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Extendability of multidimensional linear systems

Alban Quadrat
INRIA Sophia Antipolis,
2004, Route des Lucioles BP 93,
06902 Sophia Antipolis cedex, France.
Alban.Quadrat@sophia.inria.fr

***Abstract*— Within the algebraic analysis approach to multidimensional linear systems defined by linear systems of partial differential equations with constant coefficients, the purpose of this paper is to show how to use different mathematical results developed in the literature of algebraic analysis to obtain new characterizations of the concepts of controllability, in the sense of Willems and Pillai-Shankar, observability, flatness and autonomous systems in terms of the possibility to extend (smooth or distribution) solutions of the multidimensional system and of its formal adjoint. Each characterization is equivalent to a module-theoretic property that can be constructively checked by means of the packages OreModules and QuillenSuslin.**

Within the constructive algebraic analysis approach to linear systems theory (see, e.g., [1], [4], [5], [6], [7]), if $D = k[\partial_1, \ldots, \partial_n]$ is the commutative polynomial ring of partial differential (PD) operators in

$$\partial_1 = \frac{\partial}{\partial x_1}, \ldots, \partial_n = \frac{\partial}{\partial x_n},$$

with coefficients in the field $k = \mathbb{R}$ or $\mathbb{C}$, $R \in D^{q \times p}$ a $q \times p$-matrix with entries in D and $\mathcal{F}$ a D-module (e.g., $C^\infty(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$), then the multidimensional linear system or behaviour

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$$

can be studied by means of the D-module

$$M = D^{1 \times p}/(D^{1 \times q}\, R)$$

finitely presented by R. The reason why is that Malgrange's result ([3]) asserts that

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}),$$

where $\hom_D(M, \mathcal{F})$ denotes the D-module of D-homomorphisms (i.e., D-linear maps) from M to $\mathcal{F}$. Moreover, if the signal space $\mathcal{F}$ is rich enough (e.g., $C^\infty(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$), which, in module-theoretic language, means an injective cogenerator D-module ([1], [4], [5], [7]), then systemic properties of $\ker_{\mathcal{F}}(R.)$ can completely be characterized in terms module properties.

In [5], Pillai and Shankar extended Willems' definition of controllability of time-invariant linear ordinary differential systems ([8]) to the case of underdetermined linear PD systems with constant coefficients as follows.

Theorem 1 ([5]): With the previous notations and $\mathcal{F} = C^\infty(\Omega)$, where Ω is an open convex subset of $\mathbb{R}^n$, the following two assertions are equivalent:

1) $\ker_{\mathcal{F}}(R.)$ is *controllable* in the sense that, for all η_1 and $\eta_2 \in \ker_{\mathcal{F}}(R.)$ and all open subsets U_1 and U_2 of Ω such that their closures $\overline{U_1}$ and $\overline{U_2}$ do not intersect, namely, $\overline{U_1} \cap \overline{U_2} = \emptyset$, there exists $\eta \in \ker_{\mathcal{F}}(R.)$ which coincides with η_1 on U_1 and with η_2 in U_2.
2) The D-module $M = D^{1 \times p}/(D^{1 \times q}\, R)$ is *torsion-free*, namely the *torsion D-submodule* of M defined by

$$t(M) \triangleq \{m \in M \mid \exists\, d \in D \setminus \{0\} : \; d\, m = 0\},$$

is reduced to 0, i.e., $t(M) = 0$.

In the same vein of Theorem 1, the purpose of this paper is to show how to use difficult results of algebraic analysis obtained in [2], [3] to characterize systemic properties of behaviours in terms of restriction maps of certain solution space and module theory.

Definition 1: 1) The behaviour $\ker_{C^\infty(\mathbb{R}^n)}(R.)$ is *smoothly extendable* if for all bounded open convex subset Ω of $\mathbb{R}^n$, the following restriction D-homomorphism is surjective:

$$\begin{array}{rcl} \Gamma_\Omega : \ker_{C^\infty(\mathbb{R}^n)}(R.) & \longrightarrow & \ker_{C^\infty(\mathbb{R}^n \setminus \Omega)}(R.) \\ \eta & \longmapsto & \eta_{|\mathbb{R}^n \setminus \Omega}. \end{array}$$

2) The behaviour $\ker_{\mathcal{D}'(\mathbb{R}^n)}$ is *distributionally extendable* if for all bounded open convex subset Ω of $\mathbb{R}^n$, the following restriction D-homomorphism is surjective:

$$\begin{array}{rcl} \Gamma'_\Omega : \ker_{\mathcal{D}'(\mathbb{R}^n)} & \longrightarrow & \ker_{\mathcal{D}'(\mathbb{R}^n \setminus \Omega)}(R.) \\ \eta & \longmapsto & \eta_{|\mathbb{R}^n \setminus \Omega}. \end{array}$$

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We prove that the controllability of behaviours can be characterized in terms of the extendability of the solution space of the formal adjoint $\widetilde{R} \in D^{p\times q}$ of R in the sense of the distributions, i.e., $\widetilde{R} = (\theta(R_{ij}))^T \in D^{p\times q}$, where:

$$\begin{cases} \theta(\partial_i) = -\partial_i, \\ \theta(a) = a, \quad \forall\, a \in k, \\ \theta(d_1 + d_2) = \theta(d_1) + \theta(d_2), \quad \forall\, d_1,\, d_2 \in D. \end{cases}$$

Theorem 2: With the previous notations, the following conditions are equivalent:

1) The behaviour $\ker_{C^\infty(\mathbb{R}^n)}(R.)$ is controllable in the sense of Definition 1.
2) The behaviour $\ker_{C^\infty(\mathbb{R}^n)}(\widetilde{R}.)$ is smoothly extendable.
3) The behaviour $\ker_{\mathcal{D}'(\mathbb{R}^n)}(\widetilde{R}.)$ is distributionally extendable.
4) The D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ finitely presented by R is torsion-free.

The interest of Theorem 2 is that the behaviour $\ker_{\mathcal{F}}(\widetilde{R}.)$ is generally overdetermined contrary to $\ker_{\mathcal{F}}(R.)$ which is underdetermined. Hence, 2 and 3 of Theorem 2 are generally simpler to check than 1 of Theorem 1. Moreover, we show that Theorem 2 has interesting connections with some results obtained in [9]. Using [1], we also prove the following two results.

Theorem 3: With the previous notations, if R a full row rank matrix, i.e., the rows of R are D-linearly independent, then the following conditions are equivalent:

1) The behaviour $\ker_{C^\infty(\mathbb{R}^n)}(R.)$ is *differentially flat*, namely, there exist $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ such that $\ker_{C^\infty(\mathbb{R}^n)}(R.) = Q\, C^\infty(\mathbb{R}^n)^m$ and $T\, Q = I_m$.
2) The behaviour $\ker_{\mathcal{D}'(\mathbb{R}^n)}(R.)$ is differentially flat, namely, $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ exist such that $\ker_{\mathcal{D}'(\mathbb{R}^n)}(R.) = Q\, \mathcal{D}'(\mathbb{R}^n)^m$ and:
$$T\, Q = I_m.$$
3) The behaviour $\ker_{C^\infty(\mathbb{R}^n)}(R.)$ is smoothly extendable.
4) The behaviour $\ker_{\mathcal{D}'(\mathbb{R}^n)}(R.)$ is distributionally extendable.
5) The D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is a free D-module of rank $p-q$, namely, $M \cong D^{1\times(p-q)}$.

Theorem 4: With the previous notations, if we consider $R = (R_1 \quad R_2) \in D^{q\times p}$, $\widetilde{R_1}$ has full row rank, and

$$\ker_{\mathcal{F}}(R.) = \{\eta = (\eta_1^T \quad \eta_2^T)^T \in \mathcal{F}^p \mid R_1\, \eta_1 + R_2\, \eta_2 = 0\},$$

then the conditions are equivalent:

1) η_1 is observable from η_2, namely, η_1 can be expressed in terms of linear differential combinations of η_2.
2) The behaviour $\ker_{C^\infty(\mathbb{R}^n)}(\widetilde{R_1}.)$ is smoothly extendable.
3) The behaviour $\ker_{\mathcal{D}'(\mathbb{R}^n)}(\widetilde{R_1}.)$ is distributionally extendable.

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