now discuss the case that the action on the output space is a left action. Unlike in the previous case, the full characterizations for partial state observability given here will be very hard to check in practice. Therefore, we provide additional sufficient criteria for partial state observability which should be easier to verify.

As before the observability conditions depend also of the type of the action k on N. We start with k being a left action. In this case Proposition 1 yields the following characterization of partial state observability.

Proposition 4: Let h be a left action and k be a left action. System (1) is partial state observable if and only if for all triples (X_1, X_2, u) of initial states $X_1, X_2 \in G$ and an admissible input u we have for the corresponding trajectories $X_1(t), X_2(t), X_1(0) = X_1, X_2(0) = X_2$ with respect to the input u that

$$\left(\forall t \in \mathbb{R}_0^+ E_l(t) \in \mathrm{stab}_h(y_0)\right) \text{ implies } E_l(0) \in \mathrm{stab}_k(z_0).$$

We see that the partial state observability depends on the evolution of E_l for $t \geq 0$. One checks that dynamics of E_l is given by the control system, see [8],

$$\frac{d}{dt}E_l = (T_e L_{E_l} - T_e R_{E_l}) \left(F_0(e) + \sum_{j=1}^{m} u_j F_j(e) \right) \quad (3)$$

with R_X, L_X denoting the right- and left-multiplication maps on G. While this is an affine control system on G, it is not an invariant one.

Nevertheless, we can use its dynamics to obtain a characterization of partial state observability.

Proposition 5: Let h and k be left actions. The system is partial state observable if and only if for all $E_l \in \mathrm{stab}_h(y_0)$ with $E_l \notin \mathrm{stab}_k(z_0)$ the reachable set for (3) contains a point in $G \setminus \mathrm{stab}_h(y_0)$.

This characterization itself will not be very helpful in practice as it requires to compute reachable sets for an affine control system. However, we can use it to obtain a sufficient condition for partial state observability. For this, we need another general result on affine control systems.

Proposition 6: Let S be a submanifold of G and

$$\frac{d}{dt}X = H_0(X) + \sum_{j=1}^{m} u_i H_i(X) \quad (4)$$

an affine (not necessarily invariant) control system on G. For an $X \in S$ we have $\Delta(X) \not\subset T_X S$ with Δ the accessibility distribution of (3), i.e. the distribution spanned by the smallest Lie algebra of vector fields containing $H_0, \ldots, H_m$, then the set of points reachable from X contains an $Y \notin S$.

With this result we obtain the following sufficient criterion for partial state observability.

Theorem 3: Let h, k be a left actions and V the linear span $V = \mathrm{span}\{F_0(e), \ldots, F_m(e)\}$. If for all $X \in \mathrm{stab}_h(y_0)$ with $X \notin \mathrm{stab}_k(z_0)$ we have $(\mathrm{id}_{\mathfrak{g}} - \mathrm{Ad}_X)V \not\subset \mathcal{L}\,\mathrm{stab}_h(y_0)$, then the system is partial state observable.

Note, that unlike in the previous case, the condition involves the drift and control vector fields of system (1). Therefore, this criterion for partial state observability depends on the dynamics of the system (1) on G.

We give now an example for a system where Theorem 3 is applicable and which is partial state but not full state observable.

Example 3: Let $G = \mathrm{SO}(4)$ the special orthogonal group, $M = S^3$ the unit sphere in $\mathbb{R}^4$ and $N = \mathrm{St}(4,2) = \{Z \in \mathbb{R}^{4\times 2} \mid Z^T Z = I_2\}$ the compact Stiefel manifold. Here and in the following, I_n denotes the $n \times n$ identity matrix. We consider the control system on $\mathrm{SO}(4)$

$$\begin{aligned} \frac{d}{dt}X &= uXB_1 \\ y &= h(X, y_0) \end{aligned}$$

with input $u\colon \mathbb{R} \to \mathbb{R}$ and $B_1 \in \mathfrak{so}(4)$,

$$B_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The left action $h\colon \mathrm{SO}(4) \times S^3 \to S^3$ on the output space is given by $h(X, y) = Xy$ and $y_0 = e_1 = (1,0,0,0)^T$. We define the subgroup S of $\mathrm{SO}(4)$,

$$S = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & \Theta \end{pmatrix} \mid \Theta \in \mathrm{SO}(2) \right\}.$$

Note that $S \subset \mathrm{stab}_h(e_1)$. The dynamics of the left invariant error for two solutions is given by

$$\frac{d}{dt}E_l = u(E_l B_1 - B_1 E_l).$$

Thus for $E_l(t) \in S$ we have $\frac{d}{dt}E_l(t) = 0$. This implies that if we have two initial states $X_1, X_2 \in \mathrm{SO}(4)$, $X_1 \neq X_2$, with $X_2^{-1}X_1 = E_l(0) \in S$ then $E_l(t) \in S \subset \operatorname{stab}_h(e_1)$ for all $t \geq 0$. But since $E_l(t) \in \operatorname{stab}_h(e_1)$ is equivalent to $h(X_1(t), e_1) = h(X_2(t), e_1)$ this yields that this system on $\mathrm{SO}(4)$ is not full state observable. We consider partial states

$$Z = k(X, Z_0) = XZ_0 \text{ with } Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $k\colon \mathrm{SO}(4) \times \mathrm{St}(4,2) \to \mathrm{St}(4,2)$ the left action $k(X, Z) = XZ$. Note that $\operatorname{stab}_k(Z_0) = S$. For $X \in \operatorname{stab}_h(e_1)$ we have

$$X = \begin{pmatrix} 1 & 0 \\ 0 & \Theta \end{pmatrix} \text{ with } \Theta \in \mathrm{SO}(3)$$

and

$$B_1 - XB_1X^{-1} = \begin{pmatrix} 0 & -\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}(I_3 - \Theta^T) \\ (I_3 - \Theta)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & 0 \end{pmatrix}.$$

Thus for $X \in \operatorname{stab}_h(e_1)$ we have $(\mathrm{id}_{\mathfrak{g}} - \mathrm{Ad}_X)B_1 \in \operatorname{stab}_h(e_1)$ if and only if $X \in S$. By Theorem 3 this system on $\mathrm{SO}(4)$ is partial state observable with respect to $Z = k(X, Z_0)$.

Note that Theorem 3 uses only the linear span and not the Lie algebra of the vector fields $F_0, \ldots, F_m$. Since Proposition 6 actually uses the Lie algebra generated by $H_0, \ldots, H_m$ we make the following conjecture.

Conjecture 1: Let h, k be a left actions and $\mathfrak{m} = \mathrm{Lie}\{F_0, \ldots, F_m\}$. If for all $X \in \operatorname{stab}_h(y_0)$ with $X \notin \operatorname{stab}_k(z_0)$ we have $(\mathrm{id}_{\mathfrak{g}} - \mathrm{Ad}_X)\mathfrak{m} \not\subset \mathcal{L}\operatorname{stab}_h(y_0)$, then the system is partially observable.

The main challenge for this conjecture is to prove that the accessibility distribution for the left error dynamics (3) contains the distribution spanned by $T_eL_X(\mathrm{id}_{\mathfrak{g}} - \mathrm{Ad}_{X^{-1}})H(e)$, $H \in \mathfrak{m}$.

Let us now turn to the case that the action k on N is a right action. Then partial state observability is characterized as follows.

Proposition 7: Let h be a left action and k be a right action. System (1) is partial state observable if and only if for all admissible inputs and all pairs of initial states $X_1, X_2 \in G$ with corresponding solutions $X_1(t), X_2(t)$ we have

$$\left(\forall t \in \mathbb{R}_0^+ E_l(t) \in \operatorname{stab}_h(y_0)\right) \text{ implies } E_r(0) \in \operatorname{stab}_k(z_0).$$

Again, the partial state observability depends on the dynamics of E_l. Therefore, we can give a characterization of partial state observability similar to Proposition 5.

Proposition 8: Let h and k be left actions. The system is partial state observable if and only if for all $E_l \in \operatorname{stab}_h(y_0)$ with $E_l \notin \mathrm{Core}_G(\operatorname{stab}_k(z_0))$ the reachable set for (3) contains a $Y \notin \operatorname{stab}_h(y_0)$.

From this characterization and Propositon 6 we can derive a sufficient criterion for partial state observability for this case.

Theorem 4: Let h be a left action, k a right action and V the linear span $V = \mathrm{span}\{F_0, \ldots, F_m\}$. If for all $X \in \operatorname{stab}_h(y_0)$ with $X \notin \mathrm{Core}_G(\operatorname{stab}_k(z_0))$ we have $(\mathrm{id}_{\mathfrak{g}} - \mathrm{Ad}_X)V \not\subset \mathcal{L}\operatorname{stab}_h(y_0)$, then the system is partial state observable.

Again, the observability depends on the $F_0, \ldots, F_m$ and hence on the system dynamics on G. Of course, we can also make the conjecture analogous to Conjecture 1 that in Theorem 4 the linear span V can be replaced by $\mathrm{Lie}\{F_0, \ldots, F_m\}$. An investigation of these topics will be the subject of future research.

References

[1] N. Apostolou and D. Kazakos. On local observability for invariant systems on Lie groups. *Proc. R. Soc. Lond. A*, 452(1955):2801–2805, 1996.

[2] S. Bonnabel, P. Martin, and P. Rouchon. Symmetry-preserving observers. *IEEE Transactions on Automatic Control*, 53(11):2514–2526, 2008.

[3] R. W. Brockett. System theory on group manifolds and coset spaces. *SIAM J. Control*, 10(2):265–284, 1972.

[4] D. Cheng, W. P. Dayawansa, and C. F. Martin. Observability of systems on Lie groups and coset spaces. *SIAM J. Control and Optimization*, 28(3):570–581, 1990.

[5] I. M. Isaacs. *Algebra*. Brooks/Cole Publishing Company, Pacific Grove, California, 1993.

[6] V. Jurdjevic. *Geometric Control Theory*. Cambridge University Press, 1997.

[7] C. Lageman, J. Trumpf, and R. Mahony. Observers for systems with invariant outputs. In *Proceedings of the European Control Conference 2009, Budapest, Hungary*, pages 4587 – 4592, 2009.

[8] C. Lageman, J. Trumpf, and R. Mahony. Gradient-like observers for invariant dynamics on a Lie group. *IEEE Transactions on Automatic Control*, 55(2):367 – 377, 2010.

[9] H. Nijmeijer and A. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer Verlag, New York, 1996.